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Linear Difference and Differential Equations.

By Tomlinson Fort.

Let all the coefficients of a linear difference or differential equation have the same period ω; then if $y(x)$ is a solution of the equation $y(x+\omega)$ is also a solution. In Part I of this paper I set up equations where this fact is used as a basis for generalization. I treat equations of the second order and develop only a few fundamental facts and these primarily with their application to Part III of the paper in view although I hope that Part I will not be without interest on its own account.

Part II is minor in importance and is largely preparatory to Part III. In Part III I take up the self-adjoint boundary-value problems in one dimension. In the case of the differential equation in addition to generalization a number of new facts are brought out relative to the characteristic values for the problem. The method employed is applicable with little change to the difference equation. The results here are essentially the same as for the differential equation and so far as I know wholly new.

When a common notation does not seem practical the discussion for the difference equation is given in greater detail.

PART I.

§ 1. Definition of the Coefficients.

(a) *The Difference Equation.*

Consider the difference equation

$$L(i)y(i+1)+M(i)y(i)+N(i)y(i-1)=0, \tag{1}$$

where the independent variable i is restricted to integral values, $L(i)$, $M(i)$ and $N(i)$ defined when

$$a+1 \leqq i \leqq a+\omega \tag{2}$$

and $L(i)\,N(i) \neq 0$ at any point.

Let $a_{11}(i)$ and $a_{12}(i)$ be functions defined when $a \leqq i \leqq a+\omega+1$ and subject only to the restrictions that if $a_{12}(a) \neq 0$, $a_{11}(a+1)$ and $a_{12}(a+1)$ be not both zero and that if $a_{12}(a)=0$, $a_{11}(a) \neq 0$ and $a_{11}(a+1) \neq 0$. Let $b_{11}(i)$ and $b_{12}(i)$

be functions defined when $a+\omega \leqq i \leqq a+2\omega+1$ and subject only to the restrictions that they do not both vanish at the same point and that either $b_{11}(i)$ does not vanish at all or vanishes identically. We let

$$\left.\begin{aligned} u_n(i) &= a_{11}(i)\,y_n(i) + a_{12}(i)\,y_n(i-1), \\ v_n(i) &= b_{11}(i)\,y_n(i+1) + b_{12}(i)\,y_n(i). \end{aligned}\right\} \tag{3}$$

It is desired that L, M and N be so defined, not all identically zero, when $i=a$ *and when* $a+\omega+1 \leqq i \leqq a+2\omega+1$ *that if* y_n *is any solution of* (1), *when* $a \leqq i \leqq a+\omega+1$,

$$u_n(i) \equiv v_{\bar{n}}(i+\omega), \tag{4}$$

where $y_{\bar{n}}$ *is a solution of* (1) *also.*

Let y_1 and y_2 be two linearly independent solutions of (1) and adopt the notation $(r, s;\ p, q) \equiv r(p)s(q) - r(q)s(p)$. Regard

$$u_j(i) = v_j(i+\omega), \qquad j=1, 2, \tag{5}$$

as difference equations in $y_{\bar{1}}(i+\omega)$ and $y_{\bar{2}}(i+\omega)$, respectively. They are clearly solvable. We desire, moreover, that $y_{\bar{1}}(i+\omega)$ and $y_{\bar{2}}(i+\omega)$ be linearly independent. To this end define first $L(a)$, $M(a)$ and $N(a)$ so that $u_1(i)$ and $u_2(i)$ are linearly independent. This will be the case if

$$(u_1, u_2;\ a+1, a) \tag{6}$$

be different from zero.

If $a_{12}(a) \neq 0$, let a_{21} and a_{22} be two numbers such that

$$a_{11}(a+1)\,a_{22} - a_{12}(a+1)\,a_{21} = A \neq 0,$$

and let

$$\frac{M(a)}{N(a)} = \frac{a_{11}(a) - a_{22}}{a_{12}(a)}, \quad \frac{L(a)}{N(a)} = -\frac{a_{21}}{a_{12}(a)}.$$

Then using the difference equation, (6) reduces to

$$A\,(y_1(a+1)\,y_2(a) - y_2(a+1)\,y_1(a)),$$

which is different from zero. If $a_{12}(a)=0$, no definition of $L(a)$, $M(a)$ and $N(a)$ proves necessary.

When $b_{11} \not\equiv 0$ $y_{\bar{1}}(a+\omega)$ and $y_{\bar{2}}(a+\omega)$ are arbitrary. Let b_{21} and b_{22} be two numbers such that $b_{11}(a+\omega)\,b_{22} - b_{21}\,b_{12}(a+\omega) = B \neq 0$, and let

$$\begin{aligned} y_{\bar{1}}(a+\omega) &= b_{11}(a+\omega)\,u_1(a+1) - b_{21}\,u_1(a), \\ y_{\bar{2}}(a+\omega) &= b_{11}(a+\omega)\,u_2(a+1) - b_{21}\,u_2(a), \end{aligned}$$

whereupon by the use of equations (5)

$$\begin{aligned} \frac{M(a+\omega+1)}{L(a+\omega+1)} &= -\frac{(y_{\bar{1}}, y_{\bar{2}};\ a+\omega+2, a+\omega)}{(y_{\bar{1}}, y_{\bar{2}};\ a+\omega+1, a+\omega)} = \frac{b_{12}(a+\omega+1) - b_{21}}{b_{11}(a+\omega+1)}, \\ \frac{N(a+\omega+1)}{L(a+\omega+1)} &= \frac{(y_{\bar{1}}, y_{\bar{2}};\ a+\omega+2, a+\omega+1)}{(y_{\bar{1}}, y_{\bar{2}};\ a+\omega+1, a+\omega)} = -\frac{b_{22}}{b_{11}(a+\omega+1)}. \end{aligned}$$

Now from the difference equation $\bar{B} \cdot (y_{\bar 1},\, y_{\bar 2};\; a+\omega+1,\; a+\omega)$ equals

$$(v_{\bar 1},\, v_{\bar 2};\; a+\omega+1,\; a+\omega). \tag{7}$$

But from (5), (7) equals (6) which is different from zero. Hence $(y_{\bar 1},\, y_{\bar 2};\; a+\omega+1,\, a+\omega) \neq 0$, that is, $y_{\bar 1}(i+\omega)$ and $y_{\bar 2}(i+\omega)$ are not linearly dependent over $a \leqq i \leqq a+\omega+2$. If $b_{11}(i) \equiv 0$ the result that $(y_{\bar 1},\, y_{\bar 2};\; a+\omega+1,\; a+\omega)$ equals a constant different from zero times (6) and hence that $y_{\bar 1}(i+\omega)$ and $y_{\bar 2}(i+\omega)$ are not linearly dependent over $a \leqq i \leqq a+\omega+1$ is immediate. Moreover, no definition of $L(a+2\omega+1)$, $M(a+2\omega+1)$ and $N(a+2\omega+1)$ proves necessary.

When $i > a+\omega$, the required equation is

$$(y_{\bar 1},\, y_{\bar 2};\; i,\, i-1)\, y(i+1) - (y_{\bar 1},\, y_{\bar 2};\; i+1,\, i-1)\, y(i) \\ + (y_{\bar 1},\, y_{\bar 2};\; i+1,\, i)\, y(i-1) = 0, \tag{$\bar 1$}$$

which we adjoin to (1) as given. The coefficients are defined as desired so that if y_n is any solution there exists a solution $y_{\bar n}$ which satisfies (4). There is a possibility of $L(i)$ vanishing when $i > a+\omega+1$. If this should be the case we shall consider as solutions of (1) only linear combinations of $y_{\bar 1}$ and $y_{\bar 2}$ with constant coefficients.

(b) *The Differential Equation.*

Consider

$$L(x)y'' + M(x)y' + N(x)y = 0, \tag{1'}$$

where $L(x)$, $M(x)$ and $N(x)$ are defined and continuous when $a \leqq x \leqq a+\omega$ and accents denote differentiation, where, moreover, $\dfrac{M(x)}{L(x)}$ and $\dfrac{N(x)}{L(x)}$ are absolutely integrable over the interval $a \leqq x \leqq b$.

Let $g < a$ but $a-g$ arbitrarily small. Let $a_{11}(x)$ and $a_{12}(x)$ be absolutely integrable over $a \leqq x \leqq a+\omega$, $a_{11}(a)$ and $a_{12}(a)$ not both zero and $a'_{11}(a)$ and $a'_{12}(a)$ existent. Let $b_{11}(x)/b_{12}(x)$ be absolutely integrable when $a+\omega \leqq x \leqq a+2\omega$ and b_{11} and b_{12} both differentiable. Let, moreover,

$$u_n(x) = a_{11}(x)\, y'_n(x) + a_{12}(x)\, y_n(x),$$
$$v_n(x) = b_{11}(x)\, y'_n(x) + b_{12}(x)\, y_n(x).$$

We desire that L, M and N be defined when $g \leqq x < a$ and $a+\omega \leqq x < a+2\omega$, that b_{11} and b_{12} be defined as differentiable functions when $a+\omega-g \leqq x \leqq a+\omega$ and a_{11} and a_{12} when $g \leqq x < a$ so that if y_n is any solution of (1′) there exists a solution $y_{\bar n}$ such that when $g \leqq x \leqq a+\omega$

$$u_n(x) = v_{\bar n}(x+\omega). \tag{4'}$$

Let y_1 and y_2 be linearly independent solutions of (1′) and adopt the notation $[r(x), s(x); k, l] \equiv r^{(k)}(x)\, s^{(l)}(x) - r^{(l)}(x)\, s^{(k)}(x)$ where the superscripts denote differentiation. Regard

$$u_j(x) \equiv v_j(x+\omega), \qquad j=1, 2, \tag{5′}$$

as differential equations in $y_1(x+\omega)$ and $y_2(x+\omega)$ when $a \leqq x \leqq b$. They are clearly solvable. Then regard (5′) as linear algebraic equations in a_{11} and a_{12} when $g \leqq x \leqq a$ for any definition of b_{11} and b_{12} and of the coefficients. Since y_1 and y_2 are linearly independent, a_{11} and a_{12} are determined $g \leqq x \leqq a$ when $a-g$ is small enough. We desire, moreover, that $y_1(x+\omega)$ and $y_2(x+\omega)$ be linearly independent. This will be the case if $[y_1(x+\omega), y_2(x+\omega); 1, 0] \neq 0$ in the neighborhood of $x=a$. To this end we define the coefficients over $g \leqq x < a$ so that when $x \to a-0$,

$$[u_1(x), u_2(x); 1, 0] \to A[y_1(a), y_2(a); 1, 0], \tag{8}$$

where $A \neq 0$. First supposing that $a_{11}(a) \neq 0$ this is done by choosing two numbers a_{21} and a_{22} such that $a_{11}(a)a_{22} - a_{12}(a)a_{21} = A$, and then letting L, M and N be defined in such a way that as $x \to a-0$,

$$\frac{M(x)}{L(x)} \to \frac{a'_{11}(a) + a_{12}(a) - a_{21}}{a_{11}(a)} \text{ and } \frac{N(x)}{L(x)} \to \frac{a'_{12}(a) - a_{22}}{a_{11}(a)}.$$

Substitution shows that this definition is sufficient for (8). If $a_{11}(a) = 0$, no definition of the coefficients in (1′) when $x < a$ will prove necessary. We let $g = a$.

We next desire that as $x \to a+\omega+0$,

$$[v_1(x+\omega), v_2(x+\omega); 1, 0] \to B[y_1(a+\omega), y_2(a+\omega), 1, 0], \tag{9}$$

where $B \neq 0$. Here if $b_{11} \not\equiv 0$ due to the arbitrariness of $y_1(a+\omega)$ and $y_2(a+\omega)$ we can proceed in a manner exactly analogous to that employed in the case of the difference equation choosing two numbers b_{21} and b_{22} such that $b_{11}(a+\omega)\, b_{22} - b_{12}(a+\omega)\, b_{21} = B$. One can then immediately assure himself, and details are omitted, that it is possible to define $b_{11}(x)$ and $b_{12}(x)$ as differentiable functions over $a+\omega-g \leqq x \leqq a+\omega$, so that if y_n is any solution of (1′), $\dfrac{d}{dx} v_n(x)$ is continuous at $a+\omega$, and hence so that the limit expressed by (9) is the same if the approach is from above or below. Let b_{11} and b_{12} be so defined, the desired conclusions are then drawn as under (a). If $b_{11} \equiv 0$ (9) is immediate and obvious without special definition of $y_1(a+\omega)$ and $y_2(a+\omega)$ which are no longer arbitrary.

The required equation when $g < x \leqq a + 2\omega$ is

$$[y_1(x), y_2(x); 1, 0]\, y'' - [y_1(x), y_2(x); 2, 0]\, y' + [y_1(x), y_2(x); 2, 1]\, y = 0. \tag{1′}$$

If $[y_1(x), y_2(x); 1, 0]$ should vanish we proceed as under (a) considering as solutions only linear combinations of y_1 and y_2 with constant coefficients.

§ 2. THE CHARACTERISTIC EQUATION.

I shall now make the substitution of the letter "x" for i of (a) of section 1 and x of (b). The reasoning applies to either the difference or the differential equation and the following notation with use of x is to include both. Other notation will be explained.

We propose the question: *Does a solution of* (1) [(1′)] *exist such that*

$$\rho u_n(x) = v_n(x+\omega)$$

where ρ *is a constant?*

Let y_1 and y_2, as previously, be two linearly independent solutions. As y_1 and y_2 are linear combinations of y_1 and y_2 we write

$$v_1(x+\omega) = \alpha_{11} u_1(x) + \alpha_{12} u_2(x),$$
$$v_2(x+\omega) = \alpha_{21} u_1(x) + \alpha_{22} u_2(x).$$

Since from the definition of the coefficients the Wronskian determinant of $v_1(x)$ and $v_2(x)$ is different from zero at $a+\omega$, $\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21} \neq 0$.

Assume now the existence of a solution $y_3(x) = \beta_1 y_1(x) + \beta_2 y_2(x) \not\equiv 0$, such that

$$v_3(x+\omega) = \rho u_3(x).$$

Then

$$v_3(x+\omega) = \beta_1 v_1(x+\omega) + \beta_2 v_2(x+\omega) = \beta_1\{\alpha_{11} u_1(x) + \alpha_{12} u_2(x)\}$$
$$+\beta_2\{\alpha_{21} u_1(x) + \alpha_{22} u_2(x)\} = \rho u_3(x) = \rho\beta_1 u_1(x) + \rho\beta_2 u_2(x).$$

Transposing,

$$[\beta_1(\alpha_{11}-\rho) + \beta_2\alpha_{21}]u_1(x) + [\beta_1\alpha_{12} + \beta_2(\alpha_{22}-\rho)]u_2(x) = 0.$$

Since $y_1(x)$ and $y_2(x)$ are linearly independent, $u_1(x)$ and $u_2(x)$ are linearly independent and hence necessarily

$$\beta_1(\alpha_{11}-\rho) + \beta_2\alpha_{21} = 0,$$
$$\beta_1\alpha_{12} + \beta_2(\alpha_{22}-\rho) = 0.$$

But β_1 and β_2 are not both zero as $y_3(x) \not\equiv 0$, and consequently,

$$D_1 = (\alpha_{11}-\rho)(\alpha_{22}-\rho) - \alpha_{12}\alpha_{21} = 0. \tag{10}$$

This has been derived as a necessary condition. However, if ρ is so chosen that $D_1=0$ we can retrace the steps and see that a solution satisfying the relation $v_n(x) \equiv \rho u_n(x)$ does exist.

Regard (10) as an equation in ρ. We shall call it the characteristic equation of (1) [(1′)]. It is of the second degree and neither root is zero as $\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21} \neq 0$. There will always, then, exist at least one solution of (1) [(1′)], not identically zero, satisfying the relation $v_n(x+\omega) = \rho u_n(x)$.

If equation (10) has two distinct roots, ρ_1 and ρ_2, there exist at least two solutions, y_4 and y_5, not identically zero, satisfying respectively the relations

$$v_4(x+\omega)=\rho_1 u_4(x) \text{ and } v_5(x+\omega)=\rho_2 u_5(x).$$

Moreover, y_4 and y_5 are linearly independent for, assuming the contrary, there exist two constants μ_1 and μ_2 not both zero such that $\mu_1 y_4+\mu_2 y_5\equiv 0$; hence, such that

$$\mu_1 u_4(x)+\mu_2 u_5(x)\equiv 0 \quad\text{and}\quad \mu_1 v_4(x+\omega)+\mu_2 v_5(x+\omega)\equiv 0. \tag{11}$$

From the last

$$\mu_1\rho_1 u_4(x)+\mu_2\rho_2 u_5(x)\equiv 0. \tag{12}$$

For definiteness assume $\mu_1\not\equiv 0$ and eliminate $u_5(x)$ from (11) and (12). We get

$$\mu_1(\rho_1-\rho_2)u_4(x)\equiv 0,$$

but $u_4(x)\not\equiv 0$; for it is surely possible to choose a solution y_6 linearly independent of y_4, and hence, according to section 1, such that $u_6(x)$ is linearly independent of $u_4(x)$. But this is impossible if $u_4(x)\equiv 0$. It results that $\rho_1=\rho_2$, a contradiction.

§ 3. INDEPENDENCE OF THE CHARACTERISTIC EQUATION OF THE PARTICULAR FUNDAMENTAL SYSTEM OF SOLUTIONS CHOSEN.

Consider a second fundamental system of solutions y_7 and y_8. Then

$$v_7(x+\omega)=B_{11}u_7(x)+B_{12}u_8(x),$$
$$v_8(x+\omega)=B_{21}u_7(x)+B_{22}u_8(x).$$

The characteristic equation is

$$D_2=(B_{11}-\rho)(B_{22}-\rho)-B_{12}B_{21}=0.$$

But

$$y_7(x)=L_{11}y_1(x)+L_{12}y_2(x),$$
$$y_8(x)=L_{21}y_1(x)+L_{22}y_2(x),$$

where $L_{11}L_{22}-L_{12}L_{21}\not\equiv 0$. Hence,

$$v_7(x+\omega)=B_{11}[L_{11}u_1(x)+L_{12}u_2(x)]+B_{12}[L_{21}u_1(x)+L_{22}u_2(x)].$$

On the other hand, one can write

$$\begin{aligned}v_7(x+\omega)&=L_{11}v_1(x+\omega)+L_{12}v_2(x+\omega)\\&=L_{11}[\alpha_{11}u_1(x)+\alpha_{12}u_2(x)]+L_{12}[\alpha_{21}u_1(x)+\alpha_{22}u_2(x)].\end{aligned}$$

Equate and collect:

$$\begin{aligned}[(B_{11}L_{11}+B_{12}L_{21})-(L_{11}\alpha_{11}+L_{12}\alpha_{12})]u_1(x)+[(B_{11}L_{12}+B_{12}L_{22})\\-(L_{11}\alpha_{12}+L_{12}\alpha_{22})]u_2(x)=0.\end{aligned}$$

One can now conclude in various ways* that when $\lambda>0$ is sufficiently large a solution satisfying $c_1 y'(\bar{x})+c_2 y(\bar{x})=0$ has no node on $a+1<x\leqq a+\omega+1$ and that $\frac{\Delta y(a+\omega)}{y(a+\omega)}$ is positive and as large as we wish to make it, and that, moreover, when $\lambda<0$ is numerically sufficiently large not only are there the maximum number of nodes on $a+1<x\leqq a+\omega+1$ but that the last is as close to $a+\omega$ as we desire, that is, $\frac{\Delta y(a+\omega)}{y(a+\omega)}$ is negative and numerically as large as we wish to have it.

Moreover, the requirement that $y(\bar{x}+\omega)=0$ is equivalent to a requirement of the form $\bar{c}_1 K(a+\omega)\Delta y(a+\omega)+\bar{c}_2 y(a+\omega)=0$. We, consequently, conclude:

Let y be a solution satisfying $c_1 y'(\bar{x})+c_2 y(\bar{x})=0$. There exist values of λ, $\lambda_0>\lambda_1>\ldots\ldots>\lambda_k$ such that when $\lambda=\lambda_j$, $j=0,\ldots\ldots,k$, $y(\bar{x}+\omega)=0$ also; where, moreover, y has exactly j nodes on the interval $a+1<x\leqq a+\omega$. If $y(a+1)\neq 0$, $k=\omega-1$, if $y(a+1)=0$, $k=\omega-2$.

§ 7. EXTENSION OF FUNDAMENTAL BOUNDARY-VALUE THEOREM TO EQUATIONS OF THE TYPE SET UP IN SECTION 1.

(a) *The Difference Equation.*

Write equation (14) in the form

$$L(i,\lambda)\,y(i+1)+M(i,\lambda)\,y(i)+N(i,\lambda)\,y(i-1)=0. \tag{17}$$

Here $L(i,\lambda)$, $M(i,\lambda)$ are defined for all values of λ when $a+1\leqq i\leqq a+\omega$. We let $u_n(i)=y_n(i-1)$, $v_n(i+\omega)=d_{11}(i)\,K(a+\omega)\,\Delta y_n(i+\omega)+d_{12}(i)\,y_n(i+\omega)$, where $d_{11}(i)$ and $d_{12}(i)$ are real functions defined when $a\leqq i\leqq a+\omega+1$, $d_{11}(i)\neq 0$ or $d_{11}\equiv 0$. We then define L, M and N according to the method of section 1 so that if y_n is any solution there exists a solution $y_{\bar{n}}$ such that $u_n(i)=v_{\bar{n}}(i+\omega)$. *$u_n$ and $v_{\bar{n}}$ are to be considered functions of the continuous variable x, namely, those functions defined by the broken line graphs of $u_n(i)$ and $v_n(i)$, respectively.* We write moreover in place of the $a_{21}y_n(a+1)+a_{22}y_n(a)$ of section 1 $K(a)\,\Delta y_n(a)+b\,y_n(a)=\bar{u}_n(a+1)$ and in place of the $b_{21}y_n(a+\omega+1)+b_{22}y_n(a+\omega)$ of section 1 $d_{21}K(a+\omega)\,\Delta y_n(a+\omega)+d_{22}y_n(a+\omega)=\bar{v}_n(a+\omega)$ where $d_{11}d_{22}-d_{12}d_{21}=1$.

Let $a+1\leqq\bar{x}\leqq a+\omega+1$ and let y_n denote a particular solution, not identically zero, such that $u_n(\bar{x})=0$. Denote by c that integer such that $c<\bar{x}\leqq c+1$. Form the equation

$$\bar{L}(i,\lambda)\,y(i+1)+\bar{M}(i,\lambda)\,y(i)+\bar{N}(i,\lambda)\,y(i-1)=0, \tag{18}$$

---

* A satisfactory method of proving this theorem is by comparison with $\zeta\Delta^2 y(i)-P(i)\,y(i+1)=0$ where P is a constant chosen large numerically and positive or negative as we desire, and ζ is a constant such that $0<\zeta<K(i)$.

where $\bar{L}(i,\lambda)$, $\bar{M}(i,\lambda)$ and $\bar{N}(i,\lambda)$ are equal respectively to $L(i,\lambda)$, $M(i,\lambda)$ and $N(i,\lambda)$ when $c+1 \leqq i \leqq a+\omega$ and to $L(i-\omega-1,\lambda)$, $M(i-\omega-1,\lambda)$ and $N(i-\omega-1,\lambda)$ respectively when $a+\omega+1 \leqq i \leqq c+\omega+1$.

Now denote by y_{10} that solution of (14) such that $v_n(i+\omega)=u_{10}(i)$. Then there exists a solution of (18) which we denote by y_{11} such that $y_{11}(i)=y_{10}(i-\omega-1)$ over $a+\omega \leqq i \leqq c+\omega+2$. Therefore, $u_{11}(a+\omega+2)=v_n(a+\omega+1)$ and $u_{11}(a+\omega+1)=v_n(a+\omega)$. Due to the definition given $L(a,\lambda)$, $M(a,\lambda)$, $N(a,\lambda)$, $L(a+\omega+1,\lambda)$, $M(a+\omega+1,\lambda)$ and $N(a+\omega+1,\lambda)$, these equations reduce to

$$\left.\begin{aligned} \bar{u}_{11}(a+\omega+2) &= v_n(a+\omega), \\ u_{11}(a+\omega+2) &= \bar{v}_n(a+\omega). \end{aligned}\right\} \tag{20}$$

Hence since y_n satisfies a relation like (16) at $a+\omega$, y_{11} satisfies the same relation at $a+\omega+1$.

Solving (20) $y_{11}(a+\omega+2)=\bar{D}K(a+\omega)\Delta y_n(a+\omega)+Dy_n(a+\omega)$. We shall say that we have case α if $\bar{D}\neq 0$ and case β if $\bar{D}=0$. Apply the theorem of oscillation for y_n and we have that as λ increases from $-\infty$ to ∞ $y_{11}(a+\omega+2)$ vanishes exactly m times where

$$\begin{aligned} &\text{CASE } \alpha: && \text{(a)}\ \ y_n(c+1)\neq 0, && m=a+\omega-c, \\ & && \text{(b)}\ \ y_n(c+1)=0, && m=a+\omega-c-1; \\ &\text{CASE } \beta: && \text{(a)}\ \ y_n(c+1)\neq 0, && m=a+\omega-c-1, \\ & && \text{(b)}\ \ y_n(c+1)=0, && m=a+\omega-c-2. \end{aligned}$$

I shall next show that: *If λ is positive and sufficiently large y_{11} has no node on $a+\omega+2<x\leqq\bar{x}+\omega$ and if λ is negative and numerically sufficiently large, the maximum number namely; $c-a-2$.*

CASE α. Here $\dfrac{\Delta y_{11}(a+\omega+1)}{y_{11}(a+\omega+2)}$ approaches a limit as λ becomes infinite and one can immediately prove the theorem, as for example, as indicated in the third footnote under section 6.

CASE β. First, when λ is positive and sufficiently large a solution of (18) exists having no node* on $a+\omega+1\leqq x\leqq c+\omega+1$. Hence as the nodes of two solutions either separate each other or coincide, no solution can have more than one node on this interval. But under these circumstances, as $y_{11}(a+\omega+2)$ is proportional to $y_n(a+\omega)$, $y_{11}(a+\omega+1)$ which is also a linear combination of $K(a+\omega)\Delta y_n(a+\omega)$ and $y_n(a+\omega)$ can not be proportional to $y_n(a+\omega)$ also consistent with the equation $d_{11}(a)d_{22}-d_{21}d_{12}(a)=1$; and hence $y_{11}(a+\omega+1)$ vanishes one more time than $y_{11}(a+\omega+2)$ as λ increases from $-\infty$ to ∞, and as when $x \geqq a+\omega+1$ its nodes move to the left as λ decreases and to the right as λ increases when $\lambda=\infty$ there must remain one node of y_{11} on the interval $a+\omega+1<x<a+\omega+2$, and hence when λ is sufficiently large y_{11} can have no node on $a+\omega+2<x\leqq c+\omega+1$.

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* This will be the true of any solution having the same sign at $a+\omega+1$ and $a+\omega+2$.

Next, when λ is negative and numerically sufficiently large a solution exists having the maximum number of nodes* on $a+\omega+1\leqq x\leqq c+\omega+1$ conceivably possible, and hence y_{11} surely has on this interval at least as many nodes as one less than this maximum number. But as λ decreases from ∞ to $-\infty$ $y_{11}(a+\omega+1)$ vanishes one more time than $y_{11}(a+\omega+2)$, and hence when λ is negative and numerically sufficiently large y_{11} has no node on the interval $a+\omega+1\leqq x\leqq a+\omega+2$ and hence the maximum number of nodes on $a+\omega+2<x\leqq c+\omega+1$.

Now bearing in mind that as λ decreases nodes of y_{11} on the interval $a+\omega+1\leqq x\leqq c+\omega+1$ move continuously to the left, that

$$\frac{K(a+\omega)\,\Delta y_{11}(a+\omega)}{y_{11}(a+\omega)}\rightarrow -\infty$$

as $\lambda\rightarrow -\infty$ and that $y_{11}(\bar{x}+\omega)=u_{11}(\bar{x}+\omega+1)=u_{11}(\bar{x})=v_n(\bar{x}+\omega)$ a mere count gives the following theorem:

There exist values of λ, $\lambda_0>\lambda_1>\ldots\ldots>\lambda_k$ *such that when* $\lambda=\lambda_j$, $j=0$, $\ldots.$, k, *each solution of* (14) *satisfying the relation* $y_n(\bar{x}-1)=0$ *satisfies also the relation* $v_n(\bar{x}+\omega)=0$.

CASE α: (a) $k=\omega-1$,
(b) $k=\omega-2$;
CASE β: (a) $k=\omega-2$,
(b) $k=\omega-3$.

The notation "λ_j*" for any particular one of the values* $\lambda_0, \ldots., \lambda_k$ *will be generally used.*

(b) *The Differential Equation.*

We consider the equation

$$\frac{d}{dx}\left(K(x,\lambda)\,y'(x)\right)-G(x,\lambda)\,y(x)=0,$$

where the prime denotes differentiation and where K and G are defined and continuous for all values of λ when $a\leqq x\leqq a+\omega$ and where moreover $G(x,\lambda)$ is continuous in λ and continually increases going from $-\infty$ to ∞ as λ increases from $-\infty$ to ∞ and where $K>0$ never decreases.

---

* This will be the true of any solution having opposite signs at $a+\omega+1$ and $a+\omega+2$.

We write this equation in the form

$$L(x,\lambda)y''+M(x,\lambda)y'+N(x,\lambda)y=0$$

and define L, M and N over the interval $a+\omega<x\leqq a+2\omega$, according to the method of section 1, so that if y_n is any solution there exists a solution $y_{\bar n}$ such that $u_n(x)\equiv v_{\bar n}(x+\omega)$ where

$$u_n(x)=y_n(x) \quad \text{and} \quad v_{\bar n}(x+\omega)=d_{11}(x)K(a+\omega)y'_{\bar n}(x+\omega)+d_{12}(x)y_{\bar n}(x+\omega)$$

where d_{11} and d_{12} are given functions as in section 1 (b). The analogue of (a) of this section is then immediate. Form the equation

$$\bar{L}(x,\lambda)y''+\bar{M}(x,\lambda)y'+\bar{N}(x,\lambda)y=0, \tag{18'}$$

where $\bar L=L$, $\bar M=M$ and $\bar N=N$ when $\bar x\leqq x\leqq a+\omega$ and to $L(x-\omega,\lambda)$ etc., when $a+\omega\leqq x\leqq \bar x+\omega$ and reason as under (a).

There exist an infinite number of real values of λ; $\lambda_0>\lambda_1>\lambda_2>\ldots$, *such that when* $\lambda=\lambda_j$, $j=0,1,2,\ldots$, *all solutions satisfying* $u_n(\bar x)=0$ *satisfy also* $v_n(\bar x+\omega)=0$; $\bar x$ *any particular point* $a\leqq x\leqq a+\omega$.

§ 8. CONTINUITY PROOF.

(a) *The Difference Equation.*

It is apparent that the λ_j's of section 7 will, in general, be functions of the point $\bar x$. We write $\lambda_j(\bar x)$ and *shall prove that these functions, with the exception of the last* λ_k, *are continuous* $a+1\leqq\bar x\leqq a+\omega+1$ *and that* λ_k *is continuous over every interval,* $i<\bar x<i+1$, *where* i *is an integer becoming negatively infinite as* $\bar x$ *approaches* i *or* $i+1$.

Let ξ be any particular fixed value of $\bar x$ and let y_ξ be a particular solution, not identically zero, such that $u_\xi(\xi)=y_\xi(\xi-1)=0$. Let $\delta>0$ and let $0<\xi-\bar x<\delta$; and then let $y_{\bar x}$ be a solution satisfying the conditions $y_{\bar x}(\bar x-1)=0$ and $y'_{\bar x}(\bar x-1)=y'_\xi(\xi-1)$ where as previously the accent denotes the forward derivative. Let c be that integer such that $c<\xi-1\leqq c+1$ and let $\varepsilon>0$ be arbitrarily small. Then, if δ is small enough $|y_{\bar x}(c)-y_\xi(c)|<\varepsilon$ and $|y_{\bar x}(c+1)-y_\xi(c+1)|<\varepsilon$. But $y_\xi(a+\omega)$ and $y_\xi(a+\omega+1)$ are polynomials in $y_\xi(c)$ and $y_\xi(c+1)$; and moreover, $y_{\bar x}(a+\omega)$ and $y_{\bar x}(a+\omega+1)$ are the same polynomials in $y_{\bar x}(c)$ and $y_{\bar x}(c+1)$. As polynomials are continuous, if ε be sufficiently small $y_{\bar x}(a+\omega)$ and $y_{\bar x}(a+\omega+1)$ can be made to differ in absolute value as little as we please from $y_\xi(a+\omega)$ and $y_\xi(a+\omega+1)$, respectively.

Now consider equation (18) formed for c and the solutions of it $\bar{\bar y}_\xi$ and $\bar{\bar y}_{\bar x}$, bearing the relations to y_ξ and $y_{\bar x}$, respectively, expressed by (20) and hence where $\eta>0$ is arbitrary, if ε is small enough $|\bar{\bar y}_\xi(a+\omega+1)-\bar{\bar y}_{\bar x}(a+\omega+1)|<\eta$

and $|\bar{\bar{y}}_\xi(a+\omega+2)-\bar{\bar{y}}_{\bar{x}}(a+\omega+2)|<\eta$. Moreover, for larger values of i $\bar{\bar{y}}_\xi$ and $\bar{\bar{y}}_x$ are polynomials in these values.

Let $\lambda=\lambda_j(\xi)$, $j<k$ if $\xi=c+1$, otherwise $j\leqq k$; then a node which we call the $(j+1)$-st node of $\bar{\bar{y}}_\xi$ lies at $\xi+\omega$. Let $\theta>0$, then if η be small enough, the $(j+1)$-st node of $\bar{\bar{y}}_x$ lies a distance from $\bar{x}+\omega$ less than θ. Now hold $\bar{x}$ fast and vary λ. As nodes of $\bar{\bar{y}}_{\bar{x}}$ move continuously to the right as λ increases, and to the left as λ decreases; by a variation in λ as small as we please, θ sufficiently small, the $(j+1)$-st node of $\bar{\bar{y}}_{\bar{x}}$ can be made to move to $\bar{x}+\omega$. When it will have moved to this point, λ will have the value $\lambda_j(\bar{x})$. That is given a $\zeta>0$ we can choose a θ, then an η, then an ε, then a δ so that $|\lambda_j(\bar{x})-\lambda_j(\xi)|<\zeta$ when $0<\xi-\bar{x}<\delta$.

Now let $0\leqq\bar{x}-\xi<\bar{\delta}$ and let c be that integer such that $c\leqq\xi-1<c+1$. We form equation (18) for this point c. The corresponding theory developed with the relations between the y's and $\bar{\bar{y}}$'s now holds without the slightest change. In fact, it was quite immaterial whether we chose c that integer $c<\bar{x}-1\leqq c+1$, as we did or $c\leqq\bar{x}-1<c+1$. The repetition of the reasoning just gone through now shows that given a $\zeta>0$ it is possible to choose a $\bar{\delta}>0$ so that when $\bar{x}-\xi<\bar{\delta}$, $|\lambda_j(\bar{x})-\lambda_j(\xi)|<\zeta$.

Let $\bar{\bar{\delta}}\leqq\delta$, $\bar{\delta}$ then given a $\zeta>0$ it is possible to choose a $\bar{\bar{\delta}}>0$ such that when $|\bar{x}-\xi|<\bar{\bar{\delta}}$, $|\lambda_j(\bar{x})-\lambda_j(\xi)|<\zeta$. This establishes continuity.

The exceptional situation in the case of λ_k arises from the fact that if $\bar{x}-1$ is integral, there exists no value λ_k. We have case α (b) or case β (b). Let $\bar{x}-1$ approach an integer $c+1$ from below. Consider (18) formed for c. The value of λ which makes $\bar{\bar{y}}(a+\omega+2)$ vanish the $(a+\omega-c)$-th or the $(a+\omega-c-1)$-st time, as we have case α (b) or case β (b), becomes negatively infinite; but at all times λ is less than or equal to this value and hence itself becomes negatively infinite. Similarly, if $\bar{x}-1$ approaches an integer c from above we think of (18) where c is that integer $c\leqq\bar{x}-1<c+1$ with exactly the same results as when $\bar{x}-1$ approaches $c+1$ from below.

(b) *The Differential Equation.*

All the values of $\lambda_j(\bar{x})$ *are continuous* $a\leqq\bar{x}\leqq a+\omega$ *in the case of the differential equation.*

Here we make use of a few fundamental and well-known theorems for the differential equation and follow the general method employed in (a). There is less need for detail; and naturally no supplementary discussion for a particular λ_j is necessary as for the λ_k under (a). I shall again for the sake of brevity omit details of the proof.

PART III.

§ 9. STATEMENT OF PROBLEM.

(a) *The Difference Equation.*

Consider again

$$\Delta\{K(i,\lambda)\,\Delta y(i)\} - G(i,\lambda)\,y(i+1) = 0, \tag{21}$$

where $K(i,\lambda) > 0$ and $G(i,\lambda)$ are defined when

$$a \leqq i \leqq a+\omega \tag{22}$$

and

$$a \leqq i \leqq a+\omega-1 \tag{23}$$

respectively, where moreover K and G are continuous in the real parameter λ, when $-\infty < \lambda < \infty$ and such that as λ increases G always actually increases going from $-\infty$ to ∞ and K does not decrease approaching a limit as $\lambda \to \infty$.

We shall consider (21) subject to the conditions

$$\left.\begin{aligned} \alpha_{11} K(a)\,\Delta y(a) + \alpha_{12} y(a) &= \beta_{11} K(a+\omega)\,\Delta y(a+\omega) + \beta_{12}\, y(a+\omega), \\ \alpha_{21} K(a)\,\Delta y(a) + \alpha_{22} y(a) &= \beta_{21} K(a+\omega)\,\Delta y(a+\omega) + \beta_{22}\, y(a+\omega), \end{aligned}\right\} \tag{24}$$

where $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = \beta_{11}\beta_{22} - \beta_{21}\beta_{12}$.

It proves advisable to throw conditions (24) into the form

$$\left.\begin{aligned} K(a)\,\Delta y(a) + b\,y(a) &= d_{11} K(a+\omega)\,\Delta y(a+\omega) + d_{12}\, y(a+\omega), \\ y(a) &= d_{21} K(a+\omega)\,\Delta y(a+\omega) + d_{22}\, y(a+\omega), \end{aligned}\right\} \tag{25}$$

where $d_{11} \neq 0$, $d_{11} d_{22} - d_{21} d_{12} = 1$.

It will prove advantageous also to consider side by side with (25)

$$\left.\begin{aligned} K(a)\,\Delta y(a) + b\,y(a) &= -d_{11} K(a+\omega)\,\Delta y(a+\omega) - d_{12}\, y(a+\omega), \\ y(a) &= -d_{21} K(a+\omega)\,\Delta y(a+\omega) - d_{22}\, y(a+\omega). \end{aligned}\right\} \tag{26}$$

Let $u_n(i) = y_n(i-1)$ and $v_n(i+\omega) = d_{11} K(a+\omega)\,\Delta y_n(i+\omega) + d_{12}\, y_n(i+\omega)$. We then define the coefficients as in section 1 so that if y_n is any solution of (21) there exists a solution $y_{\bar n}$ such that $u_n(i) \equiv v_{\bar n}(i+\omega)$ and also $u_n(a) = K(a)\,\Delta y_n(a) + b\,y_n(a) = \bar u_n(a+1)$ and $v_n(a+\omega+1) = d_{21} K(a+\omega)\,\Delta y_n(a+\omega) + d_{22}\, y_n(a+\omega) = \bar v_n(a+\omega)$. It is only necessary to replace the a_{21}, a_{22}, b_{21} and b_{22} of section 1 by $K(a)$, $-K(a)+b$, $d_{21}K(a+\omega)$ and $-d_{21}K(a+\omega)+d_{22}$, respectively, and proceed as there.

The existence of a solution of (21) *not identically zero satisfying* (25) *now becomes necessary and sufficient for the existence of a solution not identically zero satisfying identically the relation*

$$u_n(i) = v_n(i+\omega),$$

and the existence of a solution not identically zero satisfying (26) *becomes necessary and sufficient for the existence of a solution not identically zero satisfying identically the relation*

$$u_n(i) = -v_n(i+\omega).$$

The sufficiency of these conditions is immediate. The necessity is proven thus. Suppose that there exists a solution satisfying (25). Denote this by y_1. Then

$$\begin{aligned} K(a)\Delta y_1(a)+b\,y_1(a) &= d_{11}K(a+\omega)\Delta y_1(a+\omega)+d_{12}y_1(a+\omega)\\ y_1(a) &= d_{21}K(a+\omega)\Delta y_1(a+\omega)+d_{23}y_1(a+\omega). \end{aligned}$$

We know that there exists a solution y_{I} such that $u_1(i)=v_{\mathrm{I}}(i+\omega)$. Then

$$\begin{aligned} K(a)\Delta y_1(a)+b\,y_1(a) &= d_{11}K(a+\omega)\Delta y_{\mathrm{I}}(a+\omega)+d_{12}y_{\mathrm{I}}(a+\omega)\\ y_1(a) &= d_{21}K(a+\omega)\Delta y_{\mathrm{I}}(a+\omega)+d_{22}y_{\mathrm{I}}(a+\omega), \end{aligned}$$

and consequently, y_{I} is identical with y_1 at $a+\omega$ and $a+\omega+1$ and hence at all points.

We now define $u_n(i)$ and $v_n(i)$ as functions of the continuous variable x, as has previously been done; namely, as those functions defined by the broken line graphs of $u_n(i)$ and $v_n(i)$, respectively. It results that the satisfaction of $u_n(x)\equiv v_n(x+\omega)$ by a solution of (21), not identically zero, is necessary and sufficient for the satisfaction of (25) by such a solution, and the satisfaction of $u_n(x)\equiv -v_n(x+\omega)$ is necessary and sufficient for the satisfaction of (26).

If values of λ exist such that (21) and (25) are simultaneously satisfied by a function of i, not identically zero, we shall speak of them as values l; and if values exist so that (21) and (26) are simultaneously satisfied by a function of i, not identically zero as values of l'. The solutions of (21) not identically zero satisfying (25) or (26) as the case may be will be called corresponding solutions.

(b) *The Differential Equation.*

We consider here again

$$\frac{d}{dx}\{K(x,\lambda)\,y'(x)\}-G(x,\lambda)\,y(x)=0,$$

where $K(x,\lambda)>0$ and $G(x,\lambda)$ for all values of the real parameter λ are defined and continuous when

$$a\leqq x\leqq a+\omega, \tag{22'}$$

and where K and G are continuous in λ, and such that as λ increases G increases, constantly going from $-\infty$ to ∞ and K does not ever decrease.

We consider (21′) subject to

$$\left.\begin{aligned} K(a)\,y'(a) &= d_{11}K(a+\omega)\,y'(a+\omega)+d_{12}\,y(a+\omega)\\ y(a) &= d_{21}K(a+\omega)\,y'(a+\omega)+d_{22}\,y(a+\omega), \end{aligned}\right\} \tag{25'}$$

where $d_{11}d_{22}-d_{12}d_{21}=1$ and also side by side with these subject to

$$\left.\begin{aligned} K(a)\,y'(a) &= -d_{11}K(a+\omega)\,y'(a+\omega)-d_{12}\,y(a+\omega)\\ y(a) &= -d_{21}K(a+\omega)\,y'(a+\omega)-d_{22}\,y(a+\omega). \end{aligned}\right\} \tag{26'}$$

Suppose $d_{21} \neq 0$.* Let $u_n(x) = y_n(x)$, $v_n(x+\omega) = d_{21} K(a+\omega) \Delta y_n(i+\omega) + d_{22} y(i+\omega)$.

We proceed in a manner similar to that employed under (a) of this section, defining the coefficients of the differential equation over $g \leqq x < a$ and $a+\omega < x \leqq a+2\omega$, etc., according to section 1 so that the identical satisfaction of the relation $u_n(x) = v_n(x+\omega)$ by a solution of (21′) not identically zero is necessary and sufficient for the satisfaction of (25′) by the same solution. The identical satisfaction of $u_n(x) = -v_n(x+\omega)$ is then also necessary and sufficient for the satisfaction of (26′) by the same solution. We let

$$\bar{u}_n(a) = y'_n(a) \text{ and } \bar{v}_n(a+\omega) = \frac{1}{K(a)} [d_{11} K(a+\omega) y'(a+\omega) + d_{12} y(a+\omega)].$$

§ 10. TO PROVE THAT THE MAXIMA† AND MINIMA OF THE VALUES λ_j ARE VALUES l OR l'.

Under the term maximum (minimum) we include the case that the function is a constant or has a constant value over a neighborhood.

The constant term of the characteristic equation is unity since A and B of section 1 are equal.

Suppose $\lambda_j(\bar{a})$ *a maximum. To prove that when* $\lambda = \lambda_j(\bar{a})$ $\rho = \pm 1$.

Assume that this were not the case then $\rho_1 \neq \rho_2$ as $\rho_1 \rho_2 = 1$.

Denote by y_1 and y_2 two solutions, neither identically zero, such that $\rho_1 u_1(x) = v_1(x+\omega)$ and $\rho_2 u_2(x) = v_2(x+\omega)$. Then

$$u_1(x) v_2(x+\omega) - u_2(x) v_1(x+\omega) = (\rho_1 - \rho_2) u_1(x) \cdot u_2(x).$$

Let $y_n(x)$ be a solution such that $u_n(\bar{a}) = v_n(\bar{a}+\omega) = 0$, $y_n \not\equiv 0$; in the case of the difference equation $a+1 \leqq \bar{a}$ and in the case of the differential equation $a \leqq \bar{a}$. Suppose $y_n = c_1 y_1 + c_2 y_2$, then

$$0 = c_1 u_1(a) + c_2 u_2(a),$$
$$0 = c_1 \rho_1 u_1(a) + c_2 \rho_2 u_2(a).$$

As c_1 and c_2 are not both zero $(\rho_1 - \rho_2) u_1(\bar{a}) \cdot u_2(\bar{a}) = 0$. But $\rho_1 \neq \rho_2$ and hence $u_1(\bar{a}) \cdot u_2(\bar{a}) = 0$. They are not both zero and excluding $\bar{a}$ itself in some neighborhood of $\bar{a}$ neither is zero.‡ Suppose for definiteness $u_1(\bar{a}) = 0$; then y_1 is essentially real, for write $y_1 \equiv y_j + y_k\sqrt{-1}$, where y_j and y_k are real. Then $u_j(\bar{a}) = u_k(\bar{a}) = 0$ and hence y_j and y_k are proportional. Consequently, $y_1 \equiv c y_j$. Let $y_1 \equiv y_j$. It results that ρ_1 is real, whence ρ_2 is also. Hence y_2 is essentially real and will be considered real. $u_1(x)$ changes sign at $\bar{a}$, and hence the determinant (27) changes sign at $\bar{a}$ also.

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* In case $d_{21} = 0$ we simply let $v_n(x+\omega) = D_{21}(x) y'_n(x+\omega) + D_{22}(x) y_n(x+\omega)$, where $D_{21}(a+\omega) = 0$, $D_{22}(a+\omega) = d_{22}$, $D'_{21}(a+\omega) + D_{22}(a+\omega) = \frac{d_{11}}{K(a)}$ and $D'_{22}(a+\omega) = \frac{d_{12}}{K(a)}$. Then let y_{I} and y_{II} be so chosen, first and second derivatives existent, that, if $D_{21}(x) y'_{\mathrm{I}}(x+\omega) + D_{22}(x) \bar{y}_1 = y_1$, $D_{21}(x) y'_{\mathrm{II}}(x-\omega) + D_{22}(x) y_{\mathrm{II}} = y_2$, D_{21} and D_{22} together with their derivatives at $a+\omega$ take on values as stated and are existent at other points. This can be done as is seen by solving these equations for D_{21} and D_{22}. Having made one determination of D_{21} and D_{22} regard them as fixed. The problem now is essentially the same as if $d_{21} \neq 0$.

† The existence of extremes will be discussed in section 14.

‡ In the case of the difference equation $u_n(x) = y_n(x-1)$ and in the case of the differential equation $u_n(x) = y_n(x)$.

Let y_3 and y_4 be any two real linearly independent solutions.

$$\left.\begin{aligned} y_3 &= k_{11} y_1 + k_{12} y_2, \\ y_4 &= k_{21} y_1 + k_{22} y_2. \end{aligned}\right\} \tag{28}$$

$$\left.\begin{aligned} \Delta(x) &\equiv u_3(x) \cdot v_4(x+\omega) - u_4(x) \cdot v_3(x+\omega) \\ &= (k_{11} k_{22} - k_{12} k_{21}) (u_1(x) v_2(x+\omega) - u_2(x) v_1(x+\omega)). \end{aligned}\right\} \tag{29}$$

It results that $\Delta(x)$ changes sign at $\bar{a}$ and that if $0 < \delta' < \varepsilon$, sufficiently small, $\Delta(\bar{a}+\delta') \cdot \Delta(\bar{a}-\delta') < 0$.

Under the assumption that $\lambda_j(\bar{a})$ is a maximum, I shall derive a contradiction to this statement.

Denote by y_5 and y_6, respectively, solutions such that $u_5(\bar{a}-\delta') = 0$ and $u_6(\bar{a}+\delta') = 0$ and such that at $\bar{a}-\delta'$ and $\bar{a}+\delta'$, respectively, the forward slopes of u_5 and u_6 are the same as that of u_1 at $\bar{a}$.

In the case of the differential equation let $c = \bar{a}$ and in the case of the difference equation let c be that integer, $c < \bar{a} - 1 \leq c + 1$. Form equation (18) [(18′)] of section 7 for this point c. There exist solutions $\bar{\bar{y}}_1$, $\bar{\bar{y}}_5$ and $\bar{\bar{y}}_6$ such that

$$\begin{aligned} v_1(\bar{a}+\omega) &= \bar{\bar{y}}_1(\bar{a}+\omega), \\ v_5(\bar{a}+\omega-\delta') &= \bar{\bar{y}}_5(\bar{a}+\omega-\delta'), \\ v_6(\bar{a}+\omega+\delta') &= \bar{\bar{y}}_6(\bar{a}+\omega+\delta'). \end{aligned}$$

A root of $\bar{\bar{y}}_1$ lies at $\bar{a}+\omega$. If δ' is taken sufficiently small roots of $\bar{\bar{y}}_5$ and $\bar{\bar{y}}_6$ can be made as close to $\bar{a}+\omega$ as we please. Due to the maximum, however, the situation will be somewhat as here illustrated:

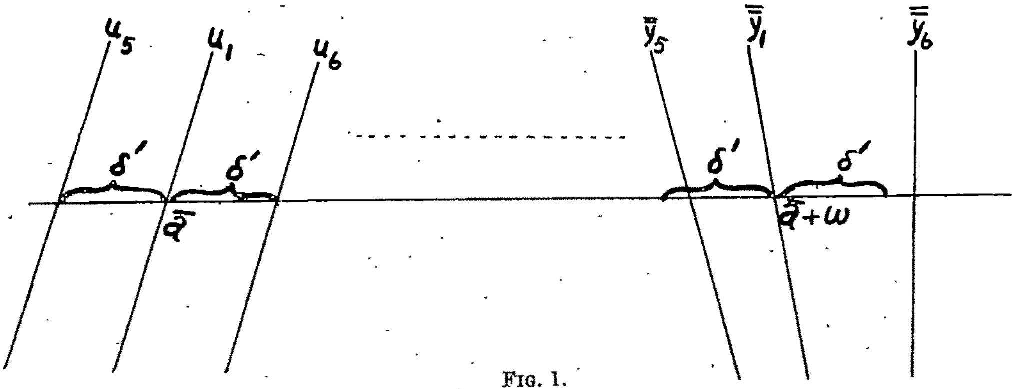

FIG. 1.

We conclude immediately

$$u_5(\bar{a}\pm\delta') v_6(\bar{a}+\omega\pm\delta') - u_6(\bar{a}\pm\delta') \cdot v_6(\bar{a}+\omega\pm\delta') \geq 0. \tag{30}$$

One can readily assure himself that another arrangement of the figure will not alter this conclusion. Refer now to (29) and we have that $\Delta(\bar{a}+\delta') \cdot \Delta(\bar{a}-\delta) \geqq 0$, the desired contradiction.

§ 11. SUPPOSE THAT WHEN $\lambda=\Lambda, \rho_1=\rho_2$ AND THAT y_7 IS A SOLUTION OF (21), [(21′)] SATISFYING (25) [(25′)] OR (26) [(26′)] SUCH THAT u_7 HAS A ROOT AT $\bar{a}$;* TO PROVE THAT Λ IS NECESSARILY AN EXTREME OF A λ_j, THE TERM EXTREME BEING USED AS PREVIOUSLY.

Under the assumption that $\rho_1=\rho_2$ there exists one solution of (21), y_7 such that $\rho u_7(x)=v_7(x+\omega)$ and a second lineary independent solution y_8 such that $u_8(x+\omega)=C_{21}u_7(x)+\rho u_8(x)$.

$$D(x)=u_7(x)v_8(x+\omega)-u_8(x)v_7(x+\omega)=C_{21}\rho^2[u_7(x)]^2,$$

which obviously never changes sign and does not vanish unless $u_7(x)=0$. We know $u_7(x)$ actually changes sign at $\bar{a}$ and hence when $\delta'>0$ is small enough $D(\bar{a}+\delta') \cdot D(\bar{a}-\delta')>0$.

Assume that Λ is not an extreme of a λ_j. Refer to the determinant (30) discussed in previous section. The corresponding figure now is

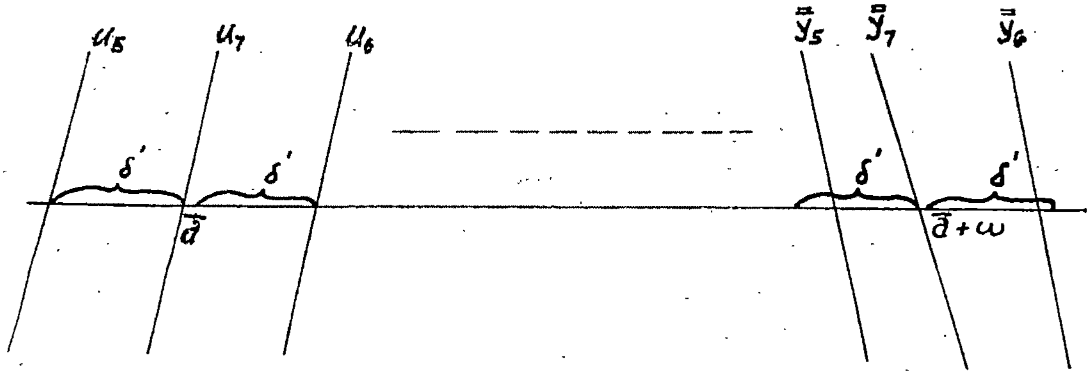

FIG. 2.

from which one concludes that D which is a constant times

$$(u_8(x)v_4(x+\omega)-u_4(x)v_3(x+\omega))$$

does not satisfy the relation $D(\bar{a}+\delta') \cdot D(\bar{a}-\delta')>0$. This is a contradiction and proves the theorem.

§ 12. λ_j A CONSTANT.

Suppose λ_j a constant over some interval $x' \leqq x \leqq x''$. Then by the proof of the previous section λ_j is a value l or a value l'; and, where $\bar{a}$ is any point

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[*] In the case of the difference equation $\bar{a}>a+v$ and in the case of the differential equation $a>a+\delta$, where δ is as in section 9 (b).

of the interval $x' \leqq x \leqq x''$, a solution satisfying $u_n(\bar{a}) = 0$ is a solution of the system consisting of (21) [(21′)] and (25) [(25′)] or of (21) [(21′)] and (26) [(26′)]. But for definite points $\bar{a}$ sufficiently close together corresponding solutions satisfying $u_n(\bar{a}) = 0$ are linearly independent. Hence, when $\lambda = \lambda_j$ all solutions of (21) [(21′)] satisfy one of these systems. λ_j is then a constant over the whole interval of its definition.

Conversely, if all solutions of (21) [(21′)] satisfy (25) [(25′)] or (26) [(26′)], the corresponding value of λ is necessarily a value λ_j. For, if $\bar{a}$ is any point on the interval $a \leqq x \leqq a + \omega$, those solutions satisfying $u_n(\bar{a}) = 0$ satisfy $v_n(\bar{a} + \omega) = 0$ also.

§ 13. WHEN λ_j IS NOT A CONSTANT TO PROVE ALL THE MAXIMA OF λ_j EQUAL AMONG THEMSELVES AND ALL THE MINIMA EQUAL AMONG THEMSELVES.

Let $\lambda_j(c)$ not be an extreme. Suppose that $\lambda_j(\bar{x})$ has two minima less than $\lambda_j(c)$. Denote these by $\bar{l}_{j-1}$ and $\bar{\bar{l}}_{j-1}$, $\bar{\bar{l}}_{j-1} < \bar{l}_{j-1}$. Let $\bar{x}$ decrease from c, λ_j can never get less than l_{j-1} for as soon as it equals $\bar{l}_{j-1}$ suppose at $\bar{c}$ then there exists a solution not identically zero of (21), [(21′)] satisfying (25) [(25′)] or (26) [(26′)] which we denote by y_7 where $u_7(c) = 0$.

Let y_9 be a solution, not identically zero, satisfying $u_9(c) = 0$. Consider $v_7(x + \omega)\, u_9(x) - v_9(x + \omega)\, u_7(x)$. This is a constant times $D(x)$ of section 11. Consequently, it vanishes only when $u_7(x)$ vanishes. But it vanishes at $\bar{c}$ a root of $u_9(x)$. Then by section 11 $\bar{c}$ being a root of $u_7(x)$ is an extreme of λ_j. From the nature of the case it is a minimum; that is, λ_j begins to increase again and the existence of $\bar{\bar{l}}_{j-1}$ is impossible.

§ 14. THE EXISTENCE OF EXTREMES OF THE λ_j'S.

(a) *The Difference Equation.*

Suppose $\lambda_j(\bar{x})$ $j \geqq 1$ not a constant. Let $\lambda = \lambda_j(a+1)$ and let y_j be a particular solution, not identically zero, satisfying the relation $y_j(a) = 0$. y_j has has at least one node on the interval $a < x \leqq a + \omega$. Let a_1 be the first such root to the right of a.

THEOREM: $\lambda_j(a_1 + 1) = \lambda_j(a+1)$ *and* λ_j *respectively increases or decreases from* a_1 *as* λ_j *increases or decreases from* a.

PROOF: Let $\lambda = \lambda_j(a+1)$. It can be immediately shown as was done in section 10 that $\rho u_j(x) \equiv v_j(x + \omega)$, $\rho \neq 0$ a constant. Consequently, $\lambda_j(a_1 + 1) = \lambda_j(a+1)$. Next let $0 < \bar{x} - a < \delta$, where δ is small, and let $y_{\bar{x}}$ be a solution such that $y_{\bar{x}}(\bar{x}) = 0$ and $y'_{\bar{x}}(\bar{x}) = y'_j(a)$, where the accent denotes the forward derivative. Suppose λ_j increases from a and let $\lambda = \lambda_j(a+1)$. As the nodes of y_j and $y_{\bar x}$ separate each other the first node of $y_{\bar x}$ to the right of $\bar x$ is to the right of a_1. Let λ increase to $\lambda_j(\bar x+1)$ and this node moves still farther to the right. Suppose that when $\lambda=\lambda_j(\bar x+1)$ it lies at $\bar a_1$ and that $\bar a_1-a_1=\eta$. η is arbitrarily small, δ being at our disposal. Now, as in the case of $a+1$ and a_1+1, $\lambda_j(\bar a_1+1)=(\lambda_j(\bar x+1)$. But $\lambda_j(\bar x+1)>\lambda_j(a+1)$. Hence, $\lambda_j(\bar a_1+1)>\lambda_j(a_1+1)$ which is what we desired to prove. The following theorem results:

$\lambda_j(\bar x)$, $j\geqq 1$ *has always at least one minimum and at least one maximum.*

As has been remarked these extremes are the roots of the solutions, y_n, not identically zero, satisfying $u_n(x)\equiv\pm v_n(x+\omega)$. The exact number of these roots on $a+1<x\leqq a+\omega$ will be discussed in the theorem of oscillation, section 17.

(b) *The Differential Equation.*

Only obvious formal changes of the treatment given under (a) are necessary.

$\lambda_j(\bar x)$, $j\geqq 1$ *has always at least one minimum and at least one maximum.*

§ 15. TO DISTINGUISH THE TWO CASES.

(a) *The Difference Equation.*

Assume $\lambda_j=\lambda_j(\bar a)$. There exists a solution, not identically zero, which we shall call y_j satisfying the relations $y_j(\bar a-1)=0$, $v_j(\bar a+\omega)=0$. Let c be that integer such that $c<\bar a-1\leqq c+1$ and consider (18) formed for this point.

When λ has any value whatever let y_n denote a solution of (21) satisfying the conditions $y_n(c)=y_j(c)$, $y_n(c+1)=y_j(c+1)$. Then according to section 7 (a) when $a+\omega<x\leqq c+\omega$, $\bar v_n(x+\omega)=y_{11}(x+\omega)$. Let $\eta>0$ be very small.

When $\lambda>0$ is sufficiently large $y_n(c+1+\eta)$ and $y_n(a+\omega+1)$ are of the same sign. Moreover, as shown under section 7, $y_{11}(a+\omega+2)$ and $y_{11}(c+\omega)$ are of the same sign; that is, $v_n(a+\omega+2)$ and $v_n(c+\omega)$ are of the same sign. *To determine, if* $y_n(a+\omega+1)$, *that is,* $u_n(a+\omega+2)$ *is of the same or opposite sign to* $v_n(a+\omega+2)$.

$v_n(a+\omega+2)=y_{10}(a+1)$ [section 7] and as $v_n(a+\omega)=K(a)\Delta y_{10}(a)+by_{10}(a)$ and $\bar v_n(a+\omega)=y_{10}(a)$, $K(a)\bar v_n(a+\omega+2)=\bar D K(a+\omega)\Delta y_n(a+\omega)+Dy_n(a+\omega)$.

CASE α: $\bar D=d_{11}+d_{21}(K(a)-b)\neq 0$. Since $\lim\limits_{\lambda=\infty}\dfrac{K(a+\omega)\Delta y_n(a+\omega)}{y_n(a+\omega)}=\infty$ if λ is sufficiently large $v_n(a+\omega+2)$ will have the sign of $\bar D K(a+\omega)\Delta y_n(a+\omega)$, that is, of $\bar D y_n(c+1+\eta)$.

CASE β: $\overline{D}=0$. $v_n(a+\omega+2)$ has the sign of

$$Dy_n(a+\omega)=[d_{12}+d_{22}(K(a)-b)]\,y_n(a+\omega),$$

that is of $Dy_n(c+1+\eta)$, for large values of λ.

Now let λ decrease. The number j is the number of times that $v_n(\bar{a}+\omega)$ has already changed sign. *It results that* $u_j(c+2+\eta)=y_j(c+1+\eta)=\pm v_j(c+\omega+2+\eta)$, *and hence,* $\rho=\pm 1$ *as follows:*

CASE α: j even $\overline{D}>0$ $\rho=-1$, $\overline{D}<0$ $\rho=1$;
j odd $\overline{D}>0$ $\rho=1$, $\overline{D}<0$ $\rho=-1$.
CASE β: j even $D>0$ $\rho=-1$, $D<0$ $\rho=1$;
j odd $D>0$ $\rho=1$, $D<0$ $\rho=-1$.

(b) *The Differential Equation.*

Proceed just as under (a). Suppose $\lambda=\lambda_j(\bar{a})$ an extreme of λ_j. Then there exists a solution y_j, not identically zero, satisfying the relations $u_j(\bar{a})=y_j(\bar{a})=0$ and $v_j(\bar{a}+\omega)=0$. Let y_n be a solution when λ has any value satisfying the relations $y_n(\bar{a})=y_j(\bar{a})$ and $y'_n(\bar{a})=y'_j(\bar{a})$. Then by section 7 when $a+\omega\leq x\leq\bar{a}+\omega$, $v_n(x+\omega)=y_{11}(x+\omega)$. Moreover, when $\lambda>0$ is sufficiently large and $\bar{\delta}>0$ small, $y_n(\bar{a}+\bar{\delta})$ and $y'_n(a+\omega)$ have the same sign, and $y_{11}(\bar{a}+\omega)$ and $y_{11}(a+\omega+\bar{\delta})$ are of the same sign; that is, $v_n(a+\omega)$ and $v_n(\bar{a}+\omega)$ are of the same sign. To determine then if $u_n(\bar{a}+\bar{\delta})$ and $v_n(\bar{a}+\omega+\delta)$ are of the same sign, is equivalent to determining if $y'_n(a+\omega)$ and $v_n(a+\omega)$ are of the same sign. This can be immediately done as $v_n(a+\omega)=d_{21}K(a+\omega)\,y'_n(a+\omega)+d_{22}y_n(a+\omega)$ and $\dfrac{K(a+\omega)\,y'_n(a+\omega)}{y_n(a+\omega)}\to\infty$ as $\lambda\to\infty$.

CASE (α) $d_{21}\neq 0$: $v_n(a+\omega)$ has the same or opposite sign as $y'_n(a+\omega)$ as $d_{21}\gtrless 0$.

CASE (β) $d_{21}=0$: $v_n(a+\omega)$ has the same or opposite sign as $y'_n(a+\omega)$ as $d_{22}\gtrless 0$.

Completing the reasoning as under (a), we have

CASE α: j even $d_{21}>0$ $\rho=-1$, $d_{21}<0$ $\rho=1$;
j odd $d_{21}>0$ $\rho=1$, $d_{21}<0$ $\rho=-1$.
CASE β: j even $d_{22}>0$ $\rho=-1$, $d_{22}<0$ $\rho=1$;
j odd $d_{22}>0$ $\rho=1$, $d_{22}<0$ $\rho=-1$.

§ 16. SOLUTIONS WITHOUT ROOTS.

(a) *The Difference Equation.*

We desire now to study the existence of values l and l', where the corresponding solutions of (21) have no node on $a+1<x<a=\omega$.

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Let $\lambda > 0$ be very large and let y_1 and y_2 be two solutions determined by the conditions

$$y_1(a) = 0, \qquad y_1(a+1) = \frac{1}{K(a)},$$

$$y_2(a) = \frac{1}{K(a)}, \qquad y_2(a+1) = 0.$$

Then (see section 2),

$$v_1(a+\omega) = a_{11}u_1(a) + a_{12}u_2(a) = a_{11}\overline{u}_1(a+1) + a_{12}\overline{u}_2(a+1) = (a_{11} - a_{12})\left(1 - \frac{b}{K(a)}\right),$$

$$v_1(a+\omega+1) = \overline{v}_1(a+\omega) = a_{11}u_1(a+1) + a_{12}u_2(a+1) = \frac{a_{12}}{K(a)}.$$

Hence,

$$a_{11} = v_1(a+\omega) + (K(a) - b)\overline{v}_1(a+\omega).$$

Similarly,

$$a_{22} = K(a)\overline{v}_2(a+\omega).$$

The characteristic equation is

$$\rho^2 - (a_{11} + a_{22})\rho + 1 = 0.$$

$$a_{11} + a_{22} = \overline{D}K(a+\omega)\Delta y_1(a+\omega) + Dy_1(a+\omega) + K(a)d_{21}K(a+\omega)\Delta y_2(a+\omega) + d_{22}K(a)y_2(a+\omega).$$

CASE (α): $\overline{D} \neq 0$.

If $\lambda > 0$ is sufficiently large, the sign of $a_{11} + a_{22}$ is that of $\overline{D}$. For, write the difference equation in the form

$$K(i+1)\Delta y(i+1) = K(i)\Delta y(i) + G(i)y(i+1),$$

and the facts, that $y_2(a+1) = 0$ and $y_1(a+1) = \frac{1}{K(a)}$, and that G becomes infinite with λ shows that by taking λ large enough we can make

$$\frac{K(a+1)\,\Delta y_1(a+1)}{|K(a+1)\Delta y_2(a+1)|}$$

as large we please. An easy induction process shows that the same thing is true of $K(a+\omega)\Delta y_1(a+\omega)$ and $|K(a+\omega)\Delta y_2(a+\omega)|$. Moreover, if y_n is any particular solution whatever, not identically zero, with fixed values at a and $a+1$, $K(a+\omega)\Delta y_n(a+\omega)/y_n(a+\omega)$ becomes positively infinite with λ. Hence, $\overline{D}K(a+\omega)\Delta y_1(a+\omega)$ will determine the sign of $a_{11} + a_{22}$. In addition when λ is large enough and $i \geq a$, $K(i)\Delta y_n(i)$ increases with i, and, consequently, is greater than zero at $a+\omega$. We conclude as desired that the sign of $a_{11} + a_{22}$ is that of $d_{11} + d_{21}K(a)$ when λ is sufficiently large.

CASE (β). $\overline{D} = 0$.

As $K(a+\omega)\Delta y_2(a+\omega)/y_2(a+\omega)$ becomes infinite with λ and as $\Delta y_2(a+\omega) < 0$ for sufficiently large values of λ and as from the fact that $d_{11}d_{22} - d_{12}d_{21} = 1$, $d_{21}D = -1$, the sign of $a_{11} + a_{22}$ is the sign of D.

Suppose now that for large values of λ $a_{11}+a_{22}>0$; that is, $\overline{D}>0$ or $D>0$ as we have cases (α) or (β). Denote the roots of the characteristic equation by ρ_1 and ρ_2. If λ is sufficiently large, $a_{11}+a_{22}$ is as large as we please, and hence, $\rho_1>0$, say, is large and $\rho_2>0$ small. Let λ decrease $(a_{11}+a_{22})$ is continuous in λ, and hence, ρ_1 and ρ_2 are continuous in λ. Consequently, $\rho_1=\rho_2=1$ before $\rho_1=\rho_2=-1$; that is, there exists a value l larger than the largest value l'. But by section 15 the maximum value of $\lambda_0(\overline{x})$, if such exists, is a value l'. *Hence, if λ_0 has extremes the solutions corresponding to this largest value l can have no node when $a \leqq x \leqq a+\omega$.*

There can not also be a value l' such that a corresponding solution has no node when $a \leqq x \leqq a+\omega$ for by the methods of section 15 solutions without roots always correspond definitely either to a value l or to a value l'.

If for large values of λ $a_{11}+a_{12}<0$ the above situation is exactly reversed and if $\lambda_0(\overline{x})$ has extremes there exists a value l' and no value l such that a corresponding solution has no node, $a \leqq x \leqq a+\omega$.

Consider now the case that $a_{11}+a_{22}>0$ for large positive values of λ. *We wish to prove that there can not be two distinct values l such that corresponding solutions have no node on $a+1 \leqq x \leqq a+\omega$.* I find it here necessary to assume K independent of λ an assumption not made in the preceding. Suppose there were two such values l and l'. Denote corresponding solutions by y and $\overline{y}$ and assume that y and $\overline{y}$ are of the same sign when $a+1<x<a+\omega$. We have from (15)

$$\begin{aligned} K(a+\omega)(y,\overline{y};\ a+\omega,\ a+\omega+1)-K(a)(y,\overline{y};\ a,\ a+1) \\ =\sum_{i=a}^{a+\omega-1}(\overline{G}(i)-G(i))\overline{y}(i+1)y(i+1). \end{aligned} \tag{31}$$

But as y and $\overline{y}$ both satisfy (25),

$$K(a+\omega)(y,\overline{y};\ a+\omega,\ a+\omega+1)=K(a)(y,\overline{y},\ a,\ a+1). \tag{32}$$

(31) *and* (32) *are inconsistent under the hypothesis that $y(x)$ and $\overline{y}(x)$ preserve the same sign when $a+1<x<a+\omega$.*

If $a_{11}+a_{22}<0$ for large values of λ, we show in exactly the same way that there can not be two values l' such that corresponding solutions have no node.

(b) *The Differential Equation.*

Let y_1 and y_2 be two solutions determined by the conditions

$$y_2(a)=0, \qquad K(a)y_2'(a)=1,$$

$$y_1(a)=\frac{1}{K(a)}, \qquad K(a)y_1'(a)=0.$$

Then
$$v_1(a+\omega) = a_{11} u_1(a) + a_{12} u_2(a) = \frac{a_{11}}{K(a)},$$
$$\bar{v}_2(a+\omega) = a_{21} \bar{u}_1(a) + a_{22} \bar{u}_2(a) = \frac{a_{22}}{K(a)}.$$

Hence,
$$a_{11}+a_{22} = [v_1(a+\omega)+\bar{v}_2(a+\omega)]K(a) = [d_{21}K(a+\omega)y_1'(a+\omega) + d_{22}y_1(a+\omega) + d_{11}K(a+\omega)y_2'(a+\omega) + d_{12}y_2'(a+\omega)]K(a).$$

Now let $\lambda=\bar{\lambda}$ be so large that y_1 and y_2 increase throughout the interval $a<x\leqq a+\omega$,
$$\frac{d}{dx}\frac{y_2}{y_1} = \frac{y_1y_2'-y_2y_1'}{y_1^2} = \frac{1}{K(a)K(x)y_1^2},$$
$$\frac{y_2(\alpha)}{y_1(\alpha)} = \frac{1}{K(a)}\int_a^\alpha \frac{dx}{K(x)y_1^2} \quad a\leqq\alpha\leqq a+\omega.$$

Now let λ increase, when $a<x\leqq a+\omega$ $\frac{1}{y_1^2}$ approaches zero as λ becomes infinite and $1/K(x)$ does not increase. Let $K(x)>\zeta$, $a\leqq x\leqq a+\omega$, when $\lambda\geqq\bar{\lambda}$.

Then $\int_a^\alpha \frac{dx}{K(x)y_1^2} \leqq \frac{1}{\zeta}\int_a^\alpha \frac{dx}{y_1^2}$. But $\frac{1}{(y_1(x))^2}\leqq 1$, and when $a<x\leqq a+\omega$ it approaches zero. It results that
$$\lim_{\lambda=\infty}\frac{y_2(\alpha)}{y_1(\alpha)} = \lim_{\lambda=\infty}\frac{1}{K(a)}\int_a^\alpha \frac{dx}{K(x)(y_1(x))^2} = 0.$$
This approach is uniform in α since $\int_a^{a+\omega}\frac{dx}{K(x)(y_1(x))^2} > \int_a^\alpha \frac{dx}{K(x)(y_1(x))^2}$ when $a+\omega<\alpha$.

Now, moreover, from the differential equation
$$\frac{K(a+\omega)y_2'(a+\omega)}{K(a+\omega)y_1'(a+\omega)} = \frac{\int_a^{a+\omega}G(x)y_2(x)\,dx+1}{\int_a^{a+\omega}G(x)y_1(x)\,dx}.$$

Hence, $\frac{y_2'(a+\omega)}{y_1'(a+\omega)} \leqq \int_a^{a+\omega}\frac{y_2(x)}{y_1(x)}\,dx + \frac{1}{\int_a^{a+\omega}G(x)y_1(x)\,dx}$ when $\lambda\geqq\bar{\lambda}$. Consequently, $\lim\limits_{\lambda=\infty}\frac{y_2'(a+\omega)}{y_1'(a+\omega)}=0$. Hence, since $\lim\limits_{\lambda=\infty}\frac{K(a+\omega)\,y_k'(a+\omega)}{y_k(a+\omega)}=\infty$; $k=1, 2$, the sign of $a_{11}+a_{22}$ is that of d_{21} if $d_{21}\neq 0$. If $d_{21}=0$ $d_{22}d_{11}=1$ and the sign of $a_{11}+a_{22}$ is that of d_{22}.

The reasoning is now continued exactly as under (a) with the same result. In order to give the uniqueness proof, using the analogous formula to (31) however, *it is necessary to assume K independent of λ.* There has been no call for this assumption earlier in the paper.

Some Singularities of a Contact Transformation.

By W. V. Lovitt.

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§ 1. *Introduction.*

A number of papers have appeared in recent years on the singularities of point transformations. This paper discusses some of the singularities of a contact transformation.

The method of treatment in this paper will follow that of Urner in the *Transactions of the American Mathematical Society*, Vol. XIII, No. 2, pp. 232–264, April, 1912, on Certain Singularities of Point Transformations in Space of Three Dimensions.

In § 2 are given the preliminary hypotheses and notation. In § 3 some preliminary formulae are established. The discussion of the singularities divides itself into three cases, viz.: when the matrix of the Jacobian of the transformation is of rank two, one, or zero. These three cases are treated in §§ 4, 5, and 6, respectively. The irregularities which appear are in the order of contact of the transformed curves. In the latter part of § 5 is given an interesting geometric interpretation of the situation when the matrix of the Jacobian is of rank 1.

§ 2. *Preliminary Hypotheses and Notation.*

Let there be given a real contact transformation

$$x_1=X(x, y, p), \quad y_1=Y(x, y, p), \quad p_1=P(x, y, p). \tag{1}$$

We assume that X, Y, and P are single-valued and continuous when $(x, y, p) =(x_0, y_0, p_0)$ and that they possess with respect to these three variables partial derivatives of the first order also continuous there. The explicit use of any derivative of X, Y, or P shall imply its existence and continuity at the point under consideration, as well as the possession of these properties by all the other derivatives of X, Y, and P of the same and lower orders.

For a proper contact transformation, that is, one which is not a mere point transformation, not both X_p and Y_p can be zero. When we speak of a contact transformation we shall be understood to mean a proper contact transformation.

From the equations (1) we obtain

$$dy_1-p_1dx_1=\rho(x, y, p)\,(dy-pdx).$$

We shall assume that

$$\rho(x_0, y_0, p_0)=0,$$

but that

$$\rho(x, y, p)\not\equiv 0.$$

A lineal-element (x_0, y_0, p_0) for which $\rho=0$ is called a singular or critical lineal-element. It will presently appear that the vanishing of ρ is equivalent to the vanishing of the Jacobian J of the transformation.

We shall study the effect of the above transformation upon the curve, or more precisely the union of lineal-elements, given by

$$x=f(t),\quad y=g(t),\quad y'=g'(t)/f'(t)\equiv h(t),\tag{2}$$

where f and g are continuous, single-valued functions of the real variable t which we assume have continuous derivatives of orders 1, 2, ..., $k+1$ at the point for which $t=t_0$. With these assumptions h is a single-valued function of the real variable t, continuous together with its derivatives of order 1, 2, ..., k at the point for which $t=t_0$, except for the zeros of $f'(t)$. We exclude from consideration those values of the parameter t for which $f'(t)=0$. Then we have certainly one of the quantities $f'(t_0)$, $g'(t_0)$, $h'(t_0)$ different from zero. The curve into which (2) is carried by the transformation is

$$\left.\begin{aligned} x_1&=X[f(t), g(t), h(t)]\equiv f_1(t),\\ y_1&=Y[f(t), g(t), h(t)]\equiv g_1(t),\\ p_1&=P[f(t), g(t), h(t)]\equiv h_1(t). \end{aligned}\right\}\tag{3}$$

Here f_1, g_1, and h_1 are single-valued and continuous and have continuous derivatives of orders 1, 2, ..., k for $t=t_0$ whenever f, g, and h are so endowed, provided X, Y, and P have continuous partial derivatives of orders 1, 2, ..., k at the point (x_0, y_0, p_0).

§ 3. *Preliminary Formulae.*

We shall need in the course of this paper to make use of some known relations between X, Y, and P when equations (1) represent a contact transformation. From Lie* we take the formulae

$$[XY]\equiv\begin{vmatrix} X_x, & X_y, & X_p\\ Y_x, & Y_y, & Y_p\\ -p, & 1, & 0\end{vmatrix}\equiv 0,\quad [PX]\equiv\begin{vmatrix} X_x, & X_y, & X_p\\ P_x, & P_y, & P_p\\ p, & -1, & 0\end{vmatrix}\equiv\rho,\quad [PY]\equiv\begin{vmatrix} Y_x, & Y_y, & Y_p\\ P_x, & P_y, & P_p\\ p, & -1, & 0\end{vmatrix}\equiv\rho P.$$

$$Y_x=PX_x-p\rho,\quad Y_y=PX_y+\rho,\quad Y_p=PX_p.\tag{4}$$

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* "Berührungstransformationen," pp. 08 and 73.

From these relations it follows at once that J and ρ vanish simultaneously. Indeed

$$J \equiv \begin{vmatrix} X_x, & X_y, & X_p \\ Y_x, & Y_y, & Y_p \\ P_x, & P_y, & P_p \end{vmatrix} \equiv \rho[PX] \equiv \rho^2.$$

Let A_1, A_2, A_3; B_1, B_2, B_3; C_1, C_2, C_3 denote, respectively, the cofactors of the elements of the first, second, and third rows in the Jacobian J. Then J may be written in any one of the three following forms:

$$\left.\begin{aligned} &X_xA_1+X_yA_2+X_pA_3, \\ &Y_xB_1+Y_yB_2+Y_pB_3, \\ &P_xC_1+P_yC_2+P_yC_3. \end{aligned}\right\} \tag{5}$$

while

$$[XY] \equiv 0, \quad [PX] \equiv \rho, \quad [PY] \equiv \rho P \tag{6}$$

may be written in the respective forms

$$pC_1-C_2 \equiv 0, \quad pB_1-B_2 \equiv -\rho, \quad pA_1-A_2 \equiv \rho P. \tag{7}$$

From (5) and (7) we obtain when $J=0$

$$\begin{cases} (X_x+pX_y)A_1+X_pA_3=\rho PX_y, \\ (X_x+pX_y)A_2+pX_pA_3=-\rho PX_x, \end{cases}$$

$$\begin{cases} (Y_x+pY_y)B_1+Y_pB_3=-\rho Y_y, \\ (Y_x+pY_y)B_2+pY_pB_3=\rho Y_x, \end{cases}$$

$$\begin{cases} (P_x+pP_y)C_1+P_pC_3=0, \\ (P_x+pP_y)C_2+pP_pC_3=0. \end{cases}$$

From equations (4), (6), and (7) and the definitions of A_i, B_i, C_i we find:

$$\begin{aligned} &B_1 \equiv X_pP_y-X_yP_p \equiv \rho_p, \\ &B_2 \equiv X_xP_p-X_pP_x=\rho+p\rho_p=\rho+pB_1, \\ &B_3 \equiv X_yP_x-X_xP_y=-\rho_x-p\rho_y, \\ &C_1 \equiv X_yY_p-X_pY_y=-\rho X_p, \\ &C_2 \equiv X_pY_x-X_xY_p=-p\rho X_p=pC_1, \\ &C_3 \equiv X_xY_y-X_yY_x=\rho(X_x+pX_y), \\ &A_1 \equiv Y_yP_p-Y_pP_y=\rho P_p-P\rho_p=\rho P_p-PB_1, \\ &A_2=p\rho P_p-P(\rho+p\rho_p)=p\rho P_p-PB_2, \\ &A_3=P(\rho_x+p\rho_y)-\rho(P_x+pP_y)=-\rho(P_x+pP_y)-PB_3. \end{aligned}$$

When $\rho=0$ we obtain

$$C_1=C_2=C_3=0$$

$$\begin{aligned} &B_1=\rho_p; \quad && B_2=p\rho_p; \quad && B_3=-(\rho_x+p\rho_y), \\ &A_1=-PB_1; \quad && A_2=-PB_2; \quad && A_3=-PB_3. \end{aligned}$$

Whence

$$B_2 = pB_1; \quad A_2 = -pPB_1; \tag{8}$$
$$X_p B_3 = -B_1(X_x + pX_y); \quad X_p A_3 = PB_1(X_x + pX_y).$$

§ 4. *Matrix of Rank 2.*

We shall first discuss the singularities of the transformation (1) when the rank of the matrix of the determinant J is 2. From the equations of transformation we compute*

$$y_1''' = \frac{[PX]}{(X_x + X_y y' + X_{y'} y'')^3} y''' + \frac{U(x, y, y', y'')}{(X_x + X_y y' + X_{y'} y'')^3},$$

where the fraction whose numerator is $U(x, y, y', y'')$ stands for all of those terms which do not contain y'''. But

$$[PX] = \rho = 0.$$

Then, in general, all curves through the point (x_0, y_0) and with a common value of y' and y'' will be transformed into curves having at the transformed point (x_1^0, y_1^0) a common value of y_1', y_1'', y_1'''. From the above formula we find

$$y_1^{(k)} = \frac{[PX]}{(X_x + X_y y' + X_{y'} y'')^3} y^{(k)} + U_k(x, y, y', \ldots, y^{(k-1)}),$$

where U_k stands for all of those terms which do not contain $y^{(k)}$. From this formula we see that, in general, all curves through the point (x_0, y_0) and with a common value for $y', y'', \ldots, y^{(k-1)}$ will be transformed into curves having at the transformed point (x_1^0, y_1^0) a common value for $y', y'', \ldots, y^{(k)}$. This gives us the

THEOREM: *If a contact transformation*

$$x_1 = X(x, y, p), \quad y_1 = Y(x, y, p), \quad p_1 = P(x, y, p)$$

satisfies the conditions

a) *X, Y, P are of class $C^{(k)}$ at the point P_0 for which $(x, y, p) = (x_0, y_0, p_0)$;*

b) *The functional determinant J vanishes for*

$$x = x_0, \quad y = y_0, \quad z = z_0,$$

but

$$J(x, y, p) \not\equiv 0;$$

c) *The matrix of the determinant J is of rank 2, then curves with contact of order $k-1$ at the point P_0 are transformed into curves with contact of order k at the transformed point $P_0'(x_1^0, y_1^0)$, i. e., the order of contact is increased by one.*

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* Lie, "Berührungstransformationen," p. 86.

It may happen, however, that the expression for y_1'',

$$y_1''=\frac{P_x+P_y y'+P_p y''}{X_x+X_y y'+X_p y''}$$

is indeterminate, whence of course $y_1^{(k)}$ becomes indeterminate for $k\geq 2$. In this case, in order to determine the curvature of the transformed curve, we proceed as follows. Let us state here that the slope of the transformed curves is always determined by the equations (1) of transformation. We have

$$dx:dy:dy'=f'(t_0):g'(t_0):h'(t_0),$$

and for the transformed curve

$$dx_1:dy_1:dy_1'=f_1'(t_0):g_1'(t_0):h_1'(t_0)$$

where

$$\left.\begin{aligned} f_1' &= X_x f'+X_y g'+X_p h', \\ g_1' &= Y_x f'+Y_y g'+Y_p h', \\ h_1' &= P_x f'+P_y g'+P_p h'. \end{aligned}\right\} \qquad (9)$$

From these formulae we can determine the slope and radius of curvature of the transformed curve unless

$$f_1'(t_0)=g_1'(t_0)=h_1'(t_0)=0.$$

Since the matrix of the determinant of J is of rank 2, there exists one direction and radius of curvature for the original curve and only one for which the determination (9) fails. This is defined by the ratios

$$f':g':h'=I_1:I_2:I_3,$$

where I stands for that one of the letters A, B, C for which $I_1:I_2:I_3\neq 0:0:0$. We shall call this the *critical curvature* for the lineal-element (x_0, y_0, p_0).

Let us now assume that the curve (2) has the critical curvature. Suppose, however, that not all of the quantities $f_1''(t_0)$, $g_1''(t_0)$, $h_1''(t_0)$ are zero. Then

$$f_1'(t)\equiv f_1''[t_0+\theta_1(t-t_0)](t-t_0), \qquad (0<\theta_1<1)$$

with similar formulae for g_1', h_1'. Hence allowing t to approach t_0, we see that

$$dx_1:dy_1:dp_1=f_1''(t_0):g_1''(t_0):h_1''(t_0).$$

Placing $f'=kI_1$, $g'=kI_2$, $h'=kI_3$ we have

$$\left.\begin{aligned} f_1'' &= X_x f''+X_y g''+X_p h''+k^2U^2(X,I), \\ g_1'' &= Y_x f''+Y_y g''+Y_p h''+k^2U^2(Y,I), \\ h_1'' &= P_x f''+P_y g''+P_p h''+k^2U^2(P,I), \end{aligned}\right\} \qquad (10)$$

where

$$U^2(FI)\equiv I_1^2F_{xx}+I_2^2F_{yy}+I_3^2F_{pp}+2I_2I_3F_{yp}+2I_3I_1F_{px}+2I_1I_2F_{xy}.$$

If we make a change of parameter, $t'=kt$, the functions f, g, h go over into such functions $\bar{f}$, $\bar{g}$, $\bar{h}$ of t' that when $t'=kt_0$, $\bar{f}=I_1$, $\bar{g}=I_2$, $\bar{h}=I_3$. Then, let us drop dash and prime, merely supposing the necessary changes already made in the functions f, g, h as they stand. This done we have a right to set k in equations (10) equal to unity and $U^2(X, I)$, $U^2(Y, I)$, $U^2(P, I)$ become constants which we shall designate, respectively, by ρ_2, σ_2, τ_2. The curvature of the transformed curve is now determined by $f_1'' : h_1''$ unless

$$f_1'' : h_1'' = 0 : 0.$$

In this case the equations

$$\left.\begin{aligned} X_x f'' + X_y g'' + X_p h'' + \rho_2 &= 0, \\ P_x f'' + P_y g'' + P_p h'' + \tau_2 &= 0, \end{aligned}\right\} \tag{11}$$

are independent as equations for the determination of f'', g'', h''. At the point under consideration

$$I_1 = f', \; I_2 = g', \; I_3 = h' = \frac{f'g'' - g'f''}{f'^2} \tag{12}$$

We may now choose $(I_1, I_2, I_3 = B_1, B_2, B_3)$. By hypothesis $f' \neq 0$. Then $B_1 \neq 0$ and, hence, it follows from equations (8) that $B_2 \neq 0$. We are thus furnished with a third equation in f'', g'', h'':

$$B_2 f'' - B_1 g'' = -B_3 B_1^2. \tag{13}$$

The three equations (11) and (13) are independent and from them can be found the values of f'', g'', h'' for which the determination of the curvature of the transformed curve from equations (10) fails.

From the last of equations (12) by successive differentiations we obtain

$$h^{(j)} = \frac{f'g^{(j)} - g'f^{(j)}}{f'^2} + H_j, \tag{14}$$

where H_j depends upon the derivatives of f, g, of orders $1, 2, \ldots, j-1$. When $f' = B_1$ and $g' = B_2$ the equation (14) may be written in the form

$$B_2 f^{(j)} - B_1 g^{(j)} = B_1^2 (H_j - h^{(j-1)}). \tag{15}$$

The General Case. Let us now suppose that there exists a curve with the lineal-element (x_0, y_0, p_0) for which $f^{(j)}$, $h^{(j)}$ have such values that

$$f_1^{(j)}(t_0) = h_1^{(j)}(t_0) = 0,$$

for $j = 1, 2, \ldots, k-1$, but $[f_1^{(k)}]^2 + [h_1^{(k)}]^2 \neq 0$. Then let us join the point $t = t_0$ on the transformed curve with the point (x_1, y_1) for which $t = t_0 + \varepsilon$, by means of a secant. Then

$$x_1 - x_1^0 = f_1(t_0 + \varepsilon) - f_1(t_0) = \frac{\varepsilon^k}{k!} f_1^{(k)}[t_0 + \theta_1 \varepsilon], \quad (0 < \theta_1 < 1),$$

with similar formulae for $y_1 - y_1^0$ and $p_1 - p_1^0$. Now, if we allow ε to approach zero, we obtain for the determination of the curvature of the transformed curve

$$dx_1 : dy_1' = dx_1 : dp_1 = f_1^{(k)}(t_0) : h_1^{(k)}(t_0).$$

If there exists a curve satisfying these requirements, then we shall say that (x_0, y_0, p_0) is a *singular lineal-element* of order $k-1$ at least. The order will be exactly $k-1$ if there is no curve for which

$$f_1^{(j)} = h_1^{(j)} = 0 \quad \text{for} \quad j = 1, 2, \ldots, k.$$

That is, the order of the singularity is precisely one unit less than the order of the highest derivatives which must be used to find the curvature of the transform of every curve with the lineal-element (x_0, y_0, p_0). Assuming for the moment that we have to do with a singular lineal-element of order $k-1$, let us designate as *critical curves* that family of curves which yield the values of $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ necessary to make

$$f_1^{(j)} = h_1^{(j)} = 0 \quad \text{for} \quad j = 1, 2, \ldots, k-1.$$

By differentiating equations (9) $j-1$ times we get

$$\left.\begin{aligned} f_1^{(j)}(t) &= X_x f^{(j)}(t) + X_y g^{(j)}(t) + X_p h^{(j)}(t) + \rho_j, \\ g_1^{(j)}(t) &= \qquad\qquad\qquad \text{``} \qquad\qquad\qquad + \sigma_j, \\ h_1^{(j)}(t) &= \qquad\qquad\qquad \text{``} \qquad\qquad\qquad + \tau_j. \end{aligned}\right\} \tag{16}$$

The functions ρ_j, σ_j, τ_j depend on the derivatives of X, Y, P of orders $2, \ldots, j$ and on the derivatives of f, g, h of orders $1, 2, \ldots, j-1$. Then the critical curves have the curvature of their corresponding transformed curves determined by $f_1^{(k)}(t_0) : h_1^{(k)}(t_0)$. In computing these we must evaluate ρ_k, τ_k and these depend upon the derivatives of f, g, h of orders $1, 2, \ldots, k-1$, taken at $t = t_0$. The values of these derivatives are determined uniquely by the equations

$$\left.\begin{aligned} &f_1^{(j)} = \quad h_1^{(j)} = 0, \\ &B_2 f^{(j)} - B_1 g^{(j)} = B_1^2 (H_j - h^{(j-1)}), \qquad (j = 1, 2, \ldots, k-1). \end{aligned}\right\} \tag{15}$$

It is clear that all of the critical curves have with each other contact of order k at least. It is also clear from equations (16), if we put $j = k$, that the critical curves which have contact of order $k+1$ are transformed into curves with second order contact. Let us now assign arbitrary values to $f_1^{(k)}$, $g_1^{(k)}$, $h_1^{(k)}$, then the first and last of equations (16) and equations (15), all written for $j = k$, are three independent equations for the determination of $f^{(k)}$, $g^{(k)}$, $h^{(k)}$. We may now summarize our results as follows:

THEOREM: *All of those curves for which* $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ *vanish, but for which* $[f^{(j+1)}]^2 + [g^{(j+1)}]^2 + [h^{(j+1)}]^2 \neq 0$ *are transformed into curves with a common* *direction and curvature. The curvatures of the transformed curves for different values of j are in general different.*

§ 5. *Matrix of Rank* 1.

We will now consider the case in which the matrix of the Jacobian:

$$J \equiv \begin{vmatrix} X_x, & X_y, & X_p \\ Y_x, & Y_y, & Y_p \\ P_x, & P_y, & P_p \end{vmatrix}$$

of the transformation (1) is of rank 1. Let X_ι, Y_ι, P_ι be a column of elements, not all of which vanish at (x_0, y_0, p_0), while i_x, i_y, i_p is a row possessing this property. The union of lineal-elements (2) is transformed by means of equations (1) into the union (3) and tangent curves go into tangent curves. If we desire to find the curvature of the transformed curve we find that it becomes indeterminate in form if, and only if, both f_1' and h_1' vanish simultaneously. Our problem is to evaluate this indeterminate form under the different circumstances which may arise.

In general, the direction and curvature of the transformed curve can be found from

$$dx_1 : dy_1 : dy_1' = f_1'(t_0) : g_1'(t_0) : h_1'(t_0),$$

where

$$\left.\begin{aligned} f_1' &= X_x f' + X_y g' + X_p h', \\ g_1' &= Y_x f' + Y_y g' + Y_p h', \\ h_1' &= P_x f' + P_y g' + P_p h'. \end{aligned}\right\} \qquad (9)$$

The determination of these ratios will fail if, and only if, such a curve be taken that

$$X_x f' + X_y g' + X_p h' = 0. \qquad (17)$$

All other curves then will be transformed into curves for which

$$dx_1 : dy_1 : dy_1' = X_\iota : Y_\iota : P_\iota,$$

and we may choose $X_\iota, Y_\iota, P_\iota = X_p, Y_p, P_p$. For $X_p \neq 0$, otherwise from the equation $Y_p - P X_p = 0$ we would have $Y_p = 0$ and the transformation would reduce to a mere point transformation which has been excluded. The equation (17) may be written in the form

$$X_p y'' + X_y y' + X_x = 0. \qquad (18)$$

Since $X_p \neq 0$, this equation determines y'' and thus a definite radius of curvature is determined for certain plane curves with the common lineal-element (x_0, y_0, p_0). *If we avoid curves with the radius of curvature thus determined, and having the common lineal-element* (x_0, y_0, p_0), *then equations* (9) *determine for us the direction and curvature of the transformed curves.*

We are considering only those values of p for which $J=0$ and the matrix of J is of rank 1. Under these conditions we find $\rho_x+p\rho_y=0$. This yields a second homogeneous relation between f', g', h':

$$f'\rho_x+g'\rho_y=0. \tag{19}$$

From equations (17) and (19) we find

$$\begin{aligned} f' &= -\lambda X_p\rho_y \equiv \lambda\alpha, \\ g' &= \lambda X_p\rho_x \equiv \lambda\beta, \\ h' &= \lambda(X_x\rho_y - X_y\rho_x) \equiv \lambda\gamma, \end{aligned}$$

where α, β, γ are defined by these equations and λ is a factor of proportionality. The curvature of the transformed curve is then determined unless

$$f':g':h'=\alpha:\beta:\gamma.$$

Any curve for which $f':g':h'=\alpha:\beta:\gamma$ will be spoken of as a curve with the *critical curvature*. If a curve has this critical curvature, then, in general, the direction and curvature of the transformed curve can be found from f_1'', g_1'', h_1''. Placing $f'=\lambda\alpha$, $g'=\lambda\beta$, $h'=\lambda\gamma$, we have

$$\left.\begin{aligned} f_1'' &= X_xf''+X_yg''+X_ph''+\lambda^2U^2(X, \alpha, \beta, \gamma), \\ g_1'' &= Y_xf''+Y_yg''+Y_ph''+\lambda^2U^2(Y, \alpha, \beta, \gamma), \\ h_1'' &= P_xf''+P_yg''+P_ph''+\lambda^2U^2(Y, \alpha, \beta, \gamma), \end{aligned}\right\} \tag{20}$$

where

$$U^2(F, \alpha, \beta, \gamma)=\alpha^2F_{xx}+\beta^2F_{yy}+\gamma^2F_{pp}+2\beta\gamma F_{yp}+2\gamma\alpha F_{px}+2\alpha\beta F_{xy}.$$

If we make a change of parameter $t'=\lambda t$, the functions f, g, h go over into such functions $f, \bar{g}, \bar{h}$ of t' that when $t'=\lambda t_0$, $\bar{f}'=\alpha$, $\bar{g}'=\beta$, $\bar{h}'=\gamma$. Then let us drop dash and prime merely supposing the necessary changes already made in the functions f, g, h as they stand. This done we have a right to set λ in equations (20) equal to unity. Let us now put

$$\begin{aligned} \rho_2 &= U^2(X, \alpha, \beta, \gamma), \\ \sigma_2 &= U^2(Y, \alpha, \beta, \gamma), \\ \tau_2 &= U^2(P, \alpha, \beta, \gamma). \end{aligned}$$

We now consider ρ_2, σ_2, τ_2 as completely determined. The variable portions of the right members of (20) as f'', g'', h'' take on all possible values, preserve throughout, the ratios $X_i:Y_i:P_i$ so that

$$\begin{aligned} f_1'' &= \lambda X_i+\rho_2, \\ g_1'' &= \lambda Y_i+\sigma_2, \\ h_1'' &= \lambda P_i+\tau_2. \end{aligned}$$

Elimination of λ will give the following three equations:

$$\begin{aligned} Y_i(f_1''-\rho_2)-X_i(g_1''-\sigma_2)&=0, \\ P_i(g_1''-\sigma_2)-Y_i(h_1''-\tau_2)&=0, \\ X_i(h_1''-\tau_2)-P_i(f_1''-\rho_2)&=0, \end{aligned}$$

only two of which are independent. Between these two, the terms independent of f_1'', g_1'', h_1'' may be eliminated to yield a homogeneous linear relation. In particular, if $Y_i\neq 0$, this equation will be

$$(P_i\sigma_2-Y_i\tau_2)(Y_if_1''-X_ig_1'')-(Y_i\rho_2-X_i\sigma_2)(P_ig_1''-Y_ih_1'')=0. \tag{21}$$

It will be noticed that (21) is satisfied by putting

$$f_1'':g_1'':h_1''=X_i:Y_i:P_i, \tag{22}$$

whatever be the values of ρ_2, σ_2, τ_2.

The curvature of the transformed curve is now determined unless $f_1''=g_1''=h_1''=0$. We state this result as follows:

THEOREM: *All curves for which f', g', h', f'', g'', h'' have values which render $f_1'=g_1'=h_1'=0$, but for which not all f_1'', g_1'', h_1'' are zero are transformed into tangent curves at the transformed point (x_1^0, y_1^0) and all the transformed curves have the same radius of curvature at the transformed point.*

The vanishing of the three quantities

$$Y_i\rho_2-X_i\sigma_2,\quad P_i\sigma_2-Y_i\tau_2,\quad X_i\tau_2-P_i\rho_2$$

is the condition that the three equations

$$\left.\begin{aligned} X_xf''+X_yg''+X_ph''+\rho_2&=0, \\ Y_xf''+Y_yg''+Y_ph''+\sigma_2&=0, \\ P_xf''+P_yg''+P_ph''+\tau_2&=0, \end{aligned}\right\} \tag{23}$$

be consistent. If, then, we have

$$Y_i\rho_2-X_i\sigma_2=P_i\sigma_2-Y_i\tau_2=X_i\tau_2-P_i\rho_2=0,$$

there exist curves having the critical curvature, such that $f_1''=g_1''=h_1''=0$, and, therefore, our determination of curvature for the transformed curve is no longer valid.

Let us suppose that f'', g'', h'' have such values that equations (23) are satisfied. No two of these equations are independent as the matrix of J is now one. The curvature of the transformed curve will be given by $f_1''':g_1''':h_1'''$, where

$$\left.\begin{aligned} f_1'''&=X_xf'''+X_yg'''+X_ph'''+\rho_3, \\ g_1'''&=Y_xf'''+Y_yg'''+Y_ph'''+\sigma_3, \\ h_1'''&=P_xf'''+P_yg'''+P_ph'''+\tau_3, \end{aligned}\right\} \tag{24}$$

where

$$\begin{aligned}\rho_3 \equiv\, & f''\,[3X_{xx}f'+3X_{xy}g'+3X_{xp}h'] \\ & +g''\,[Y_{yx}f'+Y_{yy}g'+Y_{yp}h'+2X_{yy}g'+2X_{yp}h'+2X_{xy}f'] \\ & +h''\,[P_{xp}f'+P_{yp}g'+P_{pp}h'+2X_{pp}h'+2X_{yp}g'+2X_{xp}f']+\bar{\rho}_3,\end{aligned}$$

where $\bar{\rho}_3$ is a function of f', g', h', and third derivatives of X, Y, P, with similar expressions for σ_3, τ_3, viz., expressions which are *linear* in f'', g'', h''.

Equations (24) as f''', g''', h''', take on all possible values may be written in the form

$$\begin{aligned} f_1''' &= \lambda_3 X_i + \rho_3, \\ g_1''' &= \lambda_3 Y_i + \sigma_3, \\ h_1''' &= \lambda_3 P_i + \tau_3, \quad \lambda_3 \text{ being variable.}\end{aligned}$$

Elimination of λ_3 will give the following three equations:

$$\begin{aligned} Y_i(f_1'''-\rho_3)-X_i(g_1'''-\sigma_3) &= 0, \\ P_i(g_1'''-\sigma_3)-Y_i(h_1'''-\tau_3) &= 0, \\ X_i(h_1'''-\tau_3)-P_i(f_1'''-\rho_3) &= 0,\end{aligned}$$

only two of which are independent. Between these two, the terms independent of f_1''', g_1''', h_1''', may be eliminated, to yield a homogeneous linear relation. In particular, if $Y_i \neq 0$, this equation will be

$$(P_i\sigma_3-Y_i\tau_3)\,(Y_if_1'''-X_ig_1''')-(Y_i\rho_3-X_i\sigma_3)\,(P_ig_1'''-Y_ih_1''')=0.$$

It will be noticed that this equation is satisfied by putting

$$f_1''' : g_1''' : h_1''' = X_i : Y_i : P_i \tag{25}$$

whatever be the values of ρ_3, σ_3, τ_3. If

$$Y_i\rho_3-X_i\sigma_3=P_i\sigma_3-Y_i\tau_3=X_i\tau_3-P_i\rho_3=0, \tag{26}$$

there exist curves having the given critical curvature such that $f_1'''=g_1'''=h_1'''=0$, and, therefore, our determination of curvature of the transformed curve from (24) is no longer valid. Equations (26) are *linear* equations in f'', g'', h'', and one of them is a consequence of the other two. There are two additional relations between f'', g'', h'':

$$X_xf''+X_yg''+X_ph''+\rho_2=0$$

and

$$B_2f''-B_1g''=-B_3B_1^2.$$

We have, thus, four independent equations for the determination of f'', g'', h''. In general, they cannot be satisfied. In case that these equations are satisfied f'', g'', h'' are uniquely determined and ρ_3, σ_3, τ_3 become certain fixed constants. The curvature of the transformed curve is now determined from $f_1^{(4)}$, $g_1^{(4)}$, $h_1^{(4)}$.

General Case. We now proceed to the discussion of the general case. Let us suppose that there exists a curve with the lineal-element (x_0, y_0, p_0) for which $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ have such values that

$$f_1^{(j)}(t_0)=g_1^{(j)}(t_0)=h_1^{(j)}(t_0)=0, \qquad (j=1, \ldots, k-1),$$

but

$$[f_1^{(k)}]^2+[g_1^{(k)}]^2+[h_1^{(k)}]^2 \neq 0.$$

Then the direction of the transformed curve will be given by

$$\left.\begin{aligned} f_1^{(k)}&=X_xf^{(k)}(t)+X_yg^{(k)}(t)+X_ph^{(k)}(t)+\rho_k, \\ g_1^{(k)}&=Y_xf^{(k)}(t)+Y_yg^{(k)}(t)+Y_ph^{(k)}(t)+\sigma_k, \\ h_1^{(k)}&=P_xf^{(k)}(t)+P_yg^{(k)}(t)+P_ph^{(k)}(t)+\tau_k. \end{aligned}\right\} \tag{27}$$

The functions ρ_k, σ_k, τ_k, depend upon the derivatives of X, Y, P, of orders $2, \ldots, k$, and on the derivatives of f, g, h, of orders $1, 2, \ldots, k-1$, and are *linear* in $f^{(k-1)}$, $g^{(k-1)}$, $h^{(k-1)}$.

In computing $f_1^{(k)}(t_0)$, $g_1^{(k)}(t_0)$, $h_1^{(k)}(t_0)$, we must evaluate ρ_k, σ_k, τ_k, and these depend upon the derivatives of f, g, h, of orders $1, 2, \ldots, k-1$ taken at $t=t_0$. The values of these derivatives are uniquely determined by the equations

$$\begin{gathered} Y_t\rho_j-X_t\sigma_j=P_t\sigma_j-Y_t\tau_j=X_t\tau_j-P_t\rho_j=0, \\ f_1^{(j)}=g_1^{(j)}=h_1^{(j)}=0, \qquad (j=1, 2, \ldots, k-1). \end{gathered}$$

Equations (27), as $f^{(k)}$, $g^{(k)}$, $h^{(k)}$ take on all possible values, may be written in the form

$$\begin{aligned} f_1^{(k)}&=\lambda_kX_t+\rho_k, \\ g_1^{(k)}&=\lambda_kY_t+\sigma_k, \\ h_1^{(k)}&=\lambda_kP_t+\tau_k, \quad \lambda_k \text{ being variable.} \end{aligned}$$

Elimination of λ_k will give the following three equations:

$$\begin{aligned} Y_t(f_1^{(k)}-\rho_k)-X_t(g_1^{(k)}-\sigma_k)&=0, \\ P_t(g_1^{(k)}-\sigma_k)-Y_t(h_1^{(k)}-\tau_k)&=0, \\ X_t(h_1^{(k)}-\tau_k)-P_t(f_1^{(k)}-\rho_k)&=0, \end{aligned}$$

only two of which are independent. Between these two, the terms independent of $f_1^{(k)}$, $g_1^{(k)}$, $h_1^{(k)}$, may be eliminated, to yield a homogeneous linear relation. In particular, if $Y_t \neq 0$, this equation will be

$$(P_t\sigma_k-Y_t\tau_k)(Y_tf_1^{(k)}-X_tg_1^{(k)})-(Y_t\rho_k-X_t\sigma_k)(P_tg_1^{(k)}-Y_th_1^{(k)})=0.$$

It will be noticed that this equation is satisfied by putting

$$f_1^{(k)}:g_1^{(k)}:h_1^{(k)}=X_t:Y_t:P_t \tag{28}$$

whatever be the values of ρ_k, σ_k, τ_k.

An inspection of equations (22), (25) and (28) shows that the following theorem is true:

THEOREM: *Let there be given a lineal-element* (x_0, y_0, p_0) *for which the Jacobian of the transformation is of rank* 1. *Then, all curves, possessing the common lineal-element* (x_0, y_0, p_0) *will be transformed by means of equations* (1) *into curves, which at the transformed point* (x_1^0, y_1^0) *have a common tangent, and in addition a common radius of curvature given by*

$$dx_1 : dy_1 : dy'_1 = X_i : Y_i : P_i .$$

In case the matrix of the Jacobian J, of the transformation is of rank 1, we find from the preliminary formulae

$$\rho(x, y, p) = 0, \quad \rho_p = 0, \quad \rho_x + p\rho_y = 0. \tag{29}$$

Now, $\rho(x, y, p) = 0$ is a differential equation.

Equations (29) *assure us that the critical lineal-element is one whose direction and point coincide with that of the tangent and point of tangency to the curve of the singular solution of the differential equation* $\rho = 0$.

Let us now examine more closely the equation

$$X_p y'' + X_y y' + X_x = 0. \tag{18}$$

We are talking about a particular lineal-element and hence y' is fixed. This equation determines y'', since $X_p \neq 0$. But if y' and y'' are fixed, then the radius of curvature is fixed. *We have then associated with each lineal-element of the singular solution curve a definite radius of curvature.* Denote by (α, β) the coordinates of the center of curvature and by (x, y) a point on the singular solution curve. The locus of the centers of critical curvature will be given by eliminating x, y, y', y'' from the following equations:

$$\rho(x, y, p) = 0, \quad \rho_p(x, y, p) = 0,$$
$$X_p y'' + X_y y' + X_x = 0,$$
$$\alpha = x - \frac{y'(1 + y'^2)}{y''}, \quad \beta = y + \frac{1 + y'^2}{y''}.$$

Illustration.

$$X = p,$$
$$Y = \tfrac{1}{2}(y - px)^2 - p^2(y - px),$$
$$P = -2p(y - px) - x(y - px - p^2).$$

$$J = \begin{vmatrix} 0 & 0 & 1 \\ -p(y - px - p^2), & y - px - p^2, & -x(y - px - p^2) - 2p(y - px) \\ 2p^2 + px - (y - px - p^2), & -2p - x, & x^2 + 6px - 2y \end{vmatrix} = (y - px - p^2)^2 = \rho^2$$

whence

$$\rho = y - px - p^2.$$

The singular solution of this equation is

$$x^2+4y=0.$$

The matrix of the Jacobian J is of rank 1 at any lineal-element given by

$$(x,\ -\tfrac{1}{4}x^2,\ -\tfrac{1}{2}x), \tag{30}$$

or, in particular, at $(2, -1, -1)$. For any lineal-element given by (30) we have

$$\rho=\rho_p=\rho_x+p\rho_y=0.$$

The determination of the radius of curvature of the transformed curves from equations (9) fails if, and only if,

$$X_p y''+X_y y'+X_x=0. \tag{18}$$

In the present instance this equation reduces to

$$y''=0.$$

This makes the radius of curvature of the original curves infinite. Then, the critical curves (C) which pass through any point P of the envelope curve (E) are those curves which have the tangent to (E) at P for an inflectional tangent.

§ 6. *Matrix of Rank Zero.*

This case is impossible for a proper contact transformation. For, if $X_p=0$, it follows from the preliminary formulae that $Y_p=0$ also, and the transformation reduces to a mere point transformation.

Oscillations near an Isosceles-Triangle Solution of the Problem of Three Bodies as the Finite Masses Become Unequal.

BY DANIEL BUCHANAN.

---

§ 1. *Introduction.*

In an article entitled "Oscillations near one of the Isosceles-Triangle Solutions of the Three-Body Problem," which appeared in the July (1915) number of the *Proceedings of the London Mathematical Society*, the author of the present paper discussed the periodic oscillations of an infinitesimal body about a straight line drawn through the centre of gravity of two finite bodies of equal mass and perpendicular to the plane of their motion, which was assumed to be circular. In the problem now under consideration the finite bodies are assumed to be of unequal mass and the third body is assumed to be infinitesimal. The finite bodies are started so that they move in circles, and the infinitesimal body oscillates about the straight line through the centre of mass of the finite bodies and perpendicular to the plane of their motion, as in the former article. Initial conditions are determined so that the oscillations shall be periodic. The solutions for the motion of the infinitesimal body are expansible as power series in a certain parameter ε which represents half the difference in mass between the finite bodies. When $\varepsilon=0$ the solutions reduce to the simplest case of the isosceles-triangle solutions, viz., that in which the finite bodies are of equal mass and move in circles and the third body is infinitesimal.

As we shall have frequent occasion to refer to the former paper we shall refer to it as *Proc.*, followed by the number of the equations or the section.

§ 2. *The Differential Equations.*

Let m_1 and m_2 denote the two finite bodies of mass $m-\varepsilon$ and $m+\varepsilon$, respectively, and let μ denote the infinitesimal body. Let the unit of mass be so chosen that $m=1/2$, the linear unit so that the distance from m_1 to m_2 shall be unity, and the unit of time so that the Gaussian constant shall also be unity. Let the origin of coordinates be taken at the centre of mass, the plane of motion of m_1 and m_2 as the $\xi\eta$-plane, and let the coordinates of m_1, m_2 and μ be denoted by ξ_1, η_1, ζ_1; ξ_2, η_2, ζ_2 and ξ, η, ζ, respectively. If the masses m_1 and m_2 are started from the points $1/2+\varepsilon$, 0, 0, and $-1/2+\varepsilon$, 0, 0, respectively, so that they will move in circles, then

$$\left.\begin{aligned} \xi_1 &= (\tfrac{1}{2}+\varepsilon)\cos(t-t_0), & \xi_2 &= -(\tfrac{1}{2}-\varepsilon)\cos(t-t_0), \\ \eta_1 &= (\tfrac{1}{2}+\varepsilon)\sin(t-t_0), & \eta_2 &= -(\tfrac{1}{2}-\varepsilon)\sin(t-t_0), \end{aligned}\right\} \qquad (1)$$

and the differential equations of motion for the infinitesimal body are

$$\left.\begin{aligned} \xi'' &= -\frac{\xi}{2}\left[\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right]+\frac{1}{2}\left[\frac{\xi_1}{\rho_1^3}+\frac{\xi_2}{\rho_2^3}\right]+\varepsilon\xi\left[\frac{1}{\rho_1^3}-\frac{1}{\rho_2^3}\right]-\varepsilon\left[\frac{\xi_1}{\rho_1^3}-\frac{\xi_2}{\rho_2^3}\right], \\ \eta'' &= -\frac{\eta}{2}\left[\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right]+\frac{1}{2}\left[\frac{\eta_1}{\rho_1^3}+\frac{\eta_2}{\rho_2^3}\right]+\varepsilon\eta\left[\frac{1}{\rho_1^3}-\frac{1}{\rho_2^3}\right]-\varepsilon\left[\frac{\eta_1}{\rho_1^3}-\frac{\eta_2}{\rho_2^3}\right], \\ \zeta'' &= -\frac{\zeta}{2}\left[\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right]+\varepsilon\zeta\left[\frac{1}{\rho_1^3}-\frac{1}{\rho_2^3}\right], \\ \rho_1 &= [(\xi-\xi_1)^2+(\eta-\eta_1)^2+\zeta^2]^{\frac{1}{2}}, \\ \rho_2 &= [(\xi-\xi_2)^2+(\eta-\eta_2)^2+\zeta^2]^{\frac{1}{2}}. \end{aligned}\right\} \qquad (2)$$

If we refer the motion of the system to a set of rectangular coordinates rotating about the ζ-axis with the uniform velocity unity, that is, if we transform the coordinates of μ by the substitutions

$$\left.\begin{aligned} \xi &= x\cos(t-t_0)-y\sin(t-t_0), \\ \eta &= x\sin(t-t_0)+y\cos(t-t_0), \\ \zeta &= z, \end{aligned}\right\} \qquad (3)$$

then, after (1) and (3) are substituted in (2), the differential equations become

$$\left.\begin{aligned} x''-2y'-x &= -\frac{x}{2}\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right]+\left[\frac{1}{4}+\varepsilon x-\varepsilon^2\right]\left[\frac{1}{r_1^3}-\frac{1}{r_2^3}\right], \\ y''+2x'-y &= -\frac{y}{2}\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right]+\varepsilon y\left[\frac{1}{r_1^3}-\frac{1}{r_2^3}\right], \\ z'' &= -\frac{z}{2}\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right]+\varepsilon z\left[\frac{1}{r_1^3}-\frac{1}{r_2^3}\right], \\ r_1 &= [(\tfrac{1}{2}+\varepsilon-x)^2+y^2+z^2]^{\frac{1}{2}}, \quad r_2=[(\tfrac{1}{2}-\varepsilon+x)^2+y^2+z^2]^{\frac{1}{2}}. \end{aligned}\right\} \qquad (4)$$

These equations admit the Jacobian integral

$$(x')^2+(y')^2+(z')^2=x^2+y^2+\frac{1-2\varepsilon}{r_1}+\frac{1+2\varepsilon}{r_2}+\text{const.} \qquad (5)$$

When $\varepsilon=0$ and the infinitesimal is projected along the z-axis, then $x\equiv y\equiv 0$ and the differential equations (4) reduce to

$$z''=-\frac{8z}{(1+4z^2)^{\frac{3}{2}}}.$$

The periodic solution of this equation is, *Proc.* (4),

$$z=\psi=a\sin(\tau-\tau_0)+\frac{3}{16}a^3[\sin 3(\tau-\tau_0)-\sin(\tau-\tau_0)]+\ldots,$$

$$\tau-\tau_0=\frac{(t-t_0)}{\sqrt{\frac{1}{8}(1+\delta)}},\quad \delta=\frac{9}{2}a^2-\frac{141}{32}a^4+\frac{35}{2}a^6+\ldots.$$

The constant a is a variable parameter and $a/\sqrt{\{\frac{1}{8}(1+\delta)\}}$ denotes the initial projection of μ from the origin when $\varepsilon=0$. The numerical values of t_0 and τ_0 can both be taken to be zero without loss of generality. As series similar in form to ψ occur frequently in the sequel, we shall call them *triply odd series* inasmuch as they contain only odd powers of a and odd functions of odd multiples of τ.

§ 3. *The Equations of Variation.*

We wish to show the existence of periodic solutions of (4) which are expansible as power series in ε and which for $\varepsilon=0$ reduce to $x\equiv y\equiv 0$, $z=\psi$. Let us substitute in (4)

$$(t-t_0)=(\tau-\tau_0)\sqrt{\tfrac{1}{8}(1+\delta)},\quad z=\psi+w, \tag{6}$$

where w vanishes with ε. If we denote derivatives with respect to τ by dots, and expand the right members as power series in ε, x, y and w, then the differential equations (4) become

$$\left.\begin{aligned}
&\ddot{x}-2\sqrt{\tfrac{1}{8}(1+\delta)}\,\dot{y}-\tfrac{1}{8}(1+\delta)x=x\sum_{i=0}^{\infty}X_i^{(0)}+\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}X_i^{(j)}\varepsilon^j,\\
&\ddot{y}+2\sqrt{\tfrac{1}{8}(1+\delta)}\,\dot{x}-\tfrac{1}{8}(1+\delta)y=y\sum_{i=0}^{\infty}Y_i^{(0)}+y\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}Y_i^{(j)}\varepsilon^j,\\
&w+W_0w=\sum_{i=2}^{\infty}W_i^{(0)}+\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}W_i^{(j)}\varepsilon^j.
\end{aligned}\right\} \tag{7}$$

Before characterizing the undefined terms in (7), we shall define a *triply even series*, as such series occur frequently in the sequel. A triply even series is a power series in a^2 with sums of cosines of even multiples of τ in the coefficients. The highest multiple of τ in the coefficient of a^{2k} is $2k$. Such a series is called triply even since it is even in a and only even functions of even multiples of τ enter the coefficients.

The $X_i^{(0)}$, $Y_i^{(0)}$ and $W_i^{(0)}$ are homogeneous polynomials of degree i in x, y and w, but only even powers of x, and y enter while w may enter to any power. The coefficients in $X_i^{(0)}$ and $Y_i^{(0)}$ are triply even or triply odd series, according as i is even or odd, respectively. The coefficients in $W_i^{(0)}$ are triply odd or triply even series, according as i is even or odd, respectively. As far as the computations have been carried out we have, *Proc.* (5),

$$
\begin{aligned}
W_0 &= 1-a^2\left(\frac{9}{2}-9\cos 2\tau\right)+a^4\left(\frac{687}{32}-48\cos 2\tau+\frac{177}{8}\cos 4\tau\right)+\ldots., \\
X_0^{(0)} &= 2-3a^2(1-4\cos 2\tau)+\ldots., \\
Y_0^{(0)} &= -1-\frac{3}{2}a^2(1+2\cos 2\tau)+\ldots., \\
X_1^{(0)} &= -48(1+\delta)\psi w, \quad Y_1^{(0)}=12(1+\delta)\psi w, \\
X_2^{(0)} &= x^2[16-24a^2(7-10\cos 2\tau)+\ldots.]-y^2[24-36a^2(2-5\cos 2\tau)+\ldots.] \\
&\quad -w^2[24-108a^2(4-5\cos 2\tau)+\ldots.], \\
Y_2^{(0)} &= -x^2[24-36a^2(2-5\cos 2\tau)+\ldots.]+y^2[6-3a^2(1-10\cos 2\tau)+\ldots.] \\
&\quad +w^2[6-9a^2(7-10\cos 2\tau)+\ldots.], \\
W_2^{(0)} &= -x^2[24a\sin\tau+\ldots.]+y^2[6a\sin\tau+\ldots.]+w^2[18a\sin\tau+\ldots.].
\end{aligned}
$$

The $X_i^{(j)}$, $Y_i^{(j)}$ and $W_i^{(j)}$ for $j>0$ are likewise polynomials of degree i in x, y and w, but y enters only to even degrees. If j is odd, the $X_i^{(j)}$ are even in x, and $Y_i^{(j)}$ and $W_i^{(j)}$ are odd in x. If j is even, the $X_i^{(j)}$ are odd in x, and $Y_i^{(j)}$ and $W_i^{(j)}$ are even in x. The coefficients in $X_i^{(j)}$ and $Y_i^{(j)}$ are triply even or triply odd series, according as w enters to even or odd degrees, respectively, but in $W_i^{(j)}$ the coefficients are triply odd or triply even series, according as w enters to even or odd degrees, respectively. It is observed that $X_0^{(j)}=0$ if j is even, and $Y_0^{(j)}=W_0^{(j)}=0$ if j is odd.

The expansions in the right members in (7) are valid only in a certain region of convergence. It is seen from the values of r_1 and r_2 in equations (4) that x, y and z must satisfy the inequalities

$$
\begin{aligned}
-\tfrac{1}{4} &< +\varepsilon+\varepsilon^2-x-2\varepsilon x+x^2+y^2+z^2<\tfrac{1}{4}, \\
-\tfrac{1}{4} &< -\varepsilon+\varepsilon^2+x-2\varepsilon x+x^2+y^2+z^2<\tfrac{1}{4}.
\end{aligned}
$$

The region of convergence is obtained by replacing the inequality signs with signs of equality. When referred to the fixed $\xi\eta$-axes the region is found to be the spindle formed by rotating the common portion of two intersecting circles about their common chord. These circles have their centres at $1/2+\varepsilon, 0, 0$ and $-1/2+\varepsilon, 0, 0$ and have radii $1/\sqrt{2}$.

The equations of variation are obtained by putting $\varepsilon=0$ in (7) and taking only the linear terms in x, y and w in the right members. They are

$$\left.\begin{aligned}&\ddot{x}-2\sqrt{\{\tfrac{1}{8}(1+\delta)\}}\,\dot{y}-[\tfrac{1}{8}(1+\delta)+X_0^{(0)}]x=0,\\&\ddot{y}+2\sqrt{\{\tfrac{1}{8}(1+\delta)\}}\,\dot{x}-[\tfrac{1}{8}(1+\delta)+Y_0^{(0)}]y=0,\\&\ddot{w}+W_0w=0.\end{aligned}\right\}\tag{8}$$

These equations are the same as *Proc.* (6) if we put $m_0=\frac{1}{2}$ in the latter. The solutions of the first two equations are obtained from *Proc.* (8) by putting $m_0=\frac{1}{2}$. They are

$$\left.\begin{aligned}x&=A_1e^{\sqrt{-1}\,\beta_1\tau}u_1+A_2e^{-\sqrt{-1}\,\beta_1\tau}u_2+A_3e^{\beta_2\tau}u_3+A_4e^{-\beta_2\tau}u_4,\\y&=\sqrt{-1}\,[A_1e^{\sqrt{-1}\,\beta_1\tau}v_1-A_2e^{-\sqrt{-1}\,\beta_1\tau}v_2]+A_3e^{\beta_2\tau}v_3-A_4e^{-\beta_2\tau}v_4,\end{aligned}\right\}\tag{9}$$

where A_i $(i=1,2,3,4)$ are the constants of integration, u_i and v_i $(i=1,2,3,4)$ are periodic functions of τ having the period 2π; and

$$\beta_j=\sum_{k=0}^{\infty}\beta_j^{(2k)}a^{2k}\ (j=1,2)$$

are power series in a^2 with real constant coefficients which are determined by the conditions that u_i and v_i shall be periodic. The values of $-(\beta_1^{(0)})^2$ and $(\beta_2^{(0)})^2$ are the roots of the quadratic

$$64\gamma^2-48\gamma-119=0,$$

and, therefore,

$$\beta_1^{(0)}=[\sqrt{2}-\tfrac{3}{8}]^{\frac{1}{2}},\quad \beta_2^{(0)}=[\sqrt{2}+\tfrac{3}{8}]^{\frac{1}{2}}.$$

The u_i and v_i are power series in a^2 with sums of sines and cosines of even multiples of τ in the coefficients, the highest multiple of τ in the coefficient of a^{2k} being $2k$. In u_i and v_i $(i=1,2)$ the coefficients of the sines are purely imaginary and the coefficients of the cosines are real. In u_i and v_i $(i=3,4)$ all the coefficients are real. Further,

$$\begin{gathered}u_1(\sqrt{-1})=u_2(-\sqrt{-1}),\quad u_3(\tau)=u_4(-\tau),\\v_1(\sqrt{-1})=v_2(-\sqrt{-1}),\quad v_3(\tau)=v_4(-\tau),\\u_i(0)=1\ (i=1,2,3,4).\end{gathered}$$

The determinant of the solutions (9) and their first derivatives is a constant,[*] and at $\tau=0$ its value is

$$\Delta=10\sqrt{-119}+\text{terms in }a^2.\tag{10}$$

This determinant is different from zero for $a=0$ and will remain different from zero for a^2 sufficiently small. Hence, the solutions (9) constitute a fundamental set.

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[*] Moulton's "Periodic Orbits," § 18.

The general solution of the last equation of (9) is the same as *Proc.* (7), viz.,

$$\left.\begin{aligned} w &= A_5\phi + A_6[\chi + A\tau\phi], \\ \phi &= \cos\tau + \frac{9}{16}a^2[\cos 3\tau - \cos\tau] + \ldots, \\ \chi &= \sin\tau + \frac{9}{16}a^2[\sin 3\tau + 5\sin\tau] + \ldots, \\ A &= -\frac{9}{2}a^2 + \frac{141}{16}a^4 + \ldots, \end{aligned}\right\} \qquad (11)$$

where A_5 and A_6 are the constants of integration.

§ 4. *Existence of Symmetrical Periodic Orbits.*

Let us choose as initial conditions of (7)

$$x(0) = \alpha_1,\ \dot{x}(0) = 0,\ y(0) = 0,\ \dot{y}(0) = \alpha_2,\ w(0) = 0,\ \dot{w}(0) = \alpha_3. \qquad (12)$$

With these initial conditions it can be shown from the differential equations (7) that x is even in τ and y and w are odd in τ. Hence, sufficient conditions that the solutions shall be periodic with the period 2π are

$$\dot{x} = y = w = 0 \text{ at } \tau = \pi. \qquad (13)$$

When the conditions (12) are imposed on the solutions (9) and (11), we find that $A_1 = A_2$, $A_3 = A_4$, $A_5 = 0$, and that A_1, A_3 and A_6 are linear functions of $\alpha_i (i = 1, 2, 3)$, vanishing with α_i. Now let us integrate equations (7) as power series in ε and α_i, or, which is more convenient, in ε, A_1, A_3 and A_6. In so far as the terms which are linear in A_1, A_3 and A_6 are concerned, we obtain

$$\left.\begin{aligned} x_{10} &= A_1[e^{\sqrt{-1}\beta_1\tau}u_1 + e^{-\sqrt{-1}\beta_1\tau}u_2] + A_3[e^{\beta_2\tau}u_3 + e^{-\beta_2\tau}u_4], \\ y_{10} &= \sqrt{-1}A_1[e^{\sqrt{-1}\beta_1\tau}v_1 - e^{-\sqrt{-1}\beta_1\tau}v_2] + A_3[e^{\beta_2\tau}v_3 - e^{-\beta_2\tau}v_4], \\ w_{10} &= A_6[\chi + A\tau\phi]. \end{aligned}\right\} \qquad (14)$$

When the periodicity conditions (13) are imposed on these solutions, we obtain

$$\left.\begin{aligned} 0 = {} & A_1[\sqrt{-1}\beta_1 + \dot{u}_1(0)][e^{\sqrt{-1}\beta_1\pi} - e^{-\sqrt{-1}\beta_1\pi}] + A_3[\beta_2 + \dot{u}_3(0)][e^{\beta_2\pi} - e^{-\beta_2\pi}] \\ & + \text{terms in } \varepsilon \text{ and higher degree terms in } A_1,\ A_3,\ A_6 \text{ and } \varepsilon, \\ 0 = {} & \sqrt{-1}v_1(0)A_1[e^{\sqrt{-1}\beta_1\pi} - e^{-\sqrt{-1}\beta_1\pi}] + v_3(0)A_3[e^{\beta_2\pi} - e^{-\beta_2\pi}] \\ & + \text{higher degree terms in } A_1,\ A_3,\ A_6 \text{ and } \varepsilon, \\ 0 = {} & A_6A\pi + \text{higher degree terms in } A_1,\ A_3,\ A_6 \text{ and } \varepsilon. \end{aligned}\right\} \qquad (15)$$

The determinant of the linear terms in A_1, A_3 and A_6 is

$$\begin{aligned} D = {} & [e^{\sqrt{-1}\beta_1\pi} - e^{-\sqrt{-1}\beta_1\pi}][e^{\beta_2\pi} - e^{-\beta_2\pi}] \\ & \left\{v_3(0)[\sqrt{-1}\beta_1 + \dot{u}_1(0)] - \sqrt{-1}v_1(0)[\beta_2 + \dot{u}_3(0)]\right\}. \end{aligned} \qquad (16)$$

The last factor has the form

$$\frac{5\sqrt{-119}}{7\sqrt{2}}+\text{terms in } a^2,$$

and is different from zero for a^2 sufficiently small. The second factor can not vanish as β_2 is real. Then the determinant D can vanish only when β_1 is an integer. When β_1 is not an integer, D is not zero; and hence equations (15) can be solved uniquely for A_1, A_3 and A_6 as power series in ε which vanish with ε and converge for $|\varepsilon|$ sufficiently small. Therefore symmetrical periodic solutions of (7) exist when β_1 is not an integer, and they have the form

$$x=\sum_{j=1}^{\infty} x_j\varepsilon^j,\quad y=\sum_{j=1}^{\infty} y_j\varepsilon^j,\quad w=\sum_{j=1}^{\infty} w_j\varepsilon^j, \tag{17}$$

where each x_j, y_j and w_j is separately periodic with the period 2π in τ.

From the following considerations of the differential equations (7) we show that x and y are odd in ε, and w is even in ε. The right member of the first equation in (7) is odd in x and ε, considered together, and even in y. The right member of the second equation is even in x and ε, considered together, and also even in y. Let the solutions (17) be denoted by

$$x=x(\varepsilon),\quad y=y(\varepsilon),\quad w=w(\varepsilon).$$

If the signs of x, y and ε be changed, the differential equations remain unchanged, and therefore,

$$x=-x(-\varepsilon),\quad y=-y(-\varepsilon),\quad w=w(-\varepsilon)$$

are also solutions. Since the solutions as power series in ε are unique, then

$$x(\varepsilon)=-x(-\varepsilon),\quad y(\varepsilon)=-y(-\varepsilon),\quad w(\varepsilon)=w(-\varepsilon),$$

or, x and y are odd in ε and w is even in ε. Hence, the periodic solutions of (7) have the form

$$x=\sum_{j=0}^{\infty} x_{2j+1}\varepsilon^{2j+1},\quad y=\sum_{j=0}^{\infty} y_{2j+1}\varepsilon^{2j+1},\quad w=\sum_{j=1}^{\infty} w_{2j}\varepsilon^{2j}. \tag{18}$$

The same result would be obtained in the construction of the solutions if the forms (17) were used instead of (18).

§ 5. *Existence of Symmetrical Periodic Orbits when β_1 is an Integer.*

When β_1 is an integer, the determinant (16) vanishes and the terms in (15) of higher degree in ε, A_1, A_3 and A_6 must be considered in order to establish the existence of symmetrical periodic orbits. We take the same initial conditions (12), and as in the previous section we integrate (7) as power series in A_1, A_3, A_6 and ε. In addition to the linear terms already obtained in (14), we need the quadratic terms in A_1, A_3 and A_6, and the term in A_1^3 in so far as it enters x. If these terms are denoted by x_{11}, y_{11}, w_{11} and x_{30}

respectively, then all these terms except w_{11} can be obtained from *Proc.* § 6 by putting $m_0=1/2$, $\lambda=0$ in the solutions there denoted by the same notation. Thus,

$$\left.\begin{aligned}
x_{11} &= \frac{1}{\Delta} A_1 A_6 Q_1 \tau [e^{\sqrt{-1}\beta_1\tau} u_1 - e^{-\sqrt{-1}\beta_1\tau} u_2] \\
&\qquad + \frac{\sqrt{-1}}{\Delta} A_3 A_6 Q_2 \tau [e^{\beta_2\tau} u_3 - e^{-\beta_2\tau} u_4], \\
y_{11} &= \frac{\sqrt{-1}}{\Delta} A_1 A_6 Q_1 \tau [e^{\sqrt{-1}\beta_1\tau} v_1 + e^{-\sqrt{-1}\beta_1\tau} v_2] \\
&\qquad + \frac{\sqrt{-1}}{\Delta} A_3 A_6 Q_2 \tau [e^{\beta_2\tau} v_3 + e^{-\beta_2\tau} v_4], \\
x_{30} &= \frac{1}{\Delta} A_1^3 Q_3 \tau [e^{\sqrt{-1}\beta_1\tau} u_1 - e^{-\sqrt{-1}\beta_1\tau} u_2],
\end{aligned}\right\} \quad (19)$$

where Q_1, Q_2 and Q_3 are power series in a^2 with real constant coefficients.

The term w_{11} is the solution of the differential equation

$$\ddot{w}_{11} + W_0 w_{11} = W_{11}, \qquad (20)$$

where the right member is a linear function of x_{10}^2, y_{10}^2, w_{10}^2 multiplied by a triply odd series. The complementary function of (20) is the same as the solution of the last equation of (8), and on using the method of the variation of parameters we obtain

$$w_{11} = \tau [A_1^2 \phi_1 + A_3^2 \phi_3] + A_6^2 [\text{non-periodic terms}] + \text{periodic terms},$$

where ϕ_1 and ϕ_3 are power series similar to ϕ in (11).

When the periodicity conditions (13) are imposed on the solutions

$$\begin{aligned}
x &= x_{10} + x_{11} + x_{30} + \dots, \\
y &= y_{10} + y_{11} + \dots, \\
w &= w_{10} + w_{11} + \dots,
\end{aligned}$$

we obtain the equations

$$\left.\begin{aligned}
0 &= A_3 [\beta_2 + \dot{u}_3(0)] \left\{ e^{\beta_2\pi} - e^{-\beta_2\pi} + \frac{\sqrt{-1}\pi}{\Delta} A_6 Q_2 [e^{\beta_2\pi} + e^{-\beta_2\pi}] \right\} \\
&\qquad \pm \frac{2\pi}{\Delta} A_1 [\sqrt{-1}\beta_1 + \dot{u}_1(0)] [A_6 Q_1 + A_1^2 Q_3] + \text{terms in } \varepsilon \\
&\qquad \text{and cubic and higher degree terms } A_1,\ A_3 \text{ and } A_6, \\
0 &= \sqrt{-1}\, v_3(0) A_3 \left\{ e^{\beta_2\pi} - e^{-\beta_2\pi} + \frac{\pi}{\Delta} A_6 Q_2 [e^{\beta_2\pi} + e^{-\beta_2\pi}] \right\} \\
&\qquad + \frac{\sqrt{-1}\pi}{\Delta} A_6 \left\{ \pm 2 v_1(0) A_1 Q_1 + v_3(0) A_3 Q_2 [e^{\beta_2\pi} + e^{-\beta_2\pi}] \right\} \\
&\qquad + \text{terms in } \varepsilon \text{ and cubic and higher degree terms in } A_1, \\
&\qquad A_3 \text{ and } A_6, \\
0 &= \pi [A A_6 + A_1^2 \phi_1(\pi) + A_3^2 \phi_3(\pi)] + \text{terms in } \varepsilon \text{ and higher degree} \\
&\qquad \text{terms in } A_1,\ A_3 \text{ and } A_6.
\end{aligned}\right\} \quad (21)$$

Where the double sign occurs the + is to be taken if β_1 is an even integer, and the — if it is an odd integer.

The last equation of (21) can be solved for A_6 as a power series in A_1^2, A_3^2 and ε which vanishes with A_1 and A_6 but not with ε. We shall refer to this solution as (21c). When it is substituted in the second equation of (21) we obtain an equation in A_1, A_3 and ε which vanishes with A_1 and A_3 but not with ε, the terms of lowest degree being A_1^3 and A_3. As the coefficient of A_3 in this equation is a power series in a^2 with additional terms in $1/a^2$, it will be different from zero, in general, and the equation can be solved for A_3 as a power series in A_1^3 and ε which vanishes with A_1 but not with ε. Denote this series for A_3 by (21b). After substituting (21c) and (21b) in the first equation of (21), we obtain an equation (21a) in A_1 and ε which vanishes with these terms and in which the lowest power of A_1 alone is A_1^3. The coefficient of A_1^3 in this equation is a power series in a^2 with additional terms in $1/a^2$ and will, in general, be different from zero. Hence, (21a) can be solved for A_1 as a power series in $\varepsilon^{\frac{1}{3}}$ which vanishes with ε and converges for $|\varepsilon|$ sufficiently small. There are three solutions for A_1, but only one is real; the other two being complex. When this power series for A_1 is substituted in (21b) and (21c), A_3 and A_6 are likewise power series in $\varepsilon^{\frac{1}{3}}$. Therefore, when β_1 is an integer, periodic solutions exist having the form

$$x = \sum_{j=1}^{\infty} x^{(j)} \varepsilon^{\frac{j}{3}}, \quad y = \sum_{j=1}^{\infty} y^{(j)} \varepsilon^{\frac{j}{3}}, \quad w = \sum_{j=1}^{\infty} w^{(j)} \varepsilon^{\frac{j}{3}} \tag{22}$$

where each $x^{(j)}$, $y^{(j)}$ and $w^{(j)}$ is separately periodic with the period 2π in τ. These solutions converge for ε sufficiently small numerically. There is only one set of real solutions, the other two sets being complex.

In the practical construction of the solutions it can be shown that x and y are odd in $\varepsilon^{\frac{1}{3}}$, and that w is even in $\varepsilon^{\frac{1}{3}}$. This property of the solutions can be deduced directly from the differential equations (7) by an argument similar to that used at the end of § 4 if we replace ε in that section by $\varepsilon^{\frac{1}{3}}$. Then we may consider the solutions to have the form

$$x = \sum_{j=0}^{\infty} x^{(2j+1)} \varepsilon^{\frac{2j+1}{3}}, \quad y = \sum_{j=0}^{\infty} y^{(2j+1)} \varepsilon^{\frac{2j+1}{3}}, \quad w = \sum_{j=1}^{\infty} w^{(2j)} \varepsilon^{\frac{2j}{3}}. \tag{23}$$

§ 6. *Proof that all the Periodic Orbits are Symmetrical.*

We shall now consider the existence of periodic orbits in which the infinitesimal body is projected from the xy-plane in a direction which may not be perpendicular to the x-axis, and from a point which may not lie on the x-axis. These orbits, if they exist, are called the general orbits. They have the initial values

$$x(0) = \alpha_1, \quad \dot{x}(0) = \alpha_2, \quad y(0) = \alpha_3, \quad y(0) = \alpha_4, \quad w(0) = 0, \quad \dot{w}(0) = \alpha_5. \tag{24}$$

The initial value of w can be taken to be zero without loss of geometric generality since the infinitesimal body must cross the xy-plane if the motion is to be periodic, and hence the initial time can be chosen as the time when the infinitesimal body crosses the xy-plane. Sufficient conditions that solutions having the initial values (24) shall be periodic are

$$\left.\begin{array}{lll} x(2\pi)-x(0)=0, & y(2\pi)-y(0)=0, & w(2\pi)-w(0)=0, \\ \dot{x}(2\pi)-\dot{x}(0)=0, & \dot{y}(2\pi)-\dot{y}(0)=0, & \dot{w}(2\pi)-\dot{w}(0)=0. \end{array}\right\} \qquad (25)$$

We shall now show that one of these conditions, viz., $\dot{w}(2\pi)-\dot{w}(0)=0$, can be suppressed.

So far no use has been made of the integral (5). It is by means of this integral that we show that one of the conditions in (25) is redundant. When (5) is transformed by the substitutions (6), it takes the form

$$\dot{x}^2+\dot{y}^2+(\dot{\psi}+\dot{w})^2+F(x,\, y^2,\, w;\ \epsilon)=0, \qquad (26)$$

where F is a power series in x, y^2, w and ϵ having periodic coefficients. Let us make in (26) the usual substitutions

$$\left.\begin{array}{lll} x=x(0)+\bar{x}, & y=y(0)+\bar{y}, & w=0+\bar{w}, \\ \dot{x}=\dot{x}(0)+\dot{\bar{x}}, & y=\dot{y}(0)+\dot{\bar{y}}, & \dot{w}=\dot{w}(0)+\dot{\bar{w}}, \end{array}\right\} \qquad (27)$$

where $\bar{x}$, $\dot{\bar{x}}$, $\bar{y}$, $\dot{\bar{y}}$, $\bar{w}$ and $\dot{\bar{w}}$ vanish at $\tau=0$. We shall denote the equation resulting from substituting (27) in (26) by (26a). By putting $\tau=0$ we obtain from (26a) an equation (26b) connecting the terms of (26a) which are independent of $\bar{x}, \ldots, \dot{\bar{w}}$. On substituting (26b) in (26a) we obtain an equation

$$G(\bar{x},\, \dot{\bar{x}},\, \bar{y},\, \dot{\bar{y}},\, \bar{w},\, \dot{\bar{w}})=0, \qquad (28)$$

in which at $\tau=0$ or 2π there are no terms independent of the arguments indicated. The coefficient of $\dot{\bar{w}}(2\pi)$ in this equation is $2[a+\dot{w}(0)]$, and it vanishes only when $\dot{w}(0)=-a$. If $w(0)=-a$ then $\dot{z}(0)=0$, since $z=\psi+w$ and $\dot{\psi}(0)=a$, and as $z(0)=0$ it follows that $z\equiv0$. As we desire solutions which are not identically zero we must take $\dot{w}(0)=-a$. Therefore the coefficient of $\dot{\bar{w}}(2\pi)$ in (28) is not zero at $\tau=2\pi$, and by the theory of implicit functions this equation can be solved uniquely for $\dot{w}(2\pi)$ as a power series

$$\dot{\bar{w}}(2\pi)=H\{\bar{x}(2\pi),\, \dot{\bar{x}}(2\pi),\, \bar{y}(2\pi),\, \dot{\bar{y}}(2\pi),\, \bar{w}(2\pi)\}, \qquad (29)$$

which vanishes with the arguments indicated and converges for sufficiently small values of the moduli of these arguments. When the first five conditions of (25) are satisfied it follows from (29) that the arguments of H are all zero, and therefore $\dot{\bar{w}}(2\pi)=0$. Then, since $\dot{\bar{w}}(2\pi)=\dot{w}(2\pi)-\dot{w}(0)$, it follows that the condition $\dot{w}(2\pi)-\dot{w}(0)=0$ is a consequence of the other conditions in (25) and may be suppressed.

We shall now integrate equations (7) as power series in α_i $(i=1, \ldots, 5)$ and ε. The differential equations from which the linear terms are obtained are the same as the equations of variation, and consequently the solutions are the same as (9) and (11). When the initial conditions (24) are imposed on these solutions, we find that $A_5=0$ and that the remaining A_i are linear functions of α_i which vanish with α_i. Then, as in § 4, it is more convenient to integrate (7) as power series in A_i and ε. On imposing the necessary conditions of (25) upon (9) and (11) we obtain

$$\left.\begin{aligned}
0 &= A_1[e^{2\sqrt{-1}\,\beta_1\pi}-1]+A_2[e^{-2\sqrt{-1}\,\beta_1\pi}-1]+A_3[e^{2\beta_2\pi}-1] \\
&\qquad\qquad +A_4[e^{-2\beta_2\pi}-1]+\ldots, \\
0 &= [\sqrt{-1}\,\beta_1+\dot{u}_1(0)]\left\{A_1[e^{2\sqrt{-1}\,\beta_1\pi}-1]-A_2[e^{-2\sqrt{-1}\,\beta_1\pi}-1]\right\} \\
&\qquad +[\beta_2 v_3(0)+\dot{v}_3(0)]\left\{A_3[e^{2\beta_2\pi}-1]-A_4[e^{-2\beta_2\pi}-1]\right\}+\ldots, \\
0 &= \sqrt{-1}\,v_1(0)\left\{A_1[e^{2\sqrt{-1}\,\beta_1\pi}-1]-A_2[e^{-2\sqrt{-1}\,\beta_1\pi}-1]\right\} \\
&\qquad\qquad +v_3(0)\left\{A_3[e^{2\beta_2\pi}-1]-A_4[e^{-2\beta_2\pi}-1]\right\}+\ldots, \\
0 &= [-\beta_1 v_1(0)+\sqrt{-1}\,\dot{v}_1(0)]\left\{A_1[e^{2\sqrt{-1}\,\beta_1\pi}-1]+A_2[e^{-2\sqrt{-1}\,\beta_1\pi}-1]\right\} \\
&\qquad +[\beta_2 v_2(0)+\dot{v}_3(0)]\left\{A_3[e^{2\beta_2\pi}-1]+A_4[e^{-2\beta_2\pi}-1]\right\}+\ldots, \\
0 &= 2\pi A\, A_6\,\dot{\phi}(0)+\ldots\ldots
\end{aligned}\right\} \quad (30)$$

The determinant of the coefficients of A_i is

$$2\,\pi\,\Delta A\,\dot{\phi}(0)\,[e^{2\sqrt{-1}\,\beta_1\pi}-1]\,[e^{-2\sqrt{-1}\,\beta_1\pi}-1]\,[e^{2\beta_2\pi}-1]\,[e^{-2\beta_2\pi}-1], \qquad (31)$$

and it vanishes only when β_1 is an integer since β_2 is real and Δ, A and $\dot{\phi}(0)$ are different from zero for a not zero but sufficiently small numerically. If β_1 is not an integer the determinant (31) is different from zero and equations (30) can be solved uniquely for A_i as power series in ε which vanish with ε and converge for $|\varepsilon|$ sufficiently small. Hence, when β_1 is not an integer, the general orbits exist uniquely and have the same form as the symmetrical orbits. Since the general orbits include the symmetrical and both classes of orbits are unique, then all the periodic orbits are symmetrical when β_1 is not an integer.

When β_1 is an integer the coefficients of A_1 and A_2 in the first four equations of (30) vanish and in order to discuss the existence of general orbits, it is necessary to consider the terms of higher degree in (30). In addition to the linear terms (9) we require, as in § 5, the quadratic terms in A_i $(i=1, 2, 3, 4, 6)$ which contain τ as a factor, and the terms of x containing τ as a factor and of the third degree in A_1 and A_2. These terms are found in the same way as the corresponding terms were determined in §5. They have the form

$$\left.\begin{aligned}
x_{11} &= \frac{1}{\Delta} A_6 Q_1 \tau \left[A_1 e^{\sqrt{-1}\,\beta_1 \tau} u_1 - A_2 e^{-\sqrt{-1}\,\beta_1 \tau} u_2\right] \\
&\qquad + \frac{\sqrt{-1}}{\Delta} A_6 Q_2 \tau \left[A_3 e^{\beta_2 \tau} u_3 - A_4 e^{-\beta_2 \tau} u_4\right], \\
y_{11} &= \frac{\sqrt{-1}}{\Delta} A_6 Q_1 \tau \left[A_1 e^{\sqrt{-1}\,\beta_1 \tau} v_1 + A_2 e^{-\sqrt{-1}\,\beta_1 \tau} v_2\right] \\
&\qquad + \frac{\sqrt{-1}}{\Delta} A_6 Q_2 \tau \left[A_3 e^{\beta_2 \tau} v_3 + A_4 e^{-\beta_2 \tau} v_4\right], \\
w_{11} &= \tau \left[A_1 A_2 \phi_1 + A_3 A_4 \phi_3\right] + A_6^2 \,[\text{non-periodic terms}] \\
&\qquad + \text{periodic terms}, \\
x_{30} &= \frac{1}{\Delta} A_1 A_2 Q_3 \tau \left[A_1 e^{\sqrt{-1}\,\beta_1 \tau} u_1 - A_2 e^{\sqrt{-1}\,\beta_1 \tau} u_2\right].
\end{aligned}\right\} \qquad (32)$$

When the necessary periodicity conditions of (25) are imposed on equations (32), (9) and (11), we obtain

$$\left.\begin{aligned}
0 &= A_3\left[e^{2\beta_2 \pi} - 1\right] + A_4\left[e^{-2\beta_2 \pi} - 1\right] + \frac{2\pi}{\Delta} A_6 Q_1 \left[A_1 - A_2\right] \\
&\qquad + \frac{2\sqrt{-1}\,\pi}{\Delta} A_6 Q_2 \left[A_3 e^{2\beta_2 \pi} - A_4 e^{-2\beta_2 \pi}\right] + \frac{2\pi}{\Delta} A_1 A_2 Q_3 \left[A_1 - A_2\right] \\
&\qquad + \text{terms in } \varepsilon \text{ and higher degree terms in } A_i, \\
0 &= \left[\beta_2 + \dot{u}_3(0)\right]\left\{A_3\left[e^{2\beta_2 \pi} - 1\right] - A_4\left[e^{-2\beta_2 \pi} - 1\right]\right\} \\
&\qquad + \frac{2\pi}{\Delta}\left[\sqrt{-1}\,\beta_1 + \dot{u}_1(0)\right]\left[A_6 Q_1 + A_1 A_2 Q_3\right]\left[A_1 + A_2\right] \\
&\qquad + \frac{2\sqrt{-1}\,\pi}{\Delta} A_6 Q_2 \left[\beta_2 + \dot{u}_3(0)\right]\left[A_3 e^{2\beta_2 \pi} + A_4 e^{-2\beta_2 \pi}\right] \\
&\qquad + \text{terms in } \varepsilon \text{ and higher degree terms in } A_i, \\
0 &= v_3(0)\left\{A_3\left[e^{2\beta_2 \pi} - 1\right] - A_4\left[e^{-2\beta_2 \pi} - 1\right]\right\} \\
&\qquad + \frac{2\sqrt{-1}\,\pi}{\Delta} v_1(0) A_6 Q_1 \left[A_1 + A_2\right] \\
&\qquad + \frac{2\sqrt{-1}\,\pi}{\Delta} v_3(0) A_6 Q_2 \left[A_3 e^{2\beta_2 \pi} + A_4 e^{-2\beta_2 \pi}\right] \\
&\qquad + \text{terms in } \varepsilon \text{ and higher degree terms in } A_i, \\
0 &= \left[\beta_2 v_3(0) + \dot{v}_3(0)\right]\left\{A_3\left[e^{2\beta_2 \pi} - 1\right] + A_4\left[e^{-2\beta_2 \pi} - 1\right]\right\} \\
&\qquad + \frac{2\sqrt{-1}\,\pi}{\Delta} A_6 Q_1 \left[\sqrt{-1}\,\beta_1 v_1(0) + \dot{v}_1(0)\right]\left[A_1 - A_2\right] \\
&\qquad + \frac{2\sqrt{-1}\,\pi}{\Delta} A_6 Q_2 \left[\beta_2 v_3(0) + \dot{v}_3(0)\right]\left[A_3 e^{2\beta_2 \pi} - A_4 e^{-2\beta_2 \pi}\right] \\
&\qquad + \text{terms in } \varepsilon \text{ and higher degree terms in } A_i, \\
0 &= 2\pi A A_6 \dot{\phi}(0) + 2\pi\left[A_1 A_2 \phi_1(0) + A_3 A_4 \phi_3(0)\right] \\
&\qquad + \text{terms in } \varepsilon \text{ and higher degree terms in } A_i.
\end{aligned}\right\} \qquad (33)$$

The last equation of (33) can be solved for A_6 as a power series in A_1, A_2, A_3, A_4 and ε, and the terms of lowest degree in A_i are $A_1 A_2$ and $A_3 A_4$. The determinant of the coefficients of A_3 and A_4 in the third and fourth equations is

$$2 v_3(0) [\beta_2 v_3(0) + \dot{v}_3(0)] [e^{2\beta_2 \pi} - 1] [e^{-2\beta_2 \pi} - 1],$$

and it is different from zero for a^2 sufficiently small since β_2 is real. Hence these two equations, when A_6 has been eliminated by the last equation, can be solved for A_3 and A_4 as power series in A_1, A_2 and ε in which the terms of lowest degree in A_1 and A_2 are $A_1^2 A_2$ and $A_1 A_2^2$. When these solutions for A_3, A_4 and A_6 are substituted in the second equation, the terms independent of ε contain $A_1^2 A_2$ or $A_1 A_2^2$ as a factor. Hence this equation can be solved for A_2 as a power series in A_1 and ε, and the term of lowest degree in A_1 is linear in A_1. Finally, when the solutions for A_2, A_3, A_4 A_6 are substituted in the first equation, we obtain an equation in A_1 and ε in which the terms of lowest degree in A_1 is A_1^3. As in (25a) the coefficient of A_1^3 in this equation is a power series in a^2 with additional terms in $1/a^2$ and will, in general, be different from zero; therefore this equation can be solved for A_1^3 as a power series in $\varepsilon^{\frac{1}{3}}$ which vanishes with ε and converges for $|\varepsilon|$ sufficiently small. There are three solutions as in the symmetrical orbits, but only one solution is real, the other two being complex. Since the real solutions for the general orbits and the symmetrical orbits are both unique and since the general include the symmetrical orbits, all the periodic orbits are symmetrical when β_1 is an integer. Consequently all the periodic orbits are symmetrical whether β_1 is an integer or not.

§ 7. *Constructions of the Solutions when β_1 is not an Integer.*

Let us substitute (18) in (7) and equate the coefficients of the various powers of ε. We obtain a series of differential equations which can be integrated step by step and the constants of integration arising at each step can be determined, as we shall show, so that the solutions shall be periodic and shall satisfy the symmetrical initial conditions

$$\dot{x} = y = w = 0 \text{ at } \tau = 0.$$

When these initial conditions are imposed on the solutions (18) we obtain

$$\left.\begin{aligned} \dot{x}_{2j+1}(0) = y_{2j+1}(0) = 0, &\qquad (j = 0, \ldots . \infty), \\ w_{2j}(0) = 0, &\qquad (j = 1, \ldots . \infty). \end{aligned}\right\} \tag{34}$$

The differential equations for the terms in ε are

$$\left.\begin{aligned} x_1 - 2\sqrt{\tfrac{1}{8}(1+\delta)}\,\dot{y}_1 - [\tfrac{1}{8}(1+\delta) + X_0^{(0)}]\,x_1 = X_0^{(1)}, \\ \ddot{y}_1 + 2\sqrt{\tfrac{1}{8}(1+\delta)}\,\dot{x}_1 - [\tfrac{1}{8}(1+\delta) + Y_0^{(0)}]\,y_1 = 0, \end{aligned}\right\}$$

where $X_0^{(1)}$ is the same even power series as that represented in (7) by the same notation. The complementary functions of (35) are the same as (9), and the particular integrals can be obtained by the method of the variation of parameters. The complete solutions are thus found to be

$$\left.\begin{aligned} x_1&=A_1^{(1)}e^{\sqrt{-1}\beta_1\tau}u_1+A_2^{(1)}e^{-\sqrt{-1}\beta_1\tau}u_2+A_3^{(1)}e^{\beta_2\tau}u_3+A_4^{(1)}e^{-\beta_2\tau}u_4+C_1(\tau),\\ y_1&=\sqrt{-1}\,[A_1^{(1)}e^{\sqrt{-1}\beta_1\tau}v_1-A_2^{(1)}e^{-\sqrt{-1}\beta_1\tau}v_2]+A_3^{(1)}e^{\beta_2\tau}v_3-A_4^{(1)}e^{-\beta_2\tau}v_4+S_1(\tau),\end{aligned}\right\}\tag{35}$$

where $A_i^{(1)}\,(i=1, 2, 3, 4)$ are the constants of integration, and $C_1(\tau)$ and $S_1(\tau)$ are triply even and triply odd series, respectively. As such series occur frequently in the construction, we shall denote them by $C_j(\tau)$ and $S_j(\tau)$, respectively. Since β_1 is not an integer in this case and β_2 is real, the terms of the complementary function must be suppressed in (35) by choosing $A_i^{(1)}=0$, $(i=1, 2, 3, 4)$, and therefore the desired solutions become

$$x_1=C_1(\tau), \quad y_1=S_1(\tau). \tag{36}$$

The differential equation for the terms in ϵ^2 is

$$\ddot{w}_2+W_0w_2=W_2,$$

where W_2 is a triply odd series. The complete solution of this equation is obtained by employing the method of the variation of parameters, and it is found to be

$$w_2=B_1^{(2)}\phi+B_2^{(2)}[\chi+A\tau\phi]-ap_2\tau\phi+S_2(\tau), \tag{37}$$

where $B_1^{(2)}$ and $B_2^{(2)}$ are constants of integration, and p_2 is a power series in a^2 with real constant coefficients. In order to satisfy the periodicity and the initial conditions (34), the constants of integration must have the values

$$B_1^{(2)}=0, \quad B_2^{(2)}=\frac{ap_2}{A}, \tag{38}$$

in which case the solution (37) becomes

$$w_2=\frac{1}{a^2}\bar{S}_2(\tau); \tag{39}$$

where $\bar{S}_2(\tau)$ is a triply odd power series.

The remaining steps of the integration can be carried on in the same way. The solutions for x_{2j-1} and y_{2j-1} are obtained from the particular integrals alone since β_2 is real and β_1 is assumed to be a number which is not an integer. These particular integrals are triply even and triply odd series, respectively, except for a factor $a^{-2(j-2)}$ which is introduced through the factor a^{-2} in (39). The general solution for the w_{2j} is similar to (37) and it can be made periodic by a proper choice of the constants of integration as in (38). At the steps $2j-1$ and $2j$ the solutions are

$$\left.\begin{aligned} x_{2j-1} &= \frac{1}{a^{2j-4}} C_{2j-1}(\tau), \\ y_{2j-1} &= \frac{1}{a^{2j-4}} S_{2j-1}(\tau), \\ w_{2j} &= \frac{1}{a^{2j}} S_{2j}(\tau). \end{aligned}\right\} \tag{40}$$

§ 8. *Construction of the Solutions when β_1 is an Integer.*

Let us substitute equations (23) in (7) and equate the coefficients of the various powers of $\varepsilon^{\frac{1}{3}}$. As in the previous section we obtain a series of differential equations which can be integrated step by step, and the constants of integration can be determined, as we shall show, so that the solutions shall be periodic and shall satisfy the symmetrical initial conditions

$$\dot{x}(0) = y(0) = w(0) = 0.$$

When these conditions are imposed upon (23) we obtain

$$\left.\begin{aligned} \dot{x}^{(2j+1)}(0) = y^{(2j+1)}(0) &= 0, \qquad (j=0, \ldots .\infty), \\ w^{(2j)}(0) &= 0, \qquad (j=1, \ldots .\infty). \end{aligned}\right\} \tag{41}$$

The differential equations for the terms in $\varepsilon^{\frac{1}{3}}$ are the same as the first two equations of (8) if we use the superscript 1 on x and y. Since β_1 is assumed to be an integer, the solutions of these equations which are periodic and which satisfy (41) are

$$\left.\begin{aligned} x^{(1)} &= A_1^{(1)}[e^{\sqrt{-1}\beta_1\tau}u_1 + e^{-\sqrt{-1}\beta_1\tau}u_2], \\ y^{(1)} &= \sqrt{-1}A_1^{(1)}[e^{\sqrt{-1}\beta_1\tau}v_1 - e^{-\sqrt{-1}\beta_1\tau}v_2], \end{aligned}\right\} \tag{42}$$

where $A_1^{(1)}$ is a constant which is undetermined at this step. The solutions (42) are real if $A_1^{(1)}$ is real.

The differential equation for the terms in $\varepsilon^{\frac{2}{3}}$ is

$$\ddot{w}^{(2)} + W_0 w^{(2)} = W^{(2)} = (A_1^{(1)})^2 S^{(2)}(\tau). \tag{43}$$

As functions similar to $S^{(2)}(\tau)$ occur frequently in the construction of the remaining solutions, we shall define the function $S^{2i}(\tau)$ to be a homogeneous polynomial of degree $2i$ in the exponentials $e^{\sqrt{-1}\beta_1\tau}$ and $e^{-\sqrt{-1}\beta_1\tau}$ of which the coefficients are power series in odd powers of a with sums of sines and $\sqrt{-1}$ times cosines of odd multiples of τ in the coefficients. The highest multiple of τ in the coefficient of a^{2k+1} is $2k+1$. The coefficients of $[e^{\sqrt{-1}\beta_1\tau}]^j\,[e^{-\sqrt{-1}\beta_1\tau}]^k$ and

$[e^{\sqrt{-1}\beta_1\tau}]^k[e^{-\sqrt{-1}\beta_1\tau}]^j$, $j+k=2i$, differ only in the sign of $\sqrt{-1}$. The part of $S^{(2i)}(\tau)$ which is independent of the exponentials is a triply odd power series. If the exponentials are expressed in trigonometric form, then

$$S^{(2i)}(\tau)=\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{h=0}^{i}a^{2j+1}S_{h,j,l}^{(2i)}\sin\{2h\beta_1\pm(2l+1)\}\tau,$$

where $S_{h,j,l}^{(2i)}$ are real constants: that is, $S^{(2i)}(\tau)$ is the same as a triply odd power series except that the highest multiple of τ in the coefficient of a^{2j+1} is not $2j+1$ but $2(j+i\beta_1)+1$.

The general solution of (43) is found in precisely the same way as (37) was obtained. It has the form

$$w^{(2)}=B_1^{(2)}\phi+B_2^{(2)}[\chi+A\tau\phi]-a(A_1^{(1)})^2p^{(2)}\tau\phi+(A_1^{(1)})^2S_0^{(2)}(\tau), \tag{44}$$

where $p^{(2)}$ is a power series in a^2 with real constant coefficients and $S_0^{(2)}(\tau)$ is similar to $S^{(2)}(\tau)$. On imposing the periodicity and the initial conditions (41), we have

$$B_1^{(2)}=0,\quad B_2^{(2)}=\frac{a}{A}(A_1^{(1)})^2p^{(2)}. \tag{45}$$

Then the desired solution of (43) takes the form

$$w^{(2)}=\frac{1}{a^2}(A_1^{(1)})^2\bar{S}^{(2)}(\tau),$$

where $\bar{S}^{(2)}(\tau)$ is similar to $S^{(2)}(\tau)$. Thus $A_1^{(1)}$ remains undetermined at this step also, and the terms in ϵ^3 must be considered.

The left members of the differential equations for the terms in ϵ^3 are the same as the first two equations in (8) if we use the superscript 3 on x and y. Let us denote the right members by $X^{(3)}$ and $Y^{(3)}$, respectively. Then

$$X^{(3)}=(A_1^{(1)})^3\rho^{(3)}+X_0^{(1)},\quad Y^{(3)}=\sqrt{-1}(A_1^{(1)})^3\sigma^{(3)}. \tag{46}$$

The function $X_0^{(1)}$ is the same triply even power series as that denoted in (7) by the same notation. The functions $\rho^{(2i+1)}$ and $\sigma^{(2i+1)}$ are homogeneous polynomials of degree $2i+1$ in the exponentials $e^{\sqrt{-1}\beta_1\tau}$ and $e^{-\sqrt{-1}\beta_1\tau}$, and the coefficients are power series similar in form to u_1 and u_2 in (9). In $\rho^{(2i+1)}$ the coefficients of $[e^{\sqrt{-1}\beta_1\tau}]^j[e^{-\sqrt{-1}\beta_1\tau}]^k$ and $[e^{\sqrt{-1}\beta_1\tau}]^k[e^{-\sqrt{-1}\beta_1\tau}]^j$, $j+k=2i+1$, differ only in the sign of $\sqrt{-1}$; while in $\sigma^{(2i+1)}$ the coefficients of $[e^{\sqrt{-1}\beta_1\tau}]^j[-e^{-\sqrt{-1}\beta_1\tau}]^k$ and $[e^{\sqrt{-1}\beta_1\tau}]^k[-e^{-\sqrt{-1}\beta_1\tau}]^j$ differ only in the sign of $\sqrt{-1}$.

The complementary functions of the differential equations in $x^{(3)}$ and $y^{(3)}$ are the same as (9). Since $X^{(3)}$ and $Y^{(3)}$ contain terms which have exactly the same period as the periodic parts of the complementary functions, the particular integrals will contain non-periodic terms. By the method of the variation of parameters we find that the general solutions for $x^{(3)}$ and $y^{(3)}$ are

$$\left.\begin{aligned} x^{(3)} &= A_1^{(3)} e^{\sqrt{-1}\beta_1\tau} u_1 + A_2^{(3)} e^{-\sqrt{-1}\beta_1\tau} u_2 + A_3^{(3)} e^{\beta_2\tau} u_3 + A_4^{(3)} e^{-\beta_2\tau} u_4 \\ &\quad + \tau[(A_1^{(1)})^3 P_1^{(3)} + P_2^{(3)}][e^{\sqrt{-1}\beta_1\tau} u_1 - e^{-\sqrt{-1}\beta_1\tau} u_2] + \bar{\rho}^{(3)}, \\ y^{(3)} &= \sqrt{-1}[A_1^{(3)} e^{\sqrt{-1}\beta_1\tau} v_1 - A_2^{(3)} e^{-\sqrt{-1}\beta_1\tau} v_2] + A_3^{(3)} e^{\beta_2\tau} v_3 - A_4^{(3)} e^{-\beta_2\tau} v_4 \\ &\quad + \sqrt{-1}\tau[(A_1^{(1)})^3 P_1^{(3)} + P_2^{(3)}][e^{\sqrt{-1}\beta_1\tau} v_1 + e^{-\sqrt{-1}\beta_1\tau} v_2] + \sqrt{-1}\bar{\sigma}^{(3)}, \end{aligned}\right\} \quad (47)$$

where $A_i^{(3)}$ $(i=1, 2, 3, 4)$ are the constants of integration, $P_i^{(3)}$ $(i=1, 2)$ are power series in a^2 with constant coefficients, and $\bar{\rho}^{(3)}$ and $\bar{\sigma}^{(3)}$ are similar to $\rho^{(3)}$ and $\sigma^{(3)}$, respectively. In order that the solutions shall be periodic, then

$$A_3^{(3)} = A_4^{(3)} = 0, \quad P_1^{(3)}(A_1^{(1)})^3 + P_2^{(3)} = 0.$$

The solutions for $A_1^{(1)}$ are

$$A_1^{(1)} = P_1^{(1)}, \quad \omega P_1^{(1)} \text{ or } \omega^2 P_1^{(1)},$$

where $P_1^{(1)}$ is a power series in a^2 with real constant coefficients, and ω, ω^2 are the imaginary cube roots of unity. As the imaginary solutions for $A_1^{(1)}$ lead to imaginary orbits, we retain only the real solution. Then, on imposing the initial conditions (41) on (47) we find that $A_1^{(3)} = A_2^{(3)}$, and the desired solutions for $x^{(3)}$ and $y^{(3)}$ become

$$\begin{aligned} x^{(3)} &= A_1^{(3)}[e^{\sqrt{-1}\beta_1\tau} u_1 + e^{-\sqrt{-1}\beta_1\tau} u_2] + \bar{\rho}^{(3)}, \\ y^{(3)} &= \sqrt{-1} A_1^{(3)}[e^{\sqrt{-1}\beta_1\tau} v_1 - e^{-\sqrt{-1}\beta_1\tau} v_2] + \sqrt{-1}\bar{\sigma}^{(3)}. \end{aligned}$$

The constant $A_1^{(3)}$ remains arbitrary at this step and the terms up to $\varepsilon^{\frac{4}{3}}$ must be considered before it can be determined. The solutions for $x^{(3)}$ and $y^{(3)}$ are real provided $A_1^{(3)}$ is real, which is found to be the case.

The general solutions of the differential equation in $\varepsilon^{\frac{4}{3}}$ is obtained in the same way as (44) was found, and it can be made to satisfy the periodicity and initial conditions by a choice of the constants of integration as in (45). When these conditions have been imposed, the solution for $w^{(4)}$ takes the form

$$w^{(4)} = \frac{1}{a^4}[A_1^{(3)} S^{(2)} + S^{(4)}].$$

The differential equations in $\varepsilon^{\frac{5}{3}}$ are the same as (8) in the left members if we use the superscripts 5. If we denote the right members by $X^{(5)}$ and $Y^{(5)}$, respectively, then

$$X^{(5)} = \frac{1}{a^2}[A_1^{(3)} \rho_5^{(3)} + \rho_5^{(5)}], \quad Y^{(5)} = \frac{\sqrt{-1}}{a^2}[A_1^{(3)} \sigma_5^{(3)} + \sigma_5^{(5)}],$$

where $\rho_5^{(3)}$, $\rho_5^{(5)}$ and $\sigma_5^{(3)}$, $\sigma_5^{(5)}$ are similar to $\rho^{(2k+1)}$ and $\sigma^{(2k+1)}$, respectively. The complete solutions of the equations in $x^{(5)}$ and $y^{(5)}$ are readily found in the same way as (47) were obtained. They are:

$$\left.\begin{aligned} x^{(5)} &= A_1^{(5)} e^{\sqrt{-1}\beta_1\tau} u_1 + A_2^{(5)} e^{-\sqrt{-1}\beta_1\tau} u_2 + A_3^{(5)} e^{\beta_2\tau} u_3 + A_4^{(5)} e^{-\beta_2\tau} u_4 \\ &\qquad + \frac{\tau}{a^2}\,[P_1^{(5)} A_1^{(3)} + P_2^{(5)}]\,[e^{\sqrt{-1}\beta_1\tau} u_1 - e^{-\sqrt{-1}\beta_1\tau} u_2] + \frac{1}{a^2}\rho^{(5)}, \\ y^{(5)} &= \sqrt{-1}\,[A_1^{(5)} e^{\sqrt{-1}\beta_1\tau} v_1 - A_2^{(5)} e^{-\sqrt{-1}\beta_1\tau} v_2] + A_3^{(5)} e^{\beta_2\tau} v_3 - A_4^{(5)} e^{-\beta_2\tau} v_4 \\ &\qquad + \frac{\sqrt{-1}\tau}{a^2}\,[P_1^{(5)} A_1^{(3)} + P_2^{(5)}]\,[e^{\sqrt{-1}\beta_1\tau} v_1 + e^{-\sqrt{-1}\beta_1\tau} v_2] + \frac{\sqrt{-1}}{a^2}\sigma^{(5)}, \end{aligned}\right\} \quad (48)$$

where $P_i^{(5)}$ $(i=1, 2)$ are power series in a^2 with real constant coefficients. The solutions for $x^{(5)}$ and $y^{(5)}$ can be made periodic by putting $A_3^{(5)} = A_4^{(5)} = 0$, and then choosing $A_1^{(3)}$ so that

$$P_1^{(5)} A_1^{(3)} + P_2^{(5)} = 0,$$

from which it follows that $A_1^{(3)}$ is real. From the initial conditions (41) we have $A_1^{(5)} = A_2^{(5)}$.

The remaining steps of the integration can be carried on in the same way. The solution at the step $2j$ can be made to satisfy the periodicity and the initial conditions by a proper choice of the constants of integration arising at that step. The solution for w_{2j} contains the constant $A_1^{(2j-1)}$ which is not determined until the step $2j+1$. The non-periodic parts of the complete solutions at the step $2j+1$ are similar to those in (48), and $A_1^{(2j-1)}$ enters these solutions with the same coefficient as $A_1^{(3)}$ enters (48) except for the factor $1/a^{2j-2}$. The solutions can be made to satisfy the initial and periodicity conditions by putting

$$A_1^{(2j+1)} = A_2^{(2j+1)}, \quad A_3^{(2j+1)} = A_4^{(2j+1)} = 0,$$

and then determine the $A_1^{(2j-1)}$ so that the coefficient of τ shall be zero. The desired solutions at the steps $2j$ and $2j+1$ are

$$\begin{aligned} w^{(2j)} &= \frac{1}{a^{2j}} S^{2j}(\tau), \\ x^{(2j+1)} &= A_1^{(2j+1)}\,[e^{\sqrt{-1}\beta_1\tau} u_1 + e^{-\sqrt{-1}\beta_1\tau} u_2] + \frac{1}{a^{2j-2}}\rho^{(2j+1)}, \\ y^{(2j+1)} &= \sqrt{-1} A_1^{(2j+1)}\,[e^{\sqrt{-1}\beta_1\tau} v_1 - e^{-\sqrt{-1}\beta_1\tau} v_2] + \frac{\sqrt{-1}}{a^{2j-2}}\sigma^{(2j+1)}. \end{aligned}$$

Thus the integration can be carried on to any desired degree of accuracy.

The solutions of (7), whether as power series in ε or $\varepsilon^{\frac{1}{3}}$, must satisfy the integral (5) identically in ε or $\varepsilon^{\frac{1}{3}}$. Thus this integral, besides being of use in proving that all the periodic orbits are symmetrical, serves as a check on the computations.

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The Finite Groups of Birational Transformations of a Net of Cubics.

BY LEWIS C. COX.

Introduction.

1. The classification of non-linear periodic birational transformations in the plane into finite groups is due to S. Kantor[*] and A. Wiman.[†] The former showed that all periodic birational transformations in a plane can be transformed by combinations of quadratic transformations into a finite number of types having at most eight fundamental points. In the case of seven fundamental points, he determined the different transformations, but in their groups he made errors which A. Wiman subsequently corrected.

The method of Wiman was dependent upon the fact that these groups are isomorphic with the groups of transformations of the bitangents of a plane quartic C_4 which were known.[‡]

The object of this paper is to establish a method which enables one to determine the Cremona transformations with seven fundamental points which correspond to a given linear transformation of the quartic curve.

The method used is to first find the twenty-eight bitangents of the quartic. Hence a cubic surface is determined, having the quartic as a plane section of a particular cone which is tangent to the surface and has its vertex on the surface. This cubic surface is then depicted upon the plane of the seven fundamental points. The space transformations leaving the surface invariant and corresponding to a collineation of the quartic are next found. The corresponding transformation on the plane of the seven fundamental points can then be determined. The method of determination is first developed for any collineation and is given in full for the cases in which the quartic is invariant under the groups of collineations G_{24}, G_{96}, G_{168}.

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[*] S. Kantor, "Premiers Fondaments Pour Une Theorie Des Transformations Periodiques Univoques," *Atti della Reale Accademia delle scienze fisiche e matematiche di Napoli*, Series 2, Vol. III, No. 7 and Vol. IV, No. 2 (1891), pp. 1–335.

[†] A. Wiman, "Ueber die Endlichen Gruppen von eindeutigen Transformationen in der Ebene," *Mathematische Annalen*, Vol. XLVIII (1896); pp. 194–240.

[‡] R. de Paolis, "Le trasformazioni piane doppie," *Atti delle Reale Accademia di Lincei*, Series 3a, Vol. I (1877), pp. 511–544. "Le trasformazioni piane doppia di secondo ordine, e la sua applicazione alla geometria non euclidea," *Atti d. R. Accad. di Lincei*, Series 3a, Vol. II (1878), pp. 31–50. "La trasformazione piana doppia di terzo ordine, primo genere, e la sua applicazione alle curve del quarto ordine," *Atti d. R. Accad. di Lincei*, Series 3a, Vol. II (1878), pp. 851–878.

General Case.

2. Consider the cubic surface

$$F_3 \equiv f_1(x, y, z)u^2 + 2f_2(x, y, z)u + f_3(x, y, z) = 0, \tag{1}$$

in which $f_i(x, y, z)$ is a ternary form of order i. Equation (1) can be arranged in the form

$$\{f_1(x, y, z)u + f_2(x, y, z)\}^2 = f_2^2(x, y, z) - f_1(x, y, z)f_3(x, y, z), \tag{2}$$

which represents a composite surface made up of F_3 and the plane $f_1(x, y, z) = 0$, the latter being tangent to F_3 at the point $0 \equiv (0, 0, 0, 1)$. Solving for u we get

$$u = \frac{-f_2(x, y, z) \pm \sqrt{f_2^2(x, y, z) - f_1(x, y, z)f_3(x, y, z)}}{f_1(x, y, z)}. \tag{3}$$

Hence, any line

$$\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}$$

through the point $0 \equiv (0, 0, 0, 1)$ on the surface meets F_3 in two other points. These two residual points will coincide if

$$f_2^2(\lambda, \mu, \nu) - f_1(\lambda, \mu, \nu)f_3(\lambda, \mu, \nu) = 0,$$

that is, the line is tangent to the surface and passes simply through 0.

Therefore,

$$K_4 \equiv f_2^2(x, y, z) - f_1(x, y, z)f_3(x, y, z) = 0, \tag{4}$$

is the tangent cone with its vertex at 0. The plane $u = 0$ upon which we project $F_3 = 0$ is a double plane because the residual intersections of the lines through 0 with the surface $F_3 = 0$ project into single points. The cone $K_4 = 0$ intersects the plane $u = 0$ in the quartic curve

$$C_4 \equiv \left\{\begin{array}{l} f_2^2 - f_1 f_3 = 0, \\ \quad u \;\; = 0. \end{array}\right\} \tag{5}$$

From the form of C_4 it is seen that $\left\{\begin{array}{l} f_1(x, y, z) = 0 \\ \quad u \quad\;\; = 0 \end{array}\right\}$ is a bitangent to C_4.

The Lines on F_3.

3. Let $\phi_1(x, y, z) = 0$ be the equation of the plane through 0 and one of the other twenty-seven bitangents to C_4. If one of the variables is eliminated

between $\left\{\begin{matrix}\phi_1=0\\ u=0\end{matrix}\right\}$ and (5) the remaining binary form must be a perfect square, whose roots determine the coordinates of the points of contact. Hence, eliminating z between

$$\phi_1(x, y, z)=0 \quad \text{and} \quad (f_1u+f_2)^2=f_2^2-f_1f_3$$

we obtain an expression that is rationally factorable:

$$[F_1u+F_2+\phi_2]\,[F_1u+F_2-\phi_2]=0, \tag{6}$$

in which F_1, F_2 and ϕ_2 are binary forms in x and y. Since equation (2) represents a composite surface whose components are F_3 and $f_1=0$, the equations

$$\left\{\begin{matrix}[F_1u+F_2+\phi_2]\,[F_1u+F_2-\phi_2]=0\\ \phi_1(x, y, z) \qquad =0\end{matrix}\right\} \tag{7}$$

represent the curves $\left\{\begin{matrix}\phi_1(x, y, z)=0\\ F_3 \quad =0\end{matrix}\right\}$ and $\left\{\begin{matrix}f_1(x, y, z)=0\\ \phi_1(x, y, z)=0\end{matrix}\right\}$, the latter defining an extraneous line not lying on F_3. Eliminating the factor f_1 from the factorable equation in (7) we find a remaining linear factor. This taken simultaneously with $\phi_1(x, y, z)=0$ fixes a line of F_3. The residual section of $\left\{\begin{matrix}\phi_1(x, y, z)=0\\ F_3 \quad =0\end{matrix}\right\}$ is a proper conic section.

Therefore, we conclude that the planes through 0 and the twenty-seven bitangents to C_4 meet F_3 in the twenty-seven lines on F_3 and in twenty-seven residual conic sections. Hence each part of the composite curve, consisting of a line and its residual conic which lie not only on F_3 but also in the plane through 0 and one of the twenty-seven bitangents, corresponds to one of the twenty-seven bitangents to C_4.

The plane $f_1(x, y, z)=0$ is tangent to F_3 at 0. It intersects F_3 in the cubic curve

$$\left\{\begin{matrix}f_1(x, y, z)=0,\\ 2f_2(x, y, z)u+f_3(x, y, z)=0.\end{matrix}\right\} \tag{8}$$

It also intersects the plane $u=0$ in

$$\left\{\begin{matrix}f_1(x, y, z)=0,\\ u \quad =0,\end{matrix}\right\}$$

which is the twenty-eighth bitangent to C_4. Therefore, the point 0 and the cubic curve (8) correspond to the twenty-eighth bitangent to C_4.

The question arises, which sign in the second member of the equation

$$F_1u+F_2=\pm\phi_2 \tag{9}$$

corresponds to the cubic curve. Consider the equation

$$uf_1+f_2=+\sqrt{f_2^2-f_1f_3} \tag{10}$$

which is found from (2). Substitute the coordinates of a point $P\equiv(x, y, z, u)$ which lies on the cubic (8). The equation (10) is satisfied. Hence, the plus sign in the second member of (9) corresponds to the cubic (8). The point $(0, 0, 0, 1)$ is the only point satisfying

$$uf_1+f_2=-\sqrt{f_2^2-f_1f_3}.$$

Hence, the minus sign corresponds to the point 0.

Correspondence of Transformations.

4. When C_4 is of genus 3, the only transformations which leave it invariant are collineations. The quartic curves admitting groups of such transformations are enumerated by S. Kantor.*

Let T represent any linear transformation which leaves C_4 invariant. It will permute the bitangents among themselves. From equation (2) we find

$$f_1(x, y, z)u+f_2(x, y, z)=\pm\{f_1(x', y', z')u'+f_2(x', y', z')\}. \qquad (11)$$

The relations (11) and T define two space transformations Q', Q''. Any plane through 0 and a bitangent to C_4 is changed by Q' or Q'' into a plane through 0 and some bitangent. Hence, the lines in which these planes meet F_3 are either interchanged, in which case the space transformation on F_3 is linear, or else each line is changed into the residual conic in the plane of the second line and the transformation is quadratic.

If we now use the Grassman depiction of the cubic surface F_3 upon a second plane $w=0$, the plane sections through 0 are transformed into the ∞^2 cubics through the image of 0 and having for additional basis points the images of the six special lines on F_3 which are depicted as the fundamental points. The point 0, the cubic curve (8) and the composite curves lying on F_3 in the twenty-seven planes through 0 and the lines of the surface are depicted in the plane $w=0$ as the complete fundamental system of the required Cremona transformation.

Since Q', Q'' interchange the twenty-eight special sections of F_3, one changing a line into a line, the other changing a line into a conic, the corresponding transformations T', T'' interchange the parts of the fundamental system. Hence, from T we can derive two birational Cremona transformations.

The Geiser Transformation.

5. In considering $T\equiv\begin{pmatrix}x, & y, & z\\ x', & y', & z'\end{pmatrix}$ in the plane $u=0$ we are led to two

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* S. Kantor, "Theorie der Endlichen Gruppen von eindeutigen Transformationen in der Ebene," Berlin (1895); 120 pages, see p. 86.

space transformations defined by the equations of T and (11). They are:

$$I \equiv \begin{pmatrix} x, & y, & z, & u \\ x', & y', & z', & u' \end{pmatrix},$$

and

$$\Gamma = \begin{cases} x = x' \\ y = y' \\ z = z' \\ w = -\{u' f_1(x', y', z') + f_2(x', y', z')\} - f_2(x', y', z'). \end{cases}$$

The former is the identical transformation.

The latter is a non-linear transformation possessing the following properties. It leaves all the planes and lines through 0 invariant, but interchanges the points P' and P'' which lie on F_3 and are collinear with 0. Therefore, it changes any line on F_3 into the residual conic lying on F_3 and coplanar with 0 and the line. Hence, the transformation is quadratic. The transformation Γ leaves invariant the locus of points R on F_3 which are fixed by the condition

$$f_2^2(\lambda, \mu, \nu) - f_1(\lambda, \mu, \nu) f_3(\lambda, \mu, \nu) = 0.$$

The cone $K_4 = 0$ and the surface $F_3 = 0$ have contact along a sextic space curve C_6 which is invariant point for point under the transformations I and Γ.

Hence, the transformation θ_2 in the plane $w = 0$ corresponding to Γ has the following properties: It interchanges the images of P' and P''. Therefore θ_2 is involutorial. It leaves invariant as a whole every cubic curve through the seven fundamental points. It leaves invariant point for point the image of C_6, which is a plane sextic curve Γ_6 having a double point at each of the seven fundamental points. It is of genus 3, and is non-hyperelliptic.* The transformation θ_2 interchanges the parts of every composite cubic passing through the seven fundamental points. A line through two such points and the conic through the remaining five interchange. Each fundamental point goes into a rational cubic having its double point whose image is the point. This cubic passes singly through the remaining six fundamental points. Each such fundamental cubic and its associated point are interchanged by θ_2. Hence, the transformation is the Geiser involutorial transformation.†

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* V. Snyder, "On a Special Algebraic Curve Having a Net of Minimum Adjoint Curves," *Bulletin of the American Mathematical Society*, Vol. XIV (1907), pp. 70–74.

† R. Sturm, "Die Lehre von den Geometrischen Verwandtschaften," Vol. IV, Leipzig (1908), pp. 95–98. The equations of the Geiser transformation are given by V. Snyder. "An Application of the (1, 2) Quaternary Correspondence to the Weddle and Kummer Surfaces," *Transactions of the American Mathematical Society*, Vol. II (1911), pp. 354–366.

Groups of Transformations.

6. Let $T_1, T_2, T_3, \ldots, T_n$ be a set of operations generating the group G of the quartic C_4. Let $Q'_1, Q'_2, \ldots, Q'_m$ be their corresponding space transformations. Let $T'_1, T'_2, T'_3, \ldots, T'_n$ be their corresponding Cremona transformations in the plane $w=0$. Let the symbol A, B represent a group generated by A and B. Let the symbol (r, s) be read "possesses an r to s isomorphism with."

Subgroups of G may be found

$$[T'_1, \theta_2]\,(1, 1)\,[Q'_1, \Gamma]\,(2, 1)\,[T_1],$$
$$[T'_1, T'_j, \theta_2]\,(1, 1)\,[Q'_1, Q'_j, \Gamma]\,(2, 1)\,[T_1, T_j],$$
$$[T'_1, T'_j, T'_k, \theta_2]\,(1, 1)\,[Q'_2, Q'_j, Q'_k, \Gamma]\,(2, 1)\,[T_1, T_j, T_k].$$

Similarly other subgroups may be found.

Determination of G_{48}.

7. The plane quartic curve

$$C_4 \equiv \begin{Bmatrix} x^4+y^4+z^4+k(x^2y^2+x^2z^2+y^2z^2) = 0 \\ u = 0 \end{Bmatrix} \qquad (13)$$

is invariant under a group $\overline{G}_{24}$ of linear transformations of order 24.* It is generated by the permutation $\overline{G}_6$ on the three letters x, y, z and the harmonic homology defined by changing the sign of any one of the variables.

Bitangents to C_4.

8. Certain lines in the plane $u=0$ are bitangent to C_4. In order to find them it is necessary to identify four special types. The others are found from the collineations of $\overline{G}_6$. We wish to determine, if possible, a relation between k and μ, such that $y=\mu x$ shall be a bitangent; similarly, a relation between k and λ such that $z=\pm\lambda(x\pm y)$ shall be a bitangent.

Eliminate y from $\begin{Bmatrix} y=\pm\mu x \\ u=0 \end{Bmatrix}$ and the equation of C_4 and we obtain

$$\begin{Bmatrix} x^4+k(1+\mu^2)x^2z^2+(1+k\mu^2+\mu^4)z^4=0, \\ u = 0. \end{Bmatrix} \qquad (14)$$

The line is a bitangent when the first member of (14) is a square. The condition for this is

$$k^2(1+\mu^2)^2-4k\mu^2-4k(1+\mu^4)=0. \qquad (15)$$

Solve for k and select that value for which the first member of (13) is not a perfect square. We get

$$k=\frac{4\mu^2}{(1+\mu^2)^2}-2. \qquad (16)$$

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* E. Ciani, "Contributo alla teoria del gruppo di 168 collineationi piane," *Annali di Mathematica*, Series 3, Vol. V (1901), pp. 35–55. See p. 43.

Hence $y=\pm\mu x$ must be a bitangent of the C_4, (13) having the value of k defined by (16), where μ may have any finite value.

Similarly, if we eliminate z between (13) and $z=\pm\lambda(x\pm y)$, place the discriminant of the resulting equation equal to zero, and factor out $(k-2)$ we obtain

$$(k+1)\lambda^4-k\lambda^2-1=0. \tag{17}$$

Solving for λ^2 we find $\lambda^2=+1$ or $\lambda^2=-\dfrac{1}{k+1}$. By substituting the second value of k in (16) we get

$$\lambda=\pm\frac{1+\mu^2}{1-\mu^2}. \tag{18}$$

Hence, the line $\left\{\begin{matrix} z=\pm\lambda(x\pm y) \\ u=0 \end{matrix}\right\}$ must be bitangent to C_4 for the special values of λ.

By operating on the bitangents with $\overline{G}_6$ we obtain the following bitangents:

(1) $y=-\mu z$,
(2) $z=-\mu x$,
(3) $x=-\mu y$,
(4) $x=\lambda(z-y)$,
(5) $y=\lambda(x-z)$,
(6) $z=\lambda(y-x)$,
(7) $x+y+z=0$,
(8) $x=-\lambda(y+z)$,
(9) $y=-\lambda(z+x)$,
(10) $z=-\lambda(x+y)$,
(11) $x=-\mu z$,
(12) $y=-\mu x$,
(13) $z=-\mu y$,
(14) $y=\mu z$,
(15) $z=\mu x$,
(16) $x=\mu y$,
(17) $x=\lambda(y-z)$,
(18) $y=\lambda(z-x)$,
(19) $z=\lambda(x-y)$,
(20) $x=\lambda(y+z)$,
(21) $y=\lambda(x+z)$,
(22) $z=\lambda(x+y)$,
(23) $x=\mu z$,
(24) $y=\mu x$,
(25) $z=\mu y$,
(26) $-x+y+z=0$,
(27) $x-y+z=0$,
(28) $x+y-z=0$,

The Cubic Surface.

9. The next step is to construct the cubic surface having the properties explained in the general case. Select the four bitangents

$$\left\{\begin{matrix} x+y+z=0 \\ u=0 \end{matrix}\right\},\quad \left\{\begin{matrix} -x+y+z=0 \\ u=0 \end{matrix}\right\},\quad \left\{\begin{matrix} x-y+z=0 \\ u=0 \end{matrix}\right\},\quad \left\{\begin{matrix} x+y-z=0 \\ u=0 \end{matrix}\right\}.$$

The planes passing through $0\equiv(0,0,0,1)$ and these special lines are evidently

$$x+y+z=0, \quad x-y+z=0,$$
$$-x+y+z=0, \quad x+y-z=0.$$

Take $x+y+z=0$ to be the plane tangent to F_3 at 0. The cubic surface has an equation of the form

$$F_3\equiv(x+y+z)u^2+2p(x^2+y^2+z^2)u+(-x+y+z)(x-y+z)(x+y-z)=0. \quad (19)$$

The tangent cone at $(0, 0, 0, 1)$ has the equation

$$(p^2+1)\Sigma x^4+2(p^2-1)\Sigma x^2y^2=0. \quad (20)$$

Comparing the latter with the first one of the two equations in (13) we notice that

$$\frac{2(p^2-1)}{p^2+1}=k=\frac{4\mu^2}{(1+\mu^2)^2}-2. \quad (21)$$

Hence,

$$p^2=\frac{2+k}{2-k} \text{ or } p=\frac{\pm\mu}{\sqrt{1+\mu^2+\mu^4}}.$$

Either sign may be used for p. The cubic surface

$$F_3\equiv(x+y+z)u^2+\frac{2\mu(x^2+y^2+z^2)u}{\sqrt{1+\mu^2+\mu^4}}+(-x+y+z)(x-y+z)(x+y-z)=0 \quad (22)$$

is selected for our discussion. Rearranging (19) we find

$$(x+y+z)u+\frac{2\mu(x^2+y^2+z^2)}{\sqrt{1+\mu^2+\mu^4}}=\pm(p^2+1)\sqrt{\Sigma x^4+\frac{2p^2-1}{p^2+1}\Sigma x^2y^2} \quad (23)$$

which by means of (21) can be reduced to the form

$$\{(x+y+z)v+\mu(x^2+y^2+z^2)\}^2=(1+\mu^2)\{\Sigma x^4+k\Sigma x^2y^2\}, \quad (24)$$

where $v=u\sqrt{1+\mu^2+\mu^4}$ and $k=\frac{4\mu^2}{(1+\mu^2)^2}-2.$

Equation (24) defines the composite surface F_3 and $(x+y+z)=0$.

The Lines on the Cubic Surface.

10. Each plane determined by 0 and some one of the twenty-seven bitangents to the quartic C_4 is a bitangent plane to F_3. The plane $x+y+z=0$, containing a bitangent to C_4, is tangent to F_3 at 0. It meets F_3 in a non-composite cubic curve.

We find each of the 27 lines upon F_3, together with their residual conics in the plane through 0 by solving their equations simultaneously with the equation of the composite surface (24).

The latter equation is reduced to the form

$$(x+y+z)v+\mu(x^2+y^2+z^2)=\pm(1+\mu^2)\cdot\phi_2, \tag{25}$$

where ϕ_2 corresponds to ϕ_2 in equation (6). One sign in the right member corresponds to a line on F_3 when one of the twenty-seven special planes is taken simultaneously with it. The opposite sign corresponds to a conic section lying in the bitangent plane used.

Equation (7) needs the same consideration as was given to the plane $f_1(x, y, z)=0$ in Section 3. We may show by the same reasoning as was employed there that the positive sign in the right member of (25) corresponds to the cubic curve

$$\left.\begin{aligned} 2\mu(x^2+y^2+z^2)u+(-x+y+z)(x-y+z)(x+y-z)&=0,\\ x+y+z&=0.\end{aligned}\right\} \tag{26}$$

The following list of equations represents the lines and the residual conic sections unless otherwise stated. The equations are numbered to correspond with the bitangents given previously. The first two equations in each trio fix a line, the first and third fix the residual conic sections in the plane connecting the line with 0. The third equation is added for a purpose which will be explained later.

(1)
$$y+\mu z=0$$
$$v+(1+\mu+\mu^2)(x-\overline{1-\mu}\cdot z)=0$$
$$(x+y+z)v+\mu(x^2+y^2+z^2)$$
$$=+\{(1+\mu^2)x^2-(1+\mu^4)z^2\}$$

(4)
$$x=\lambda(z-y)$$
$$v+\mu\{(1-\lambda)z+(1+\lambda)y\}=0$$
$$(x+y+z)v+\mu(x^2+y^2+z^2)$$
$$=-2\mu\cdot\lambda^2\left\{z^2+y^2-\frac{2\lambda^2+1}{\lambda^2}\cdot yz\right\}$$

(7)
$x+y+z=0$ corresponds to $0\equiv(0,0,0,1)$
$$(x+y+z)v+\mu(x^2+y^2+z^2)$$
$=+2\mu(x^2+z^2+xz)$ corresponds to the cubic curve in the tangent plane.

(8)
$$x=-\lambda(y+z)$$
$$(1-\lambda)v+\mu(1+3\mu^2)(y+z)=0$$
$$(x+y+z)v+\mu(x^2+y^2+z^2)$$
$$=+2\mu\cdot\lambda^2\left\{y^2+z^2+\frac{2\lambda^2+1}{\lambda^2}yz\right\}$$

$$(26)\qquad \begin{gathered} -x+y+z=0 \\ v=0 \\ (x+y+z)v+\mu(x^2+y^2+z^2) \\ =-2\mu\{y^2+z^2+yz\} \end{gathered}$$

$$(10)\qquad \begin{gathered} z=-\lambda(x+y) \\ (1-\lambda)v+\mu(1+3\mu^2)(x+y)=0 \\ (x+y+z)v+\mu(x^2+y^2+z^2) \\ =+2\mu\cdot\lambda^2\cdot\left\{x^2+y^2+\frac{2\lambda^2+1}{\lambda^2}xy\right\} \end{gathered}$$

$$(11)\qquad \begin{gathered} x=-\mu z \\ v+(1+\mu+\mu^2)(y-\overline{1-\mu}\,z)=0 \\ (x+y+z)v+\mu(x^2+y^2+z^2) \\ =+\{(1+\mu^2)y^2-(1+\mu^4)z^2\} \end{gathered}$$

$$(12)\qquad \begin{gathered} y=-\mu x \\ v+(1+\mu+\mu^2)(z-\overline{1-\mu}\,x)=0 \\ (x+y+z)v+\mu(x^2+y^2+z^2) \\ =+\{(1+\mu^2)z^2-(1+\mu^4)x^2\}. \end{gathered}$$

Apply $\begin{pmatrix} x\,y\,z \\ y\,z\,x \end{pmatrix}$ to (1), (2), (4), (5), (8), (9) to get (2), (3), (5), (6), (9), (10). Write $-\mu$ for μ, $-v$ for v in (1), (2), (3) to get (14), (15), (16) and $-\lambda$ for λ in (4), (5), (6), (8), (9), (10) to get (17), (18), (19), (20), (21), (22). Interchange x and y in (3), (16), (26) to get (12), (24), (27); y and z in (1), (14), (27) to get (13), (25), (28); z and x in (2), (15) to get (11), (23).

Arrangement of the 27 *Lines on* F_3.

11. A well-known property of a non-singular cubic surface F_3 is that each line on F_3 is contained in a set of five planes each of which contains two other lines on F_3. By testing the equations in Section 10, we find that the line (26) forms a triangle with each pair of lines [(14), (25)], [(20), (8)], [(17), (4)], [(1), (13)], [(28), (27)] and, similarly, (28) with [(16), (24)], [(22), (10)], [(19), (6)], [(3), (12)], [(26), (27)], and (27) with [(15), (23)], [(21), (9)], [(18), (5)], [(2), (11)], [(28), (26)].

If we call $(26)\equiv C_{14}$* and $(16)\equiv C_{12}$, then (27) is a c line which we shall designate by c_{25}; (24) by c_{45} and (28) by c_{36}.

---

* Schläfli's notation has been adopted. F. Schläfli, "An Attempt to Determine the Twenty-seven Lines upon a Surface of the Third Order and to Divide Such Surfaces into Species in Reference to the Reality of the Lines upon the Surface," *Quarterly Journal*, Vol. II (1858), pp. 55–65 and 110–120; see p. 115. K. Doehlemann, "Geometrische Transformationen," Teil II (1907), pp. 303–305.

The only four lines skew to (26) and (24) but intersecting (16) and (27) are (21), (23), (2) and (18), which must consist of two c lines, one a and one b line. Call them c_{34}, c_{46}, a_2, b_2, respectively. The only four lines skew to (16) and (27), but intersecting (26) and (24) are (14), (20), (4) and (13). Of these (14) is skew (28) and, therefore, must be c_{23}. Similarly, (20), (4) and (13) must be c_{26}, a_4, b_4, respectively. We see now that $(25) \equiv c_{56}$, $(8) \equiv c_{35}$, $(17) \equiv b_1$, $(1) \equiv a_1$, $(15) \equiv c_{13}$, $(9) \equiv c_{16}$, $(5) \equiv a_5$, $(11) \equiv b_5$, leaving unlettered the lines [(22), (10)], [(19), (6)], [(3), (12)].

Consider the double six

$$\begin{array}{cccccc} c_{14}, & c_{45}, & c_{34}, & c_{24}, & a_6, & b_6, \\ c_{16}, & c_{56}, & c_{36}, & c_{26}, & a_4, & b_4, \end{array}$$

(25) meets (10), (6) and (12) which must be c_{24}, a_6, b_6 in the same order, but (20) is skew to (10), (6) to (4), (12) to (13); hence, the order is correct. This completely determines the lettering of the lines which is as follows:

$$\begin{array}{llll} (1) \equiv a_1 & (8) \equiv c_{35} & (15) \equiv c_{13} & (22) \equiv c_{15} \\ (2) \equiv a_2 & (9) \equiv c_{16} & (16) \equiv c_{12} & (23) \equiv c_{46} \\ (3) \equiv a_3 & (10) \equiv c_{24} & (17) \equiv b_1 & (24) \equiv c_{45} \\ (4) \equiv a_4 & (11) \equiv b_5 & (18) \equiv b_2 & (25) \equiv c_{56} \\ (5) \equiv a_5 & (12) \equiv b_6 & (19) \equiv b_3 & (26) \equiv c_{14} \\ (6) \equiv a_6 & (13) \equiv b_4 & (20) \equiv c_{26} & (27) \equiv c_{25} \\ & (14) \equiv c_{23} & (21) \equiv c_{34} & (28) \equiv c_{36} \end{array}$$

Instead of choosing (26) as a c line we might have taken it for an a_1 or b_1 line in which cases (16) would be b_1 or a_1, respectively. Hence, we may name the lines on F_3 in a variety of ways.

Identification of the Generators of G_{45}.

12. Select a transformation $T_3 \equiv \begin{pmatrix} x, & y, & z \\ y', & z', & x' \end{pmatrix}$ which leaves C_4 invariant. (The subscript in T_3 denotes the period.) Choose the transformation Q'_3 which is defined by the equations

$$\begin{cases} x = y' \\ y = z' \\ z = x' \\ v = \dfrac{-\mu(x'^2+y'^2+z'^2)+(x'+y'+z')v'+\mu(x^2+y^2+z^2)}{x+y+z} \end{cases}$$

which in our case reduces to the linear space transformation

$$Q'_3 = \begin{pmatrix} x, & y, & z, & v \\ y', & z', & x', & v' \end{pmatrix}.$$

It is one of the two space transformations which correspond to T_3. The transformation Q'_3 permutes the twenty-eight planes through 0 among themselves. It leaves invariant the left member of each second degree equation listed in Section 10. It may or may not change the right member. The same is true of those second degree equations from which those planes are found, which with the bitangent planes fix the lines on F_3. The transformation Q'_3 is operated as follows: Take for example the line a_1. The transformation $T_3 \equiv \begin{pmatrix} x, & y, & z \\ y', & z', & x' \end{pmatrix}$ transforms the bitangent $\begin{Bmatrix} y = -\mu z \\ u = \quad 0 \end{Bmatrix}$ into the bitangent $\begin{Bmatrix} z = -\mu x \\ u = \quad 0 \end{Bmatrix}$; hence, Q'_3 changes the plane $y = -\mu z$ into the plane $z = -\mu x$. The left member of the third equation under (1) remains invariant under Q'_3. The right changes but it is changed only by the equations $\begin{Bmatrix} x = y' \\ y = z' \\ z = x' \end{Bmatrix}$ which define T_3. Therefore, change the sign of the right member of the quadratic equation given; as,

$$-\{(1+\mu^2)x^2 - (1+\mu^4)z^2\}.$$

This corresponds to the line a_1 itself. Operate on this expression by T_3 and obtain

$$-\{(1+\mu^2)y^2 - (1+\mu^4)x^2\};$$

the latter corresponds to the line a_2. Hence, we conclude that a_1 goes into a_2.

In applying this process we find that Q'_3 transforms a_1, a_2, a_3, a_4, a_5, a_6 and the point 0 into a_2, a_3, a_1, a_5, a_6, a_4 and the point 0, respectively.

The images of a_1, a_2, a_3, a_4, a_5, a_6 and the point 0 in the plane $w = 0$ are the fundamental points 1, 2, 3, 4, 5, 6, 7.[*] The images of the b_i lines are the conics $[i7]$. The image of the residual of b_i is the line $(i, 7)$. The image of the line c_{ik} is the line (i, k). The image of its residual is the conic $[ik]$. The image of the residuals of 0, a_i are the cubics $[7]$ $[i]$.

Hence, the corresponding Cremona transformation is

$$T'_3 \equiv \begin{pmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ 2, & 3, & 1, & 5, & 6, & 4, & 7 \end{pmatrix}.$$

This notation means that the image of 1 is 2, and so on for the other fundamental points. Since a transformation is fixed when the images of its fundamental points are known, we have our required transformation which is linear and of period 3.

If we had taken the transformation Q''_3 which also corresponds to T_3, we would have been led to a new transformation T''_3. This can also be found by operating on T'_3 by θ_2, the Geiser transformation. It transforms the point 1 into the residual of its image in the transformation T'_3. Similarly for the other fundamental points. Therefore,

---

[*] The symbol (ij) denotes the line ij, $[ij]$ the conic through the remaining five fundamental points, and $[i]$ the cubic with a double point at i which passes through the other six fundamental points.

$$T''_3 \equiv \begin{Bmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ [2], & [3], & [1], & [5], & [6], & [4], & [7] \end{Bmatrix}$$

which is of order 8.

The transformation $T_2 \equiv \begin{Bmatrix} x, & y, & z \\ -x', & y', & z' \end{Bmatrix}$ leads to the transformation

$$T'_2 \equiv \begin{Bmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ 1, & [13], & [12], & [17], & [16], & [15], & [14] \end{Bmatrix}.$$

The transformation $T''_2 \equiv T'_2 \cdot \theta_2$ is

$$T''_2 \equiv \begin{Bmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ [1], & (13), & (12), & (17), & (16), & (15), & (14) \end{Bmatrix}$$

which is a Jonquières transformation* of order 4.

Use the same fundamental process on $\overline{T}_2 \equiv \begin{pmatrix} x, & y, & z \\ y, & x, & z \end{pmatrix}$. We are led to

$$\overline{T}'_2 \equiv \begin{Bmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ [57], & [47], & [67], & [27], & [17], & [37], & [7] \end{Bmatrix}$$

which is of order 5 and

$$\overline{T}''_2 \equiv \begin{Bmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ (57), & (47), & (67), & (27), & (17), & (37), & [7] \end{Bmatrix}$$

which is a Jonquières transformation of order 4.

G_{48} *with Some of its Subgroups.*

13. Using the notation of Section 6 we can arrive at different groups:

$$\begin{aligned}
G_4 &\equiv [T'_2, \theta_2]\,(1,1)\,[Q'_2, \Gamma]\,(2,1)\,[T_2] \equiv \overline{G}_2 \text{ on } C_4, \\
G_6 &\equiv [T'_3, \theta_2]\,(1,1)\,[Q'_3, \Gamma]\,(2,1)\,[T_3] \equiv \overline{G}_3 \text{ on } C_4, \\
G_4 &\equiv [\overline{T}'_2, \theta_2]\,(1,1)\,[\overline{Q}'_2, \Gamma]\,(2,1)\,[T'_2] \equiv \overline{G}_2 \text{ on } C_4, \\
G_{12} &\equiv [T'_3, T'_2, \theta_2]\,(1,1)\,[Q'_3, Q'_2, \Gamma]\,(2,1)\,[T_3, T_2] \equiv \overline{G}_6 \text{ on } C_4, \\
G_{16} &\equiv [\overline{T}'_2, T'_2, \theta_2]\,(1,1)\,[\overline{Q}'_2, Q'_2, \Gamma]\,(2,1)\,[T_2, T_2] \equiv \overline{G} \text{ on}_8\ C_4, \\
G_{24} &\equiv [T'_3, T'_2, \theta_2]\,(1,1)\,[Q'_3, Q'_2, \Gamma]\,(2,1)\,[T_3, T_2] \equiv \overline{G}_{12} \text{ on } C_4, \\
G_{48} &\equiv [T'_3, T'_2, \overline{T}'_2, \theta_2]\,(1,1)\,[Q'_3, Q'_2, \overline{Q}'_2, \Gamma]\,(2,1)\,[T_3, T_2, T_2] \equiv \overline{G}_{24}.
\end{aligned}$$

On account of the isomorphism existing between the groups G_{2n} and the groups $\overline{G}_n$ many other groups may be found such that groups G_m are simply isomorphic to $\overline{G}_m$.

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* K. Doehlemann, "Geometrische Transformationen," Teil II (1907), p. 150.

14. Since the value of μ in our previous discussion may have any finite value, we may expect to find groups of higher order than 48 by restricting the value of μ.

Determination of G_{192}.

15. Let $k=0$, hence from the equation $\frac{4\mu^2}{(1+\mu^2)^2}-2=0$, we deduce

$$\mu^4+1=0. \tag{27}$$

Therefore, the quartic curve (13) has an equation of the form

$$\left\{\begin{matrix} x^4+y^4+z^4=0 \\ u \quad =0 \end{matrix}\right\}, \tag{28}$$

which has been studied by Dyck.* This curve is invariant under the group $\overline{G}_{96}$ of order 96. The octahedral group $\overline{G}_{24}$ and the transformation $T_4\equiv\begin{pmatrix} x, & y, & z \\ ix', & y', & z' \end{pmatrix}$. generate it. Hence, in order to determine the corresponding group G_{192} all that remains is to determine a transformation in the plane $w=0$ corresponding to T_4.

Repeat the argument precisely as given in the previous case understanding that μ is subject to the condition

$$\mu^4+1=0$$

in all of the previous equations.

We are led to the transformations T'_4 of order 6,

$$\left\{\begin{matrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ [1], & [46], & [67], & [36], & (15), & [5], & [26] \end{matrix}\right\}$$

$$\text{and } T''_4\equiv T'_4\cdot\theta_2\equiv\left\{\begin{matrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ (1), & (46), & (67), & (36), & [15], & (5), & (26) \end{matrix}\right\}$$

is a Jonquière transformation of order 3.

T'_4 and T''_4 correspond to $T_4\equiv\left\{\begin{matrix} x, & y, & z \\ ix', & y', & z' \end{matrix}\right\}$.

Hence, the group $G_{192}\equiv[G_{24}, T'_4, \theta_2]$ (2,1) the group $\overline{G}_{96}\equiv[\overline{G}_{24}, T_4]$.

Determination of G_{336}.

16. Consider the case where μ satisfies the equation

$$\mu^2-\mu+2=0. \tag{29}$$

The value of k subject to this restriction is -3μ. Hence, the quartic C_4 reduces to the form

$$C_4\equiv\left\{\begin{matrix} x^4+y^4+z^4-3\mu(x^2y^2+x^2z^2+y^2z^2)=0 \\ u \quad =0 \end{matrix}\right\}. \tag{30}$$

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* W. Dyck, "Notiz über eine reguläre Riemannische Fläche vom Geschlechte drei und die zugehörige Normalcurve vierter Ordnung," *Mathematische Annalen*, Vol. XVIII (1881), pp. 510–516; see p. 512.

Ciani[*] states that this quartic, subject to the condition (29), can be reduced by a suitable linear transformation to the form

$$\left\{\begin{matrix} x^3 z + y^3 x + z^3 y = 0 \\ u \hfill = 0 \end{matrix}\right\}. \tag{31}$$

Klein[†] has shown that the latter admits a group of collineations $\overline{G}_{168}$ which is generated by an octahedral group $\overline{G}_{24}$ and a certain cyclic group $\overline{G}_7$.

Hence, the quartic in the form given by Ciani is invariant under a group $\overline{G}_{168}$ which is generated by our octahedral group $\overline{G}_{24}$ and some $\overline{T}_7$. Such a transformation

$$T_7 \equiv \left\{\begin{matrix} x = -[2x' - \mu(y' - z')] \\ y = 2x' + \mu(y' - z') \\ z = \mu^2(y' + z') \end{matrix}\right\}$$

is given by Sharpe.[‡]

Hence, we repeat the argument which is given for G_{48} with the restriction that μ satisfies the equation

$$\mu^2 - \mu + 2 = 0$$

in order to determine a transformation T_7' corresponding to T_7. Since some care is necessary in determining the images of $a_1, a_2, a_3, a_4, a_5, a_6$ and the point 0, the details in finding the image of a_1 are given.

First operate on the quartic

$$\left\{\begin{matrix} x^4 + y^4 + z^4 - 3\mu(x^2 y^2 + x^2 z + y^2 z^2) = 0 \\ u \hfill = 0 \end{matrix}\right\}$$

with the transformation T_7. Select the special term containing x^4 which we find to be

$$16(2+k)x^4.$$

Since
$$k = \frac{4\mu^2}{(1+\mu^2)^2} - 2,$$

therefore,
$$16(2+k)x^4 = \frac{(16)(4)\mu^2}{(1+\mu^2)^2} \cdot x^4.$$

Hence, in extracting the root we find the factor

$$\sqrt{16(2+k)} = \pm \frac{8\mu}{1+\mu^2},$$

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[*] E. Ciani, "I Varii Tipi Possibili di Quartiche Piane più Volte Omologico Armoniche," *Rendiconti del circolo Matematico di Palermo*, Vol. XIII (1899), pp. 347–373; see p. 365.

[†] F. Klein, "Ueber die Transformation siebenter Ordnung der Elliptischen Functionen, *Mathematische Annalen*, Vol. XIV (1879), pp. 428–471; see p. 446.

[‡] F. R. Sharpe, "Conics through Inflections of Self-Projective Quartics," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVII (1915), pp. 66–72; see p. 71.

but since $\mu^2-\mu+2=0$,

$$\sqrt{16(2+k)}=-\frac{8\mu}{1+\mu^2}=4\mu^2.$$

Hence, in making certain comparisons the factor $4\mu^2$ must be taken into account.

We now operate with the Q_7' corresponding to T_7 as follows: The line a_1 is determined by

$$y=-\mu z$$
$$(x+y+z)v+\mu(x^2+y^2+z^2)=-\{(1+\mu^2)x^2-(1+\mu^4)z^2\},$$

or

$$y=-\mu z$$
$$(x+y+z)v+\mu(x^2+y^2+z^2)=-(\mu-1)\{x^2+3z^2\}.$$

Operate on $y=-\mu z$ *by* T_7 and we get the equation

$$z=\frac{\mu^2}{4}(y-x). \qquad (32)$$

Hence, a_1 goes into the line a_6 or its residual. To decide which, operate on

$$-(\mu-1)\{x^2+3z^2\} \qquad (33)$$

by

$$x=-[2x'-\mu(y'-z')],\quad z=\frac{\mu^2}{4}(y'+z'), \qquad (34)$$

where $z'=\frac{\mu^2}{4}(y''-x'')$ as in equation (32).

The coefficient of x''^2 reduces to $-\frac{1}{2}\mu^7$, but a_6 is defined by

$$z=\frac{\mu^2}{4}(y-x),(x+y+z)v+\mu(x^2+y^2+z^2)=\frac{+2\mu^5}{16}\left\{y^2+x^2-\frac{2\lambda^2+1}{\lambda^2}\cdot xy\right\}. \qquad (35)$$

Compare the x''^2 term with the corresponding term in (35) remembering the former contains an extra factor $4\mu^2$, due to operating with T_7. Hence, we conclude that a_1 goes into the residual of a_6. Similarly, we show that a_3 and 0 go into the lines c_{36}, c_{56} while a_2, a_4, a_5, a_6 go into the residuals of c_{12}, b_6, c_{24}, c_{14}, respectively. Hence, in the w plane we get the transformations T_7' of order 5,

$$\left\{\begin{matrix}1, & 2, & 3, & 4, & 5, & 6, & 7\\ [6], & [12], & (36), & (67), & [24], & [14], & (56)\end{matrix}\right\}$$

and

$$T_7'\cdot\theta_2\equiv T_7''\equiv\left\{\begin{matrix}1, & 2, & 3, & 4, & 5, & 6, & 7\\ [6], & (12), & [36], & [67], & (24), & (14), & [56]\end{matrix}\right\}.$$

Hence, the group $C_{336}\equiv[G_{24}, T_7', \theta_2](2,1)$, the group $\overline{G}_{168}\equiv[\overline{G}_{24}, T_7]$.

CORNELL UNIVERSITY, *July*, 1915.

On the Determination of a Certain Class of Surfaces.[*]

By Archer Everett Young.

Introduction.

The second fundamental form for a surface S referred to any set of conjugate lines can be written[†]

$$D\,d\bar{u}^2+D''\,d\bar{v}^2.$$

If the surface in question has positive total curvature there exists an infinity of conjugate systems whose choice as lines of reference, the parameters being properly chosen, reduce this form to

$$D\,(d\bar{u}^2+d\bar{v}^2).$$

These particular conjugate systems have been named[‡] isotherm-conjugate. If, on the other hand, the surface has negative total curvature, there exists an infinity of conjugate systems whose choice as lines of reference permit the second form to be written

$$D\,(d\bar{u}^2-d\bar{v}^2).$$

We shall, for convenience, refer to such as associate isotherm-conjugate lines. Many well-known surfaces of positive total curvature have *lines of curvature* which belong to the class defined as isotherm-conjugate, while others of negative total curvature have lines of curvature which are associate isotherm-conjugate.

We have discussed in this paper the *problem of determining all surfaces having lines of curvature which are either isotherm-conjugate or associate isotherm-conjugate.*

We show in Article I that on the surfaces having negative total curvature, the asymptotic lines form a network of infinitesimal rhombi, while on those of positive the characteristic lines[§] form a similar network. We may say, therefore, that our ***problem is that of determining all surfaces on which either the asymptotic or the characteristic lines form a network of infinitesimal rhombi.***

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[*] Presented to the American Mathematical Society, December 31, 1915.

[†] Bianchi, "Lezioni di Geometria Differenziale," p. 113.

[‡] Bianchi, *loc. cit.*, p. 168.

[§] A name given by Pucci to that system of lines of positive total curvature, for which when it is parametric, $\frac{D}{E}=\frac{D''}{G}$, $\frac{D}{e}=\frac{D''}{g}$, $D'=0$ ("Dell' angolo caratteristico e delle linee caratteristiche di una superficie," *Rom. Acc. L. Rend.*, IV (1889), pp. 501–507).

The problem is treated in Article II from the standpoint of the spherical representation of the asymptotic lines for surfaces having negative total curvature, and characteristic lines for surfaces having positive total curvature. After the derivation of the system of equations, on the solution of which the general problem depends, certain particular solutions are considered in Article III. In the closing Article IV, we have reduced the problem to the solution of the fundamental equations used in the Bonnet form, with the addition of another equation in the same functions. We are enabled thus to point out a general method for determining all surfaces of the problem which are solutions of the general "Problem of the Spherical Representation".

I. *The Characteristic Geometric Properties of the Surfaces.*

Let $d\bar{s}^2 = Edu^2 + 2F du dv + G d\bar{v}^2$ be the expression for the linear element of a surface S of negative total curvature referred to asymptotic lines. The equation of the characteristic lines, which in general is[*]

$$\begin{gathered}[D(GD-ED'')-2D'(FD-ED')]\,d\bar{u}^2+[2D'(GD+ED'')-4FDD'']\,du\,dv \\ +[2D'(GD'-FD'')-D''(GD-ED'')]\,d\bar{v}^2=0,\end{gathered}$$

reduces to

$$E d\bar{u}^2 + G d\bar{v}^2 = 0,$$

since the lines of reference are asymptotic. They are, therefore, imaginary on a surface of negative total curvature.

If the asymptotic lines on the surface chosen, form a network of infinitesimal rhombi, the expression above by a proper choice of parameters reduces to

$$d\bar{s}^2 = \lambda\,(d\bar{u}^2 + 2\cos\theta\, du\,dv + d\bar{v}^2), \tag{I}$$

where θ is the angle between the asymptotic lines.

The equations of the characteristic lines and of the lines of curvature on the surface corresponding to (I) are, respectively, $d\bar{u}^2 + d\bar{v}^2 = 0$, and $d\bar{u}^2 - d\bar{v}^2 = 0$.

Referring the surface to lines of curvature by writing $du + dv = du_1$, $du - dv_1 = dv_1$, (I) becomes[†] $d\bar{s}^2 = \frac{\lambda}{2}\Big((1+\cos\theta)\,d\bar{u}_1^2 + (1-\cos\theta)\,d\bar{v}_1^2\Big)$, and the equation of the asymptotic lines is now $d\bar{u}_1^2 - d\bar{v}_1^2 = 0$. Hence the theorem:

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[*] Eisenhart, *Transactions of A. M. S.*, Vol. V, pp. 421–437.

[†] Knoblauch, "Einleitung in die Allgemeine Theorie der Krummen Flächen," p. 12.

If the asymptotic lines on a surface S of negative total curvature form on the surface a network of infinitesimal rhombi, the lines of curvature must be associate isotherm-conjugate. The converse is easily proven.

From the form of the expression for the linear element just above, it appears that the corresponding surface will be *isothermic*; that is, *have isothermal lines of curvature*, when and only when

$$\tan\frac{\theta}{2}=\frac{f(u_1)}{\phi(v_1)},$$

where, of course, θ is the angle between the asymptotic lines, and hence the theorem:

A surface of negative total curvature which has associate isotherm-conjugate lines of curvature will be isothermic when and only when the tangent of one-half the angle between the asymptotic lines is equal to a function of one parameter divided by a function of the other, the lines of reference being lines of curvature.

Turning now to the consideration of surfaces having positive total curvature, let $\overline{ds}_1^2=E_1\overline{du}^2+2F_1 dvdu+G_1\overline{dv}^2$ be the expression for the linear element of a surface referred to characteristic lines.

Since $\dfrac{D_1}{E_1}=\dfrac{D_1''}{G_1}$, $D_1'=0$, the equation of the asymptotic lines may be written thus:

$$E_1\,du^2+G_1\,\overline{dv}^2=0.$$

If the characteristic lines form a network of rhombi, then, by a proper choice of parameters, the expression for the linear element becomes

$$\overline{ds}_1^2=\lambda_1(du^2+2\cos\theta_1\,dudv+\overline{dv}^2), \tag{I'}$$

where θ_1 is the angle between the characteristic lines. The equations of the asymptotic lines and of the lines of curvature on a surface which corresponds to (I′) are, respectively,

$$\overline{du}^2+\overline{dv}^2=0,\ \text{and}\ \overline{du}^2-\overline{dv}^2=0. \tag{II}$$

Taking the lines of curvature for lines of reference, and following the reasoning of the preceding case, we easily prove the following theorems:

If a surface of positive total curvature has characteristic lines which divide the surface into a network of infinitesimal rhombi, its lines of curvature are isotherm-conjugate; and, conversely.

The necessary and sufficient condition that a surface of positive total curvature, which has isotherm-conjugate lines of curvature, shall be isothermic, is that the tangent of one-half the angle between the characteristic lines shall be equal to a function of one parameter over a function of the other, the lines of reference being lines of curvature.

II. *Discussion of the General Problem.*

If (I) is the first form for a surface S of negative total curvature, referred to asymptotic lines, the third form may be written thus:[*]

$$d\bar{\sigma}^2=\mu\,(d\bar{u}^2-2\cos\theta\,du\,dv+d\bar{v}^2), \tag{III}$$

where $\mu=\dfrac{\lambda}{-K}$, K being the total curvature.

Likewise, if (I′) is the first form for a surface S_1 of positive total curvature, referred to characteristic lines, the third form[†] is

$$d\bar{\sigma}_1^2=\mu_1\,(d\bar{u}^2-2\cos\theta_1\,du\,dv+d\bar{v}^2), \tag{III′}$$

where $\mu_1=\lambda_1/K_1$, K_1 as before being the total curvature.

If (III) is the third form for a surface S, referred to asymptotic lines, the functions involved must satisfy the following equation:[‡]

$$\frac{\partial}{\partial u}\left[\frac{\dfrac{\partial\log\mu}{\partial v}+\cos\theta\dfrac{\partial\log\mu}{\partial u}}{\sin^2\theta}\right]=\frac{\partial}{\partial v}\left[\frac{\partial\log\mu}{\partial u}+\cos\theta\frac{\partial\log\mu}{\partial v}\right]; \tag{1}$$

and, similarly, if (III′) is the third form for a surface S_1 referred to characteristic lines,[§]

$$\frac{\partial}{\partial u}\left[\frac{\dfrac{\partial}{\partial u}(\mu_1\cos\theta_1)+\cos\theta_1\dfrac{\partial(\mu\cos\theta_1)}{\partial v}}{\mu_1\sin^2\theta_1}\right]=\frac{\partial}{\partial v}\left[\frac{\dfrac{\partial}{\partial v}(\mu_1\cos\theta_1)+\cos\theta_1\dfrac{\partial(\mu_1\cos\theta_1)}{\partial u}}{\mu_1\sin^2\theta_1}\right]. \tag{2}$$

For the purpose of comparison we write equations (1) and (2) in the following forms, respectively:

$$\begin{aligned}\frac{\partial^2\log\mu}{\partial u^2}-\frac{\partial^2\log\mu}{\partial v^2}&+\frac{\partial\log\mu}{\partial u}\left(\frac{\partial}{\partial u}\log\left(\frac{\cos\theta}{\sin^2\theta}\right)+\frac{\partial}{\partial v}\log\tan^2\frac{\theta}{2}\right)\\&-\frac{\partial\log\mu}{\partial v}\left(\frac{\partial}{\partial v}\log\left(\frac{\cos\theta}{\sin^2\theta}\right)+\frac{\partial}{\partial u}\log\tan^2\frac{\theta}{2}\right)=0;\end{aligned} \tag{1′}$$

$$\begin{aligned}\frac{\partial^2\log\mu_1}{\partial u^2}-\frac{\partial\log\mu_1}{\partial v^2}&+\frac{\partial\log\mu_1}{\partial u}\left(\frac{\partial}{\partial u}\log\frac{\cos\theta_1}{\sin^2\theta_1}+\frac{\partial}{\partial v}\log\tan^2\frac{\theta_1}{2}\right)\\&-\frac{\partial\log\mu_1}{\partial v}\left(\frac{\partial}{\partial v}\log\frac{\cos\theta_1}{\sin^2\theta_1}+\frac{\partial}{\partial u}\log\tan^2\frac{\theta_1}{2}\right)\\&+\frac{\partial^2}{\partial u^2}\log\tan\frac{\theta_1}{2}-\frac{\partial^2}{\partial v^2}\log\tan\frac{\theta_1}{2}=0.\end{aligned} \tag{2′}$$

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[*] Bianchi, *loc. cit.*, p. 156.

[†] Bianchi, *loc. cit.*, p. 168.

[‡] *Ibid.*, p. 156.

[§] *Ibid.*, p. 169.

The Gauss equation which the functions appearing in (1′) and (2′), respectively, must satisfy, may be written thus:

$$\csc^2\frac{\theta}{2}\left(\frac{\partial^2\log\mu}{\partial u^2}+\frac{\partial^2\log\mu}{\partial v^2}\right)-2\left(1-\cot^2\frac{\theta}{2}\right)\frac{\partial^2\log\mu}{\partial u\partial v}-\frac{\partial}{\partial u}\log\tan^2\frac{\theta}{2}\frac{\partial\log\mu}{\partial u}$$
$$-\frac{\partial}{\partial v}\log\tan^2\frac{\theta}{2}\frac{\partial\log\mu}{\partial v}+8\cos^2\frac{\theta}{2}\mu-4\cot\frac{\theta}{2}\frac{\partial^2\theta}{\partial u\partial v}=0, \qquad (3)$$

where μ and θ are to be replaced, respectively, by μ_1 and θ_1 when used with (2′).

The general problem, then, is reduced to the solutions of the pairs of equations (1′), (3) *and* (2′), (3); *the first leading to the surfaces of negative total curvature which are solutions of the problem, and the second to those of positive.*

As the general solution of these systems of equations can not be obtained, the general solution of the problem, from this standpoint, at least, is denied us. Particular solutions are, however, easily obtained, and the method of procedure is as follows:

Having obtained a solution of the pair of equations (1′), (3) or (2′), (3), we have the third form for the corresponding surfaces referred, in the first case to asymptotic lines, and in the second to characteristic lines. The first forms, or the expressions for the linear elements on the surfaces, are then obtained by quadrature from one or the other of the following systems of equations:[*]

$$\left\{\begin{aligned}\frac{\partial\log\rho}{\partial u}&=\frac{-\cos\theta\dfrac{\partial\log\mu}{\partial v}-\dfrac{\partial\log\mu}{\partial u}}{\sin^2\theta}\\ \frac{\partial\log\rho}{\partial v}&=\frac{-\cos\theta\dfrac{\partial\log\mu}{\partial u}-\dfrac{\partial\log\mu}{\partial v}}{\sin^2\theta}\end{aligned}\right\}, \qquad (4)$$

where $K=-1/\rho^2$;

$$\left\{\begin{aligned}\frac{\partial\log\rho_1}{\partial u}&=\frac{\cos\theta_1\dfrac{\partial}{\partial u}(\mu_1\cos\theta_1)+\dfrac{\partial}{\partial v}(\mu_1\cos\theta_1)}{\mu_1\sin^2\theta_1}\\ \frac{\partial\log\rho_1}{\partial v}&=\frac{\cos\theta_1\dfrac{\partial}{\partial v}(\mu_1\cos\theta_1)+\dfrac{\partial}{\partial u}(\mu_1\cos\theta_1)}{\mu_1\sin^2\theta_1}\end{aligned}\right\} \qquad (4')$$

where $K_1=1/\rho^2$.

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[*] Bianchi, *loc. cit.*, p. 155 and p. 168.

The Cartesian coordinates may then be obtained, following the general method, by the integration of a Ricatti equation or by some special device, depending upon the functions involved.

Before closing this article we shall call attention to some particular solutions which are suggested by the form of the equations (1), (2) and (3).

CASE 1. If in particular the function μ reduces to a constant, the corresponding surfaces are pseudo-spherical. All pseudo-spherical surfaces are included here. Likewise, if the function μ_1 reduces to a constant, the integration of the equations (2′) and (3) leads to the surfaces of constant positive total curvature.

CASE 2. If $\cos\theta = f(u+v)$, equation (1′) may be written in the form

$$\frac{\partial^2 \log\mu}{\partial u^2} - \frac{\partial^2 \log\mu}{\partial v^2} - \phi(u+v)\left(\frac{\partial \log\mu}{\partial u} - \frac{\partial \log\mu}{\partial v}\right) = 0, \tag{5}$$

where ϕ being a function of θ must be a function of $u+v$ as indicated. The general solution of (5) is $\log\mu = F(u+v)\psi(u-v) + \chi(u+v)$, where ψ and χ are arbitrary functions, and $\phi = 2F'/F$.

Substituting this value for $\log\mu$ in the Gauss equation (3), changing parameters by writing $u+v=u_1$, $u-v=v_1$, differentiating, and separating functions of u_1 from functions of v_1, we show that the function ψ must reduce to a constant. It follows easily that the corresponding surfaces are surfaces of revolution. All surfaces of revolution having negative total curvature are included here. In a similar way, we obtain the surfaces of revolution having positive total curvature, from the solution of (2′) and (3), assuming that the $\cos\theta_1$ is a function of the sum of the two variables.

If, in particular, θ is a constant, the integration of (1′), (3) and (4) gives as the linear element of the corresponding surfaces the following:

$$d\bar{s}^2 = \left[\frac{(1 - b e^{c(u+v)})^2}{e^{c(u+v)}}\right]^{\tan^2\frac{\theta}{2}} (d\bar{u}^2 + 2\cos\theta\, dudv + d\bar{v}^2), \tag{I}$$

where b and c are constants and θ is the angle between the asymptotic lines. The surfaces corresponding to (I) are the surfaces of revolution of negative total curvature which have the same spherical representation of their lines of curvature as the catenoid or minimal surface of revolution. To every value of θ from 0 to π, there corresponds a separate surface. Beside the catenoid corresponding to $\theta = \frac{\pi}{2}$ we may mention a surface generated by revolving the parabola about a line perpendicular to its axis, and also a surface generated by a fourth degree curve having a conjugate point, as being included in the set.

Likewise, if θ_1 is a constant, we have as the first form for the corresponding surfaces

$$d\bar{s}_1^2 = \left[\frac{(1 - be^{c(u+v)})^2}{e^{c(u+v)}}\right]^{-\tan^2 \frac{\theta_1}{2}} (du^2 + 2\cos\theta_1\, dudv + d\bar{v}^2), \tag{I$'$}$$

where θ_1 is now the angle between the characteristic lines of course.

These compose the class of surfaces of revolution, of positive total curvature, which have the same spherical representation of their lines of curvature as the catenoid.

CASE 3. An examination of equations (1′) and (2′) shows that if $\theta_1 = \theta$, and the function θ is chosen so that

$$\frac{\partial^2 \log \tan \theta/2}{\partial u^2} = \frac{\partial^2 \log \tan \theta/2}{\partial v^2},$$

the equations which μ and μ_1 must satisfy are one and the same; and, hence the theorem:

To every surface S of negative total curvature which is a solution of the problem and which has the tangent of one-half the angle between the asymptotic lines equal to a function of one of the parameters divided by a function of the other, the lines of reference being lines of curvature, there exists another S_1 of positive total curvature, having the same spherical representation of its characteristic lines as S of its asymptotic.

It is easily shown that any two surfaces S and S_1 which are related in this way are isothermic surfaces, connected by the Bour-Darboux theorem.* It is easily proven: *That all isothermic surfaces S and S_1 connected by the Bour-Darboux theorem are associates of one another, the asymptotic lines of one corresponding to the characteristic lines on the other, and vice versa.*† The converse of this theorem is true.

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* Darboux, "Théorie des Surfaces," Vol. II, p. 243.

† We have discussed in detail in a previous paper the problem of determining such isothermic surfaces.

IV. *Discussion of the Problem from the Standpoint of the Spherical Representation of the Lines of Curvature.*

Taking the Bonnet form* for the fundamental equation of a surface referred to lines of curvature, we have

$$\left.\begin{aligned} \frac{\partial M}{\partial v}-\frac{\partial N}{\partial u}-PQ &= 0,\\ \frac{\partial P}{\partial v}+MQ &= 0,\\ \frac{\partial Q}{\partial u}-NP &= 0, \end{aligned}\right\} \tag{6}$$

where

$$\left.\begin{aligned} M &= -\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v},\\ N &= \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}, \end{aligned}\right\} \tag{7}$$

the first and third form for the surface being respectively,

$$d\bar{s}^2 = Ed\bar{u}^2+Gd\bar{v}^2, \tag{I}$$
$$d\bar{\sigma}^2 = P^2\,du^2+Q^2\,dv^2. \tag{III}$$

For the surfaces sought, the parameters being properly chosen, $D=\pm D''$, and hence,

$$\frac{E}{G}=\frac{Q^2}{P^2} \text{ or } \frac{\sqrt{E}}{\sqrt{G}}=\pm\frac{Q}{P}, \tag{8}$$

the plus and minus signs corresponding respectively to surfaces of positive and negative curvature.

Eliminating G from (7) by means of (8), we have

$$\left.\begin{aligned} \frac{\partial\log\sqrt{E}}{\partial v} &= \mp\frac{P}{Q}M,\\ \frac{\partial\log\sqrt{E}}{\partial u} &= -\frac{\partial\log\left(\frac{P}{Q}\right)}{\partial u}\pm N\frac{Q}{P}. \end{aligned}\right\} \tag{9}$$

The condition for the integrability of (9) may be written thus

$$\frac{\partial^2\log\left(\frac{P}{Q}\right)}{\partial u\partial v}\mp\frac{\partial}{\partial u}\left(\frac{MP}{Q}\right)\mp\frac{\partial}{\partial v}\left(N\frac{Q}{P}\right)=0, \tag{10}$$

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* *Journal de l'École Polytechnique*, Vol. XLII, pp. 132–151.

where the upper signs in both (9) and (10) are for surfaces of positive total curvature, the lower for those of negative.

This equation determines the lines on the Gauss sphere which permit of surfaces of positive or negative total curvature of the class sought, having them as the spherical representation of their lines of curvature.

Evidently the two cases reduce to one and the same when $\dfrac{\partial^2 \log}{\partial u \partial v}\left(\dfrac{P}{Q}\right) = 0$; that is, when $\dfrac{P}{Q} = \dfrac{f(u)}{\phi(v)}$.

The corresponding surfaces are the isothermic surfaces discussed in the preceding article.

If we make the substitution

$$\frac{P}{Q} = \frac{1}{\phi}(\mp \int N\phi\, du + V),$$

(10) transforms into the Laplace equation

$$\frac{\partial^2 \phi}{\partial u \partial v} - \frac{\partial \log M}{\partial u}\frac{\partial \phi}{\partial v} + MN\phi = 0. \tag{11}$$

Eliminating Q from the last two equations of (6) we have again equation (11) with ϕ replaced by P.

The problem, then, is reduced to the simultaneous solution of the following system of equations:

$$\left.\begin{aligned} &\frac{\partial M}{\partial v} - \frac{\partial N}{\partial u} - PQ = 0, \\ &\frac{\partial Q}{\partial u} - NP = 0, \\ &\frac{\partial^2 P}{\partial u \partial v} - \frac{\partial \log M}{\partial u}\frac{\partial P}{\partial v} + MNP = 0, \\ &\frac{P}{Q} = \frac{1}{P_1}(\mp \int NP_1\, du + V), \end{aligned}\right\} \tag{12}$$

where P_1 is any solution of the preceding Laplace equation. Given any set of values M, N, P, Q, satisfying this system of equations, we have, by quadrature, from (8) and (9), the functions appearing in the first fundamental form of surfaces which are solutions of the problem. The Cartesian coordinates for any point on the same can be obtained by quadrature as soon as the Cartesian coordinates of the corresponding point on the Gauss sphere are known.

The problem of solving the first three equations of (12) is identical with that of the "Problem of the Spherical Representation" as treated by us in a previous article.[*] If we add any fourth equation to these three we can expect, of course, a corresponding solution as the number of equations then only equals the number of functions to be determined. Considering the third equation as a Laplace equation for the determination of P we are led to choose as the new equation some one of the following expressions equated to zero.

$$\begin{aligned} h &= -MN, \\ k &= -\frac{d^2 \log M}{dudv} - MN, \\ k_1 &= -\frac{d^2 \log (Mk)}{dudv} - k, \\ k_2 &= -\frac{\partial^2 \log (M k k_1)}{\partial u \partial v} - k_1, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ k_i &= -\frac{\partial^2 (\log Mk \ldots . k_{i-1})}{\partial u \partial v} - k_{i-1}, \end{aligned}$$

where i is an arbitrarily large positive integer.

The general solution of the Laplace equation can be found when and only when h or k_i vanishes.[†] If h vanishes, neglecting the case where the corresponding surfaces are developable, we may say that M vanishes. The corresponding surfaces are those on which the lines $v=$const. are geodesics. The corresponding lines on the Gauss sphere are great circles. The problem of determining these surfaces, then, is solved as soon as one determines the unit sphere referred to systems of great circles and their orthogonal trajectories.

If, however, h does not vanish but k does, the corresponding surfaces are either surfaces for which one set of the lines of curvature, the lines $v=$const., have constant geodesic curvature, or certain surfaces having the same spherical representation of their lines of curvature as these.[‡] Moreover, all these surfaces can be obtained by algebraic methods and quadrature from the general class corresponding to the case where h vanishes.[§]

And, in general, the surfaces corresponding to the case $k_{i+1}=0$ can always be obtained by algebra and quadrature from those corresponding to the preceding case $k_i=0$.

The totality of surfaces found in this way, starting with the case $M=0$ and continuing to the case $k_i=0$, where i is arbitrarily large make up the general class belonging to the problem of the spherical representation.

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[*] A. E. Young, "On the Problem of the Spherical Representation," etc., AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII, No. 1.

[†] See previous reference, p. 42.

[‡] A. E. Young, *loc. cit.*, p. 47.

[§] *Loc. cit.*, p. 43.

The surfaces of this class belonging to our present problem can be found as follows:

Taking the general class corresponding to the case $M=0$, we choose the corresponding functions E, G, P and Q so as to satisfy the relation[*]

$$\frac{E}{G}=\frac{UQ^2}{VP^2}, \tag{13}$$

where U and V are functions of u and v, respectively, to be determined. The original class with these limitations on the functions are solutions of the problem, and, in fact, compose that system of surfaces belonging to the problem on which one set of the lines of curvature are geodesics.

Again, taking the general class of surfaces corresponding to $k=0$, we may choose the functions P and Q corresponding, so that the last equation of (12) is satisfied, P_1 now being a general solution of the Laplace equation and hence actually known, and thus obtain at once the surface which we seek from the general class. Or, following the preceding method, we may choose the new functions E, G, P and Q corresponding to the case $k=0$ so that they satisfy (13) and thus obtain the desired surfaces. The surfaces in this case will be those of the problem having one set of lines of curvature characterized by geodesic curvature, and certain other surfaces having the same spherical representation of their lines of curvature as these.

In general, therefore, the surfaces of the problem, involved in the solution of the "Problem of the Spherical Representation," can be obtained from the separate systems as classified above by a process which involves nothing more difficult than the limitation by the simple algebraic equation (13) of the four functions appearing in the first and third fundamental quadratic forms, the lines of reference being lines of curvature.

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[*] The introduction of the functions U and V is due to the fact that no particular choice of parameters has been made, as was done when equation (8) was derived.

On Orthoptic and Isoptic Loci.

By Prof. Harold Hilton and Miss R. E. Colomb.

Section 1.

The nature of the orthoptic locus (the locus of the intersection of two perpendicular tangents) of a given real plane algebraic curve does not seem to have been investigated with any fulness, except that the degree of the orthoptic locus has been obtained in certain cases and a few of the more elementary properties of the locus have been given, at any rate when the given curve does not pass through the circular points at infinity.*

In the following discussion of the locus we shall denote the degree, class, number of nodes, number of cusps, number of bitangents, number of inflexions, deficiency of the given curve by $n, m, \delta, \varkappa, \tau, \iota, D$; and the corresponding quantities for the orthoptic locus by $n', m', \delta', \varkappa', \delta', \iota', D'$.

We take ω, ω' as the circular points at infinity. If we choose any points E, F on $\omega\omega'$ so that $(EF, \omega\omega')$ is harmonic, the orthoptic locus will be the locus of the intersection of tangents from E and F to the given curve.

It must be understood throughout that, when we discuss the orthoptic locus of some special type of curve, we imply that the curve is the most general curve of the special type discussed. Otherwise the orthoptic locus may degenerate or be simplified. For instance, the orthoptic locus of a central conic is in general a circle; but it becomes a pair of lines through ω and ω', if the conic divides $\omega\omega'$ harmonically.

It must be understood also that the line $\omega\omega'$ is never counted as part of the orthoptic locus.

Section 2.

First suppose (§§ 2–4) that the given curve has m^2 distinct ordinary foci; *i. e.*, no inflexional tangent or bitangent of the curve goes through ω or ω', the curve does not touch $\omega\omega'$ or go through ω or ω', etc.

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* Bassett, "Elementary Treatise on Cubic and Quartic Curves," §§ 68, 157; Zimmermann, *Crelle* 126 (1903), p. 183; Taylor, *Proc. Royal Soc.*, 37 (1884), p. 138, and *Messenger Math.*, 16 (1886), p. 1; Jonquières, *Nouv. Annales Math.*, 20 (1861); Loucher, *ibid.*, II, 11 (1892); Ljungh, Diss. "Ueber Isoptische und Orthoptische Kurven" (Lund, 1895). We have not been able to peruse this last paper.

Suppose two tangents are drawn to the curve from E and F meeting at P. Let E, and therefore F, approach ω. Then, if the tangents at E and F do not become consecutive, P will approach ω, the pencil $P(EF, \omega\omega')$ being harmonic. Hence, ultimately the orthoptic locus has one branch through ω for each pair of tangents from ω to the curve, and the tangent to the branch and the line $\omega\omega'$ form a harmonic pencil with the pair of tangents from ω to the curve.

Hence, through ω $\frac{1}{2}m(m-1)$ branches of the orthoptic locus pass, and similarly for ω', while there is no other point of the orthoptic locus on $\omega\omega'$. Hence, $n' = m(m-1)$.

Moreover, if ωH, $\omega' H$ and ωK, $\omega' K$ are two conjugate pairs of tangents from ω and ω' (so that H, K are real foci) and C is the middle point of HK, it follows from the harmonic properties of a quadrilateral that $\omega(C\omega', HK)$ and $\omega'(C\omega, HK)$ are harmonic pencils. Therefore, $C\omega$ and $C\omega'$ touch the orthoptic locus at ω and ω', *i. e.*, C is a *singular focus* of the orthoptic locus. Hence,

The orthoptic locus of a curve of class m is in general of degree $m(m-1)$, *and its singular foci are the* $\frac{1}{2}m(m-1)$ *middle points of the segments joining any two of the m real foci of the given curve.*

The intersections of a curve with its orthoptic locus are given by the following theorem, the proof of which we leave to the reader:

A curve of degree n and class m in general cuts its orthoptic locus at the $2m$ *points of contact of tangents from the circular points to the curve, touches it at the feet of the* $m(m+n-4)$ *normals which are also tangents to the curve, and cuts it in the* $m(m-3)(n-2)$ *points from which two perpendicular tangents can be drawn to the curve neither of which coincides with the tangent at the point.*

Section 3.

We now consider the *class* of the orthoptic locus. It will be sufficient to find the number of tangents which can be drawn from ω' to the locus.

First consider the tangents from ω' to the locus whose points of contact are not on $\omega\omega'$.

Suppose PH, $P'H$ to be two consecutive tangents to the curve and PK, $P'K$ the consecutive tangents perpendicular to PH, $P'H$. Suppose, moreover, that PP' passes through ω'. Then, because $P(HK, \omega\omega')$ and $P'(HK, \omega\omega')$ are harmonic, HK passes through ω. But H and K are ultimately the points of contact of the tangents PH and PK. Hence,

If a tangent from ω' *to the orthoptic locus touches it at a finite point P, the line joining the points of contact of the two perpendicular tangents from P to the curve passes through* ω.

Now suppose any line whatever through ω meets the given curve in H and K. Let the tangents at H and K meet at T, and consider the envelope of the line TV where $T(HK, \omega V)$ is harmonic. By what has just been said, every tangent from ω' to the envelope which does not coincide with $\omega\omega'$ will be a tangent to the orthoptic locus. Now, in general, the envelope will not touch $\omega\omega'$, and hence the number of tangents from ω' to the orthoptic locus with a finite point of contact is equal to the class of the envelope.

To find the class of the envelope, we find the number of tangents from ω to the curve. The line TV can not pass through ω unless ωHK touches the given curve at H but not at K (or vice versa). If this is the case, however, ωHK will be a tangent to the envelope at a point U such that $(\omega U, HK)$ is harmonic. Hence, each tangent from ω to the curve is a $(n-2)$-ple tangent to the envelope; so that the class of the envelope is $m(n-2)$.

Now the only tangents from ω' to the orthoptic locus with their points of contact on $\omega\omega'$ are the $m(m-1)$ tangents at the $\frac{1}{2}m(m-1)$-ple point ω' itself. Hence,

$$m' = m(n-2) + m(m-1) = m(m+n-3).$$

Section 4.

We now consider the double points and cusps of the orthoptic locus.

Let PH, $P'H$ be two consecutive tangents to the given curve and PK, $P'K$ the consecutive tangents perpendicular to PH, $P'H$. Then P, P', H, K are concyclic, and therefore the angle $KPP' =$ the angle KHP'. Hence in the limit:

If PH, PK are two perpendicular tangents to a curve, the angle between PK and the tangent to the orthoptic locus at P is equal to the angle KHP.

If PH is a bitangent to the given curve, there are two points of contact H on PH, and therefore there are two tangents to the orthoptic locus at P; *i. e.*, P is a double point of the orthoptic locus. It will be readily seen that, if PH is an inflexional tangent, P is a cusp of the orthoptic locus. Hence,

The orthoptic locus has a double point at each of the $m\tau$ intersections of a bitangent of the given curve with a perpendicular tangent, and has a cusp at each of the $m\iota$ intersections of an inflexional tangent of the given curve with a perpendicular tangent.

It will readily be seen that all the cusps of the orthoptic locus are given in this way. Hence,

$$\kappa' = m\iota.$$

We have now found n', m', κ' and can deduce δ', τ', ι' from Plücker's equations. We find

$$n' = m(m-1), \quad m' = m(m+n-3), \quad \delta' = \tfrac{1}{2}m\{(m+1)(m-2)^2+2\tau\},$$
$$\kappa' = m\iota, \quad \tau' = \tfrac{1}{2}m\{m(m+n)^2-(6m^2+6mn+n^2)-m+22+2\delta\},$$
$$\iota' = m(3m+\kappa-6), \quad D' = \tfrac{1}{2}(m-1)(m-2)+mD.$$

Of the $\frac{1}{2}m\{(m+1)(m-2)^2+2\tau\}$ nodes of the orthoptic locus $m\tau$ are the intersections of the bitangents of the given curve with perpendicular tangents, and $\frac{1}{4}m(m+1)(m-1)(m-2)$ are accounted for by the $\frac{1}{2}m(m-1)$-ple points of the orthoptic locus at ω and ω'. There remain $\frac{1}{4}m(m+1)(m-2)(m-3)$ others. Hence,

There are in general $\frac{1}{4}m(m+1)(m-2)(m-3)$ *points from which two pairs of mutually perpendicular tangents can be drawn to a given curve of class* m.

The results of §§ 2–4 are illustrated by the conic whose orthoptic locus is a circle through the intersection of the conic and the directrices.

Section 5.

In §§ 5–7 we give very briefly without proof the properties of the orthoptic locus of curves bearing some special relation to the line $\omega\omega'$.

If the given curve touches $\omega\omega'$ at Y, the orthoptic locus has a branch through Z, where $(YZ, \omega\omega')$ is harmonic, corresponding to each tangent from Z to the given curve other than $\omega\omega'$.

We find that, if the given curve touches $\omega\omega'$ at k points (which may or may not be cusps of the curve),

$$n' = (m-k)(m-1), \quad m' = (m-k)(m+n-3-k), \quad \kappa' = (m-k)\iota.$$

If the given curve has $\omega\omega'$ as inflexional tangent at Y, the branches of the orthoptic locus through Z are all linear and touch $\omega\omega'$ at Z (more generally, if the curve has r-point contact with $\omega\omega'$ at Y, each branch of the orthoptic locus has $(r-1)$-point contact with $\omega\omega'$ at Z), and that

$$n' = (m-1)(m-2), \quad m' = (m-2)(m+n-4), \quad \kappa' = (m-2)(\iota-1).$$

As examples of the results of §§ 2–5 consider

(i) A parabola, whose orthoptic locus is its directrix.

(ii) $3(x+y) = x^3$, whose orthoptic locus is

$$81y^2(x^2+y^2)-36(x^2-2xy+5y^2)+128=0.$$

Here we have

$$n=3, \quad m=3, \quad \delta=0, \quad \kappa=1, \quad \tau=0, \quad \iota=1,$$
$$n'=4, \quad m'=4, \quad \delta'=2, \quad \kappa'=2, \quad \tau'=1, \quad \iota'=2.$$

The given curve touches $\omega\omega'$ at its cusp $(0, \infty)$, and therefore the orthoptic locus has a double point at $(\infty, 0)$ the tangents there being $9y^2=4$, which are also tangents to the given curve.*

The intersections $(\mp\frac{2}{3}\sqrt{2}, \pm\frac{2}{3}\sqrt{2})$ of the inflexional tangent of the given curve with the perpendicular tangents are cusps of the orthoptic locus.

(iii) $y^2=x^3$, whose orthoptic locus is $729y^2=108x-16$.

The curve has $\omega\omega'$ as inflexional tangent at $(0, \infty)$, and therefore the orthoptic locus touches $\omega\omega'$ at $(\infty, 0)$.

The orthoptic locus meets the curve where $x=-4/9$, which gives the points of contact of the tangents from ω, ω' to the curve. The orthoptic locus touches the curve where $x=2/9$, *i. e.*, at the feet of the two normals of the curve which are also tangents.

Section 6.

In § 5 we considered a curve for which $\omega\omega'$ was a multiple tangent, tacitly assuming that no two of the points of contact divided $\omega\omega'$ harmonically.

If this were the case, the branches of the orthoptic locus at ω and ω' are as in § 5, but the nature of the other points of the locus on $\omega\omega'$ is altered, and the degree of the locus is in general lowered. We shall suppose that $\omega\omega$ touches the curve at Y and Z only, so that $(\omega\omega', YZ)$ is harmonic.

(A) Let the curve have ordinary contact at Y and Z. The locus has $(m-3)$ branches through Y due to two perpendicular tangents to the curve, one of which touches near Z and the other passes close to Y but does not touch near Y or Z. Similarly, it has $(m-3)$ branches through Z. It has also in addition a branch cutting $\omega\omega'$ due to two perpendicular tangents, one of which touches near Y and the other near Z.

Example. The tricuspidal quartic $x^2y^2-4(x^3+y^3)+18xy-27=0$ with orthoptic locus $x+y+2=0$.

(B) Let the curve have ordinary contact at Y and a cusp at Z.

The locus has $(m-3)$ branches through Y and $(m-4)$ through Z, and in addition another branch touching $\omega\omega'$ at Z.

(C) Let the curve have ordinary contact at Y and an inflexion at Z.

The locus has $(m-4)$ branches touching $\omega\omega'$ at Y and $(m-4)$ branches through Z, and in addition a branch touching $\omega\omega'$ at Y.

Example. The curve with tangential equation $\lambda^4+2\mu^4=\lambda^2\mu$ and orthoptic locus $2x^2+y+3=0$.

(D) Let the curve have inflexions at Y and Z.

---

* Because $\omega\omega'$ is a *cuspidal* tangent of the given curve.

The locus has $(m-5)$ branches touching $\omega\omega'$ at Y and similarly for Z. It has in addition a branch touching $\omega\omega'$ elsewhere.

Example. $\lambda^2\mu^2=\lambda^5+\mu^5$ with orthoptic locus

$$(x-y)^2+2(x+y)+4=0.$$

and so on in all similar cases.

Section 7.

We now notice briefly some cases in which the given curve passes through ω and ω'.

If the curve has a linear branch through ω and ω', the orthoptic locus has a branch through ω touching the curve at ω, and a superlinear branch of order 2 at ω for each of the $m-2$ tangents from ω to the curve not touching at ω. The line $\omega\omega'$ and the tangent to the superlinear branch form a harmonic pencil with the tangent at ω and the tangent from ω to the given curve. The orthoptic locus has also a linear branch through ω corresponding to each of the $\frac{1}{2}(m-2)(m-3)$ pairs of tangents from ω to the given curve not touching at ω (see §2). We find

$$n'=m(m-1),\quad m'=m(m+n-5)+4,\quad \kappa'=2(m-2)+m\iota.$$

The reader may illustrate the above statements on the circle whose orthoptic locus is a concentric circle, or on the cissoid $(x+1)^3+xy^2=0$ whose orthoptic locus is

$$16(x^2+y^2)^3+108(x^2+y^2)(x^2+2y^2)+108x^3+729y^2=0.$$

If the given curve has two linear branches through ω, each branch contributes its quota to the form of the orthoptic locus at ω as stated above. The orthoptic locus has also two superlinear branches of order 2 given by the intersections of two tangents to the curve, one of which touches one branch and the other touches the other branch near ω. The two superlinear branches have a common tangent, and this tangent and $\omega\omega'$ form a harmonic pencil with the tangents to the curve at ω.

Similarly, if the given curve has more than two branches through ω.

If the curve touches $\omega\omega'$ at ω and ω', the orthoptic locus has $m-2$ linear branches through ω, of which $m-3$ touch $\omega\omega'$ at ω in addition to the $\frac{1}{2}(m-3)(m-4)$ branches through ω obtained as in § 2, so that

$$n'=2+4(m-3)+(m-3)(m-4)=(m-1)(m-2).$$

Example. The three-cusped hypocycloid whose orthoptic locus is a circle.

The reader may consider the case in which the given curve has cusps at ω and ω', and illustrate by the cardiode whose orthoptic locus consists of a circle and a limaçon, all three curves having a common singular focus.

Section 8.

If an inflexional or multiple tangent of the given curve passes through ω, this tangent is part of the orthoptic locus. In this section n', m', ... refer to the remainder of the orthoptic locus excluding such tangents.

If a bitangent passes through ω, the orthoptic locus has $2(m-2)$ linear branches through ω touching each other in pairs, and $\frac{1}{2}(m-2)(m-3)$ other linear branches through ω. We have

$$n'=(m+1)(m-2), \quad m'=m(m+n-3)-4, \quad \kappa'=m\iota.$$

If an inflexional tangent passes through ω, the orthoptic locus has $(m-2)$ superlinear branches of order 2 at ω, and $\frac{1}{2}(m-2)(m-3)$ other linear branches through ω. We have

$$n'=(m+1)(m-2), \quad m'=m(m+n-4), \quad \kappa'=m\iota-4.$$

Example. $(2x+1)^3=27(x^2+y^2)$ with orthoptic locus

$$(16y^4-16xy^2+4x^2+4y^2-4x-1)(x^2+y^2)=0.$$

If the curve has an inflexion at ω and ω', the inflexional tangents count twice over as part of the locus, and $n'=m(m-1)-4$.

Section 9.

The nature of the *locus of the intersection of two perpendicular normals* of a given curve is readily obtained, for it is the orthoptic locus of its evolute.

Taking the general curve of § 2, its evolute is of degree $n_1=3m+\kappa$, is of class $m_1=n+m$, has $\iota_1=0$ inflexions, etc. The evolute has n cusps at which $\omega\omega'$ is the tangent.

Hence, putting n_1, m_1, ... for n, m, ... in § 5, and putting $k=n$, we see that the locus of the intersection of two perpendicular normals is of degree $n'_1=m(m+n-1)$, is of class $m'_1=m(4m+\kappa-3)$, and has $\kappa'_1=0$ cusps.

If the given curve touches $\omega\omega'$,

$$n'_1=(m-1)(m+n-2), \quad m'_1=(m-1)(4m+\kappa-6), \quad \kappa'_1=0.$$

For example, the evolute of a parabola is a semi-cubical parabola whose orthoptic locus is a parabola.

If the given curve has $\omega\omega'$ as inflexional tangent,

$$n'_1=(m-2)(m+n-3), \quad m'_1=(m-2)(4m+\kappa-9), \quad \kappa'_1=0.$$

The reader may illustrate on the semi-cubical parabola.

Section 10.

We here enumerate the types of real curve whose orthoptic locus is of the first or second degree.

I. *Orthoptic locus a straight line.*
  - (i) Parabola.
  - (ii) $n=4$, $m=3$. Curve touches $\omega\omega'$ in two points dividing $\omega\omega'$ harmonically.

II. *Orthoptic locus a circle.*
  - (i) Circle.
  - (ii) Conic.
  - (iii) $n=4$, $m=3$. Curve touches $\omega\omega'$ at ω and ω'.

III. *Orthoptic locus a parabola.*
  - (i) $n=3$, $m=3$. Curve has $\omega\omega'$ as inflexional tangent.
  - (ii) $n=5$, $m=4$. Curve has $\omega\omega'$ as an ordinary and as an inflexional tangent, the points of contact dividing $\omega\omega'$ harmonically.
  - (iii) $n=6$, $m=5$. Curve has two inflexions dividing $\omega\omega'$ harmonically, at which $\omega\omega'$ is the tangent.

IV. *Orthoptic locus a conic.*
  - (i) $n=4$, $m=3$. $\omega\omega'$ is a bitangent.
  - (ii) $n=6$, $m=4$. $\omega\omega'$ is a triple tangent, two of the points of contact dividing $\omega\omega'$ harmonically.
  - (iii) $n=8$, $m=5$. $\omega\omega'$ is a quadruple tangent, two pairs of points of contact dividing $\omega\omega'$ harmonically.

As an example the reader may enumerate the fourteen types of curve whose orthoptic locus is a cubic, and the thirty-eight types whose orthoptic locus is a quartic.

Section 11.

We now state the properties of the *isoptic locus* of a given curve; *i. e.*, the locus of the intersection of two tangents inclined at any given angle α.*

To each pair of tangents to the curve from ω correspond two branches of the isoptic locus through ω. We have

$$n'=2m(m-1),\quad m'=2m(n+m-2),\quad \varkappa'=2m\iota.$$

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* We suppose $\alpha\not\equiv 0$ and $\alpha\not\equiv\frac{1}{2}\pi$. It is not in general possible to distinguish the case in which two tangents cut at an angle α, from that in which they cut at an angle $\pi-\alpha$. But consider the case of the circle or parabola.

The curve cuts the isoptic locus at the $2m(m-3)(n-2)$ points from which two tangents can be drawn inclined at an angle α and neither of them touching at the point. It touches the locus at the $2m$ points of contact of tangents from ω and ω' * and at the $2m(m+n-4)$ points such that a line making an angle α with the tangent at the point touches the curve elsewhere.

If the curve touches $\omega\omega'$ at Y, the locus has $(m-1)(m-2)$-ple points at ω and ω', and has an $(m-1)$-ple point at Z where $(Y\omega\, Z\omega')$ has the cross-ratio $e^{\pm 2i\alpha}$. We find

$$n'=2(m-1)^2, \quad m'=2(m-1)(m+n-3), \quad \varkappa'=2(m-1)\iota.$$

If the curve has $\omega\omega'$ as inflexional tangent at Y, the locus has $(m-2)(m-3)$-ple points at ω and ω' and has $(m-2)$ linear branches touching $\omega\omega'$ at Z. We find

$$n'=2(m-1)(m-2), \quad m'=2(m-2)(m+n-3), \quad \varkappa'=2(m-2)(\iota-1).$$

As examples consider (we write k for $\tan^2\alpha$):

(i) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with isoptic locus $k(x^2+y^2-a^2-b^2)^2=4a^2b^2\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1\right)$.

(ii) $y^2=4ax$ with isoptic locus $y^2-4ax=k(x+a)^2$.

(iii) $y^2=x^3$ with isoptic locus

$$3^{12}k(x^2-ky^2)^2-3^9 2^3\{(k^2+k+2)x^3+(k^3+3k^2)xy^2\}$$
$$+3^6 2^4\{(k^3+4k^2+9k)x^2+(2k^3+27k^2+54k+27)y^2\}-3^3 2^7(k^3+3k^2)x+2^8k^3=0,$$

touching the curve where $x=-4/9$ and $k(9x-2)^2=81x$, and touching $\omega\omega'$ where $x^2=ky^2$.

To find the nature of the locus of the intersection of two tangents, one drawn to each of two given curves and inclined at a constant angle, we have only to take away the isoptic locus of each curve considered separately from the isoptic locus of the (degenerate) curve, consisting of the two curves taken together.†

The case in which the given curves are both circles is well known.

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* Contrast with the orthoptic locus.

† Taylor, *loc. cit.*

A New Canonical Form for Systems of Partial Differential Equations.

By L. B. Robinson.

Introduction.

Some years ago Delassus obtained a canonical form useful for the study of systems of partial differential equations.[*] It is possible to reduce to this form a very large class of differential systems, but that the form is not absolutely general was discovered almost simultaneously by the author[†] and by M. Gunter.[‡] Were the integration of the given differential system the only question of interest there would be no need to improve the canonical form given by Delassus, for when the given differential system is of the first order the above-mentioned canonical form is always valid, and it has been shown by Riquier that it is always possible to reduce a differential system to the first order.[§]

But since canonical forms are often useful in the study of the comitants of either differential or algebraic systems, it seems desirable to give a canonical form which has no exceptional cases and will apply equally to a differential system or a system of polynomials. In the following paper the author will construct such a canonical form. It will be noted that the symbols $X_1, X_2 \ldots.$ which are used to represent differential operators could likewise be interpreted as the variables of a system of polynomials.[||]. As the new canonical form is closely analogous to that of Delassus, it seems that it will be equally useful in studying the invariants or the characteristics of differential systems.[¶] In fact this turns out to be the case in some simple cases considered by the author.

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[*] *Annales de l'école normale supérieure,* Vol. XIII, 3d Ser., p. 421.

[†] "Notes from the Mathematical Seminary," J. H. U., 1913. *Comptes Rendus,* 15 Juillet 1913.

[‡] *Comptes Rendus,* 14 Avril 1913.

[§] Riquier, "Sur les systèmes d'équations aux dérivées partielles," *Annales de l'école normale,* Vol. X, 3d Ser., p. 359.

[||] Delassus, *Annales de l'école normale supérieure,* Vol. XIV, 3d Ser., p. 21.

[¶] For further bibliography consult Gunter, "On the Theory of the Characteristics of Systems of Partial Differential Equations" (Russian), *Comptes Rendus,* 13 Octobre 1913; 23 Mars 1914; 20 Avril 1914. Janet, *Comptes Rendus,* 13 Janvier 1913; 27 Octobre 1913. In the three above-mentioned notes of M. Gunter in the *Comptes Rendus,* results somewhat similar to those of the author seem to have been obtained.

§ 1. *A Property of Differential Systems with Regular Initial Conditions.*

Consider a system S of partial differential equations with independent variables $x_1, x_2, \ldots, x_q$ and unknown functions $u_1, u_2, \ldots, u_s$. The equations are supposed to be solved for different derivatives of the u's in such a way that no second member can contain a derivative of order higher than that of the corresponding first member. A derivative which is a first member of S, or a derivative obtained from a first member by differentiation is called a principal derivative. All others are named parametric.* Let the values of the unknowns u and of a suitably chosen number of their parametric derivatives be given as functions of some of the variables x for certain initial values of the remaining variables x. We shall suppose that these initial conditions are all regular in the sense of Riquier.† Let an integer or 'cote' be associated with each of the variables x and the unknowns u in some convenient way,‡ the 'cotes' of the x's being unity, and let the 'cote' of a derivative of an unknown u_p be obtained by adding the order of the derivative to the 'cote' of u_p.

Let Γ be the maximum 'cote' of the first members of the initial conditions. We shall now consider the group of equations $S_{\Gamma+1}$ obtained from the system S, prolonged by repeated differentiations, by selecting those equations in which the 'cote' of the first members is $\Gamma+1$. To simplify writing we shall consider derivatives with respect to four independent variables only as the reasoning would be the same in the general case, and shall use the following abbreviation for a derivative of a typical variable u

$$\frac{\partial^{n_1+n_2+n_3+n_4}u}{\partial x_1^{n_1}\partial x_2^{n_2}\partial x_3^{n_3}\partial x_4^{n_4}}=(n_1, n_2, n_3, n_4).$$

We shall now examine the three derivatives

$$\alpha=(n_1, n_2, n_3, n_4),$$
$$\beta=(n_1, n_2, n_3+1, n_4),$$
$$\gamma=(n_1, n_2, n_3+1, n_4-1).$$

Suppose that the first of these is a principal derivative occurring in the group $S_{\Gamma+1}$. The second is obviously obtained from the first by differentiation. Hence, if the first is principal, so is the second. It is to be shown, assuming as we have done, that the initial conditions are regular, that γ is also principal.

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* Riquier, "Les systèmes d'équations aux derivées partielles, Chapter VI.

† *Loc. cit.*, Chapter XII.

‡ Riquier, Chapter VII.

For suppose γ to be parametric. Then it is derived from some parametric derivative of 'cote' Γ, obtained by subtracting unity from the exponent of one of its independent variables. Hence there are four possible primitives of γ, viz.:

$$(n_1, n_2, n_3+1, n_4-2) \quad =\phi_1(x_2, x_3, x_4, x_5 \ldots .), \tag{1}$$

$$(n_1, n_2, n_3, n_4-1) \quad =\phi_2(x_2, x_3, x_4, x_5 \ldots .), \tag{2}$$

$$(n_1, n_2-1, n_3+1, n_4-1) =\phi_3(x_2, x_3, x_4, x_5 \ldots .), \tag{3}$$

$$(n_1-1, n_2, n_3+1, n_4-1) =\phi_4(x_2, x_3, x_4, x_5 \ldots .). \tag{4}$$

We have written the above initial conditions in the most general form. For in the ϕ's one of the variables, to fix ideas say x, can not occur as an argument because if it did no derivatives of u could be principal, therefore u would be a parameter just as in the following system:

$$\frac{\partial v}{\partial x}=\phi(x, y, u), \quad \frac{\partial w}{\partial x}=\psi(x, y, u).$$

In the above system u is an arbitrary unknown. Let us write $u=\theta(x, y)$ where θ is an arbitrary function of x and y. The derivatives of u are obtained by differentiating θ. They are evidently parametric.

As for $x_2, x_3, \ldots.$, we do not assert that they all occur actually in the ϕ's, but when we say the system has regular initial conditions we mean that if x_i occurs actually in one of the functions ϕ then $x_{i+1}, x_{i+2}, \ldots, x_q$ also occur actually in the same ϕ.

Now returning to the equations (1), (2), (3), (4) we observe first of all that x_4 can not occur actually in (1), for if it did β could not be a principal derivative. For the same reason x_3 can not appear actually in (2) nor x_2 actually in (3). And, therefore, from the hypotheses that the initial conditions are regular, it follows that x_2 and x_3 do not occur in (1), and x_2 does not occur in (2).

So then we see that we can not obtain γ from the group of initial conditions under the hypothesis that they have the regular form. Therefore γ must be a principal derivative. In exactly the same way we can show that the two derivatives

$$(n_1+1, n_2, n_3, n_4-1),$$
$$(n_1, n_2+1, n_3, n_4-1),$$

must be principal also. We may now announce the following result:

Suppose given a system S where the initial conditions are regular. Let Γ be the maximum 'cote' of the first members of the initial conditions. Let

$$(n_1, n_2, \ldots, n_p)$$

be a typical first member of the group $S_{\Gamma+1}$. Then all derivatives obtained by differentiating the above expression once, with respect to one of the independent variables $x_1, x_2, \ldots, x_p$ which are engaged therein, and then reducing by unity the exponent of the last independent variable involved, are also principal.

Call this property "K." As a simple example we may suppose $\frac{\partial^2 u}{\partial x_2^2}$ to be a first member of 'cote' $\Gamma+1$ belonging to a system S involving only two independent variables x_1 and x_2. Differentiate with respect to both of these independent variables. We obtain the derivatives

$$\frac{\partial^3 u}{\partial x_1 \partial x_2^2}, \quad \frac{\partial^3 u}{\partial x_2^3}.$$

Then, if the initial conditions are regular, the group of first members of $S_{\Gamma+1}$ must also contain $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ as well as $\frac{\partial^2 u}{\partial x_2^2}$.

§ 2. *Proof that Systems with the Property K Are Regular.*

Firstly, we shall show that if the given system has the property K, the prolonged system has this property also. For, let

$$(n_1, n_2, \ldots, n_q)$$

be a derivative belonging to the group of first members of the system non-prolonged. Differentiate an arbitrary number of times with respect to each variable and we obtain

$$(n_1+\lambda_1, n_2+\lambda_2, \ldots, n_q+\lambda_q).$$

Differentiate this last derivative with respect to x_l and integrate once with respect to x_q. We obtain

$$(n_1+\lambda_1, n_2+\lambda_2, \ldots, n_l+\lambda_l+1, \ldots, n_q+\lambda_q-1),$$

and this last expression can clearly be derived from

$$(n_1, n_2, \ldots, n_l+1, \ldots, n_q-1),$$

while this last derivative surely exists among the group of first members if we assume that the system has the property K.

Select one of the unknowns involved, for example u, and arrange all its derivatives whatsoever whose 'cote' is Γ in the order* adopted by Delassus, and call this set of derivatives (A). From now on, following the example of Delassus, we shall use the notation

$$X_1^{a_1},\ X_2^{a_2},\ X_3^{a_3},\ \ldots,\ X_p^{a_p} \equiv \frac{\partial^{a_1+a_2+a_3,\ \ldots,\ a_p}u}{\partial x_1^{a_1},\ \partial x_2^{a_2},\ \partial x_3^{a_3},\ \ldots,\ \partial x_p^{a_p}}\,.$$

Let $X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}$, which we shall call (a), be a member of the group (A). Differentiate with respect to x_1 and we obtain

$$X_1^{n_1+1},\ X_2^{n_2},\ X_3^{n_3},\ \ldots,\ X_p^{n_p}.$$

Among the group of the derivatives of u there are a certain number from which we can obtain the last one by a single differentiation. One of these is

$$X_1^{n_1+1},\ X_2^{n_2},\ X_3^{n_3},\ \ldots,\ X_p^{n_p-1},$$

which is anterior to (a) in the set (A). Differentiate (a) with respect to x_2 and we obtain

$$X_1^{n_1},\ X_2^{n_2+1},\ X_3^{n_3},\ \ldots,\ X_p^{n_p},$$

which can be obtained from

$$X_1^{n_1},\ X_2^{n_2+1},\ X_3^{n_3},\ \ldots,\ X_p^{n_p-1},$$

which belongs to the set (A). And continuing thus we can show that any derivative of (a) taken once with respect to $x_1, x_2, \ldots, x_{p-1}$ can be obtained by differentiating some member anterior to (a) with respect to x_p if $n_p \geqq 1$, which is the case by hypothesis.

Now differentiate twice with respect to x_1 and we obtain

$$X_1^{n_1+2},\ X_2^{n_2},\ X_3^{n_3},\ \ldots,\ X_{p-1}^{n_{p-1}},\ X_p^{n_p}.$$

If $n_p > 1$ this can be obtained from $X_1^{n_1+1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}-2}$ by differentiating twice with respect to x_p. If $n_p = 1$ it can be obtained from

$$X_1^{n_1+2},\ X_2^{n_2},\ X_3^{n_3},\ \ldots,\ X_{p-1}^{n_{p-1}-1}$$

by differentiating with respect to x_{p-1}, x_p. This course of reasoning is general and we can say:

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* Delassus, "Extension du Théorème de Cauchy aux systemès les plus generaux des équations aux dérivées partielles," *Annales de l'école normale supérieure* (1896), Vol. IX.

Let (A) be the totality of the derivatives of u with respect to $x_1, x_2, \ldots, x_q$ *which have the 'cote'* Γ. *Let*

$$(a) = X_1^{n_1}, \ldots, X_p^{n_p}, \qquad (n_p \overline{>} 1)$$

be a member of the set. If we differentiate (a) once with respect to one of the variables $x_p, x_{p+1}, \ldots, x_q$, *and perform similar differentiations upon the other elements of (A), the totality of derivatives found includes all the derivatives of u of 'cote'* $\Gamma+1$. *By differentiating similarly K times, a set is formed which includes all derivatives of 'cote'* $\Gamma+K$.

Suppose now that our system has the property K, and consider again a derivative of the set (A):

$$(a) = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}.$$

If it is principal, all its derivatives are principal. If it is parametric, we make the following argument:

Differentiate with respect to $x_p, x_{p+1}, \ldots, x_q$. We obtain

$$\left.\begin{array}{l} X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p+1}, \\ X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}, X_{p+1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}, X_{p+j}, \end{array}\right\} \qquad (I)$$

$$\left.\begin{array}{l} X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}, X_{p+j+1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}, X_q. \end{array}\right\} \qquad (II)$$

If any one of the derivatives belonging to (I) or (II) is principal, then any preceding one is so, on account of the property K, which is assumed to hold good for our system. Hence there are three possibilities:

(α) All derivatives obtained from (a) by a single differentiation with respect to $x_p, x_{p+1}, \ldots, x_q$ are principal.

(β) They are all parametric.

(γ) They are divisible into sets (I) and (II); of which the first is composed of principal derivatives, the second of parametric.

Since (I) is composed of principal derivatives, any expression obtained from it by differentiation is also principal. As for the derivatives of (II), any one obtained by operating with $x_{p+j+1}, x_{p+j+2}, \ldots, x_q$ only may or may not be principal. But if we differentiate with respect to any members of the set $x_1, x_2, \ldots, x_{p+j}$ only, once or a greater number of times, we obtain a derivative which is principal because we could obtain it from (I) also.

We shall try to satisfy the requirements by initial conditions of the following types. If a certain group of first members extracted from our systems satisfies condition (α), we shall write down as initial condition corresponding to this type

$$(a) = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p} = a \text{ constant.}$$

If condition (β) is satisfied, we shall write down as corresponding initial condition,

$$(a) = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p} = \phi(x_p, x_{p+1}, \ldots, x_q).$$

Finally, if condition (γ) is satisfied, we shall write the initial condition thus:

$$(a) = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p} = \phi(x_{p+j+1}, x_{p+j+2}, \ldots, x_q).$$

Note that $x_1, x_2, \ldots, x_{p-1}$ are excluded from ϕ in all three cases. In fact we agree that this exclusion shall make part of our hypotheses.

Having assumed that the above types of initial conditions probably satisfy requirements, we shall now proceed to demonstrate rigorously that our supposition is correct.

We proceed as follows confining our attention to the case (γ), because as a result of the exclusion of $x_1, x_2, \ldots, x_{p-1}$ from the ϕ, the argument would not be altered if conditions (α) or (β) were satisfied.

If we differentiate (a) we shall obtain the initial conditions corresponding to (II) and its derivatives with respect to $x_{p+j+1}, x_{p+j+2}, \ldots, x_q$. The right-hand members are

$$\frac{\partial\phi}{\partial x_{p+j+1}}, \frac{\partial\phi}{\partial x_{p+j+2}}, \ldots, \frac{\partial\phi}{\partial x_q},$$

and their derivatives with respect to the variables involved effectively. Hence it is clear that if ϕ contains actually all the variables in the range

$$x_{p+j+1}, x_{p+j+2}, \ldots, x_q,$$

we should certainly obtain by prolongation all the initial conditions corresponding to the parametric derivatives of 'cote' higher than Γ. The only questions which can now arise are the following:

(1) Can any derivative of the ϕ's correspond to a principal derivative? For in that case the ϕ's could not possibly contain all the variables of the range just mentioned actually.

(2) Can we, by prolonging the set of initial conditions constructed in the manner described in the preceding paragraphs,* obtain two different expressions for the same parametric derivative?

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* The description of the construction of the initial conditions runs from the top of p. 100 to the bottom of p. 101.

If we can answer these two questions in the negative, we shall have demonstrated that the initial conditions constructed in the way just described, and which are obviously regular, correspond exactly to the type of system under consideration, *i. e.*, systems with the property K.

Solution of the First Question.

Firstly, we agree that from now on systems which have the property K shall be called canonical. Then let

$$\alpha = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s}$$

be a derivative whose 'cote' is Γ, where

$$n_1 + n_2 + n_3 + \ldots + n_{p-1} + S = n - 1.$$

Furthermore, let

$$\beta = X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-1}^{m_{p-1}}, X_p^{s+i}$$

be a derivative whose 'cote' is $\Gamma + 1$, where

$$m_1 + m_2 + m_3 + \ldots + m_{p-1} + S + i = n,$$

and where i is an integer positive or negative.

Suppose that the first of these two quantities is parametric. In fact, were it not so, it would not figure among the group of derivatives corresponding to arbitrary initial conditions, and we should not be obliged to consider it at all. Our object is to prove that if we differentiate the expression

$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s} = \phi(x_p, x_{p+1}, \ldots, x_q)$$

with respect to $x_p, x_{p+1}, \ldots, x_q$, we shall not obtain any derivative identical with β or a derivative of β.*

To accomplish this we proceed as follows: Firstly, suppose one of the m's to be greater than the n with the same subscript. We then write side by side

$$(\alpha) \quad X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s} = \phi(x_p, x_{p+1}, \ldots),$$
$$(\beta) \quad X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-1}^{m_{p-1}}, X_p^{s+i} = M.$$

If we differentiate the first of these with respect to $x_p, x_{p+1}, \ldots, x_q$ only, the exponents of $X_1, X_2, X_3, \ldots, X_{p-1}$ will not be increased, and therefore one of the m's will always be superior to the n with the same subscript. Therefore, in this case, a derivative of (α) with respect to the specified variables, can not be identical with β or a derivative of β. But if

---

* It may happen that $x_p, x_{p+1}, \ldots, x_{p+j}$ do not occur actually in ϕ. Then we differentiate with respect to $x_{p+j+1}, x_{p+j+2}, \ldots, x_q$.

$$n_r \geqq m_r, \qquad (r=1, 2, \ldots, p-1)$$

for all values of r, we differentiate β as many times as is necessary to make the exponent of X_1 equal to n_1, the exponent of X_2 equal to n_2, and in general the exponent of X_r equal to n_r, where $r=1, 2, \ldots, p-1$. Since $i \geqq 1$, we may now differentiate (α) with respect to x_p until s becomes equal to $s+i$. Now we can write the two expressions derived from α and β,

$$\alpha' = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s+i} = \phi_{x_p}^{(i)}(x_p, x_{p+1}, \ldots),$$
$$\beta' = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s+i} = M^{(i-1)}.$$

Now if ϕ contains x_p actually, α' is parametric, while β' is principal, since it is derived from β which is principal. But $\alpha'=\beta'$, therefore if $\phi_{x_p}^{(i)}$ contains x_p actually, we are led to a contradiction. But we shall now show that x_p can not occur actually in ϕ which is the primitive function from which $\phi_{x_p}^{(i)}$ is derived, if the system has the property K.

For we have assumed that

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-1}^{m_{p-1}}, X_p^{s+i}$$

is principal. Therefore,

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}+1}, X_p^{s+i}$$

is also principal. Then from the property K it follows that

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}+1}, X_p^{s+i-1}$$

is principal. A repeated application of the same rule shows us that

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}+2}, X_p^{s+i-2},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-2}^{n_{p-2}}, X_{p-1}^{n_{p-1}}, X_p^{s+1},$$

are all principal. Now let us consider the group derived from

$$\alpha = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s} = \phi(x_p, x_{p+1}, \ldots),$$

by differentiating it once with respect to $x_p, x_{p+1}, \ldots, x_q$. We write it

$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s+1},$$
$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s}, X_{p+1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s}, X_q.$$

The first of these we have just shown to be principal, but in any case it follows from the fact that the first member of the above group is principal that we must equate α to $\phi(x_{p+1}, x_{p+2}, \ldots, x_q)$, where ϕ may not contain actually all the arguments written down, but surely contains all those posterior to the first one which occurs actually. Then, if we differentiate α with respect to $x_{p+1}, x_{p+2}, \ldots, x_q$, we shall not obtain any derivative identical with β. For we have assumed in this case that $n_i \geqq m_i$ for all values of i. Also the order of β is greater than the order of α by unity. Hence, the exponent of X_p in β is greater than the corresponding exponent in α. And if we prolong α by differentiating with respect to $x_{p+1}, x_{p+2}, \ldots, x_q$, we shall not increase the exponent of X_p. Therefore by that operation we can never obtain a derivative identical with β or with a derivative of β.

The above is a special case given to illustrate the method employed. Let us pass to the consideration of the most general case. Let us compare

$$
\begin{aligned}
&(\gamma)\ \ X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-2}^{n_{p-2}}, X_{p-1}^{n_{p-1}}, X_p^{s},\\
&(\delta)\ \ X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}}, X_p^{r}, X_{p+\lambda}^{t}, X_{p+\lambda+1}^{j}, \ldots, X_q^{K},
\end{aligned}
$$

where

$$n_1+n_2+n_3+\ldots+n_{p-2}+n_{p-1}+S=n-1,$$

and

$$m_1+m_2+m_3+\ldots+m_{p-2}+m_{p-1}+r+t+j+\ldots+K=n.$$

Assume that (γ) is parametric and (δ) principal. In the case where at least one of the m's is greater than the n with the same subscript, it is obvious that no derivative of (γ) with respect to $x_p, x_{p+1}, \ldots, x_q$ is identical with (δ) or a derivative of (δ). But, suppose,

$$n_i \geqq m_i, \qquad (i=1, 2, \ldots, p-1).$$

There are two cases to be disposed of. Firstly, let $r>S$. Since (δ) is principal,

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}+1}, X_p^{f}, X_{p+\lambda}^{t}, X_{p+\lambda+1}^{j}, \ldots, X_q^{K}$$

is surely principal, and it follows from the property K that

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}+1}, X_p^{f}, X_{p+\lambda}^{t}, X_{p+\lambda+1}^{j}, \ldots, X_q^{K-1}$$

is principal. And a repeated application of the above process shows us that

$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-2}^{n_{p-2}}, X_{p-1}^{n_{p-1}}, X_p^{s+1}$$

is certainly principal. Hence x_p can not occur actually in the arbitrary function ϕ which corresponds to (γ). And since $r>S$ no derivative of (γ) with respect to $x_{p+1}, x_{p+2}, \ldots, x_q$ is identical with (δ) or with a derivative of (δ).

Now let us consider the case where $r \leq S$. Then

$$X_1^{m_1},\ X_2^{m_2},\ X_3^{m_3},\ \ldots,\ X_{p-2}^{m_{p-2}},\ X_{p-1}^{m_{p-1}},\ X_p^r,\ X_{p+1},\ X_{p+\lambda}^t,\ \ldots,\ X_q^{K-1}$$

is principal and continuing the process, we show that

$$X_1^{n_1},\ X_2^{n_2},\ X_3^{n_3},\ \ldots,\ X_{p-2}^{n_{p-2}},\ X_{p-1}^{n_{p-1}},\ X_p^s,\ X_{p+1}$$

is principal. In a similar fashion we can show that

$$\begin{gathered} X_1^{n_1},\ X_2^{n_2},\ X_3^{n_3},\ \ldots,\ X_{p-2}^{n_{p-2}},\ X_{p-1}^{n_{p-1}},\ X_p^s,\ X_{p+2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_1^{n_1},\ X_2^{n_2},\ X_3^{n_3},\ \ldots,\ X_{p-2}^{n_{p-2}},\ X_{p-1}^{n_{p-1}},\ X_p^s,\ X_{p+\lambda}, \end{gathered}$$

are principal, and we write our initial condition

$$X_1^{n_1},\ X_2^{n_2},\ X_3^{n_3},\ \ldots,\ X_{p-1}^{n_{p-1}},\ X_p^s = \phi(x_{p+\lambda+1},\ \ldots).$$

Comparing this with

$$X_1^{m_1},\ X_2^{m_2},\ X_3^{m_3},\ \ldots,\ X_q^K = M,$$

and prolonging the two equations, we shall arrive at no incompatibility since the second expression contains a factor $X_{p+\lambda}^f$ where $f \neq 0$. Thus we have given a negative response to the first question.

Solution of the Second Question.

Let us write

$$\left.\begin{aligned} X_1^{n_1},\ X_2^{n_2},\ \ldots,\ X_p^{n_p} &= \phi(x_p,\ x_{p+1},\ \ldots), \quad n_p \neq 0, \\ X_1^{m_1},\ X_2^{m_2},\ \ldots,\ X_p^{m_p} &= \psi(x_p,\ x_{p+1},\ \ldots), \quad m_p \neq 0, \end{aligned}\right\} \qquad \text{(III)}$$

where

$$\begin{aligned} n_1 + n_2 + \ldots + n_p &= n - 1, \\ m_1 + m_1 + \ldots + m_p &= n - 1. \end{aligned}$$

At least one of the m's differs from the corresponding n, for the two expressions are assumed to be different. Then since the sum of the m's equals the sum of the n's for one value of i at least, $m_i > n_i$, and for at least one value of j, which letter is different from i, we have the inequality $n_j > m_j$. It follows that the exponents of at least two of the X's in each of the expressions of (III) are different. Therefore, we have the inequality $n_i \neq m_i$ where $i \neq p$. Therefore, if we differentiate (III) an arbitrary number of times with respect to x_p, $x_{p+1}, \ldots, x_q$, we shall never arrive at incompatibilities. In case $m_p = 0$, $n_p \neq 0$, the course of reasoning is not different, for we write

$$\begin{gathered} X_1^{n_1},\ X_2^{n_2},\ \ldots,\ X_{p-1}^{n_{p-1}},\ X_p^{n_p} = \phi(x_p,\ x_{p+1},\ \ldots), \\ X_1^{m_1},\ X_2^{m_2},\ \ldots,\ X_{p-1}^{m_{p-1}} = \psi(x_{p-1},\ x_p,\ \ldots). \end{gathered}$$

Then at least one of the m's is greater than the corresponding n, and again we see that no incompatibility can arise.

All that we have now proven is only a generalization of section 197 of Riquier's book.

Section 3.

The facts which are now at our disposal will be used to rectify a defect in the work of Delassus. This author, in the paper we have cited,[*] has reduced any set of equations to a canonical form of which, however, there are exceptional cases. The determinant

$$\Delta=\frac{\partial(\xi_1,\xi_2,\ldots,\xi_{p-1},\zeta_1)}{\partial(\xi'_1,\xi'_2,\ldots,\xi'_p)},$$

mentioned on page 436 of his work actually does vanish for certain exceptional cases. The following example makes this clear. Indeed, the reasoning of the last paragraphs of page 436, and the first paragraphs of the following page is a little defective.

It is easy to show that if we are given a set of partial differential equations, there is no unique canonical form which is at the same time orthonomic.

To prove this let us take a system of partial differential equations of the second order containing three equations. When there are only one unknown and three independent variables, the maximum number of derivatives which may be involved in this system is six. Denote these by X_1^2, X_1X_2, X_1X_3, X_2^2, X_2X_3, X_3^2, and take them three at a time. There will be twenty different combinations.

Let us suppose that X_1^2, X_1X_2, X_1X_3 are the first members of our system. In other words let us write the system thus:

$$X_1^2=f_1,\quad X_1X_2=f_2,\quad X_1X_3=f_3, \tag{I}$$

the second members being functions of the independent variables x_1, x_2 and x_3 only. Let us now consider the system

$$X_1^2=\phi_1,\quad X_1X_2=\phi_2,\quad X_2^2=\phi_3, \tag{II}$$

the second members being of the same type as those of the first system. Furthermore, let them satisfy the conditions for the passivity of system (II). We can easily show that no linear transformation of variables will send the set (II) into a set of type (I). For, make the change,

$$\begin{aligned}x'_1&=ax_1+bx_2+cx_3,\\ x'_2&=a'x_1+b'x_2+c'x_3,\\ x'_3&=a''x_1+b''x_2+c''x_3.\end{aligned}$$

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[*] *Annales de l'école normale supérieure*, Vol. IX (1896).

Then,

$$\begin{aligned}X_1^2&=a^2X_1'^2+2aa'X_1'X_2'+2aa''X_1'X_3'+\ldots\ldots,\\ X_1X_2&=abX_1'^2+(ab'+a'b)X_1'X_2'+(ab''+a''b)X_1'X_3'+\ldots\ldots,\\ X_2^2&=b^2X_1^2+2bb'X_1'X_2'+2bb''X_1'X_3'+\ldots\ldots,\end{aligned}$$

and it is easy to see that the determinant

$$\begin{vmatrix} a^2, & 2aa', & 2aa'', \\ ab, & ab'+a'b, & ab''+a''b, \\ b^2, & 2bb', & 2bb'', \end{vmatrix}$$

vanishes.

We next consider the possibility of transforming (II) into a system whose first members are X_1^2, X_1X_2, X_2X_3. We easily see that no choice of 'cotes' will make these three quantities normal to all three of the expressions X_1X_3, X_2^2, X_3^2, and therefore a change of variables followed by a resolution, leads to a set which is non-orthonomic. And if we examine the eighteen cases which remain, we shall see that we shall obtain similar results each time.

The above simple examples show us that cases exist where it is impossible to make a reduction to a fixed canonical form which is also orthonomic.

Section 4.

In what follows we shall show that it is always possible, by the aid of a linear homogeneous transformation of the independent variables, to put any system of equations into a form which shall be a special case of what we have called the canonical form. This form, indeed, varies with the system of equations, but is so defined that when the original system is given, we can say what the canonical form shall be. Likewise we shall see that we can determine whether or not the system is passive by a finite number of operations, since our canonical form is regular. From now on we shall call the canonical form, defined in the first section, form (A), and the new form about to be defined, form (B). (A) contains (B), that is if a system enters into form (B), it certainly belongs also to (A).

We shall, to simplify writing, define our new form in the case of three independent variables only. It is characterized by the following property:

If $X_1^{a_1}X_2^{a_2}X_3^{a_3}$ *is a first member of the set, then*

$$X_1^{a_1}X_2^{a_2+1}X_3^{a_3-1},\quad X_1^{a_1+1}X_2^{a_2}X_3^{a_3-1},\quad X_1^{a_1+1}X_2^{a_2-1}X_3^{a_3},$$

are also first members. In fact any quantity is a first member which possesses the double property of (*a*) *being obtained from* $X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}$ *by adding unity to one exponent, and subtracting it from another.*

(*b*) *being anterior to* $X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}$ *according to the definition of anterior given by Delassus.*

Let us call this double property (J). It is easy to see that this new form is less general than the canonical form (A). For if a system possesses the property (K), the existence of $X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}$ among the group of first members involves the existence of $X_1^{\alpha_1}X_2^{\alpha_2+1}X_3^{\alpha_3-1}$ and $X_1^{\alpha_1+1}X_2^{\alpha_2}X_3^{\alpha_3-1}$ in the same group, but not the existence of $X_1^{\alpha_1+1}X_2^{\alpha_2-1}X_3^{\alpha_3}$.

In the second section we have shown that if a system possesses the property (K), the prolonged system possesses the same property. By the same method we can show that if a system possesses the property (J) the prolonged system possesses it also.

Section 5.

We shall now proceed to develop a method by which we can transform the most general system of partial differential equations into canonical form (B). Firstly, we shall demonstrate the following lemma:

Given an equation

$$\delta=f(x, y, \ldots, \sigma, \ldots, \tau, \ldots),$$

belonging to the most general system S, where τ *and* σ *represent the derivatives of a class*[*] *greater and less than* δ, *respectively. Suppose that it is desirable to solve the system according to the general method employed for the solution of implicit functions, and that the initial values are* $x_0, y_0, \ldots, \sigma_0, \ldots, \tau_0, \ldots, \delta_0$. *It is always possible to solve the system S in such a way that the initial values of all derivatives of the type* $\frac{\partial f}{\partial \tau}$ *shall vanish.*

For, consider the system

$$\delta_1=f_1(x, y, \ldots, \sigma, \ldots, \tau, \ldots), \delta_2=f_2, \ldots, \delta_n=f_n.$$

Suppose that $\frac{\partial f_1}{\partial \tau_1} \neq 0$. Then we can solve the first equation of the system with respect to τ_1 and eliminate τ_1 from the second members of the remaining equations of the set.[†] Since τ_1 does not occur among the first members of these remaining equations, nor $\delta_2, \delta_3, \ldots, \delta_n$ among the arguments of the first equation, or of the first equation solved with respect to τ_1 the system can now be written:

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[*] For the meaning of this word see Riquier, *loc. cit.*, p. 208.

[†] τ_1 is any one of the τ's.

$$\tau_1=\bar{f}_1(xy, \ldots, \delta_1, \sigma, \ldots, \tau, \ldots), \delta_2=\bar{f}_2, \delta_3=\bar{f}_3, \ldots, \delta_n=f_n.$$

And so we can repeat this process until we exhaust all possibilities and obtain a system of the type desired.

We will now suppose, in order to fix ideas, that our system is composed of equations all of order n, and involving only one unknown u. Furthermore, let it be of the type described in the lemma. Finally we shall write it

$$S\begin{cases}\xi_1 \;\; +\phi_1 \;\; (\eta\ldots\ldots)=0,\\ \xi_2 \;\; +\phi_2 \;\; (\eta\ldots\ldots)=0,\\ \ldots\ldots\ldots\ldots\ldots\ldots,\\ \xi_{p-1}+\phi_{p-1}(\eta\ldots\ldots)=0,\\ \zeta \;\; +\phi_p \;\; (\eta\ldots\ldots)=0.\end{cases}$$

If we suppose all the derivatives of order n written down in the order adopted by Delassus,* ξ_2 shall be supposed posterior to ξ_1, ξ_3 posterior to ξ_2, etc. But we do not assert that ξ_2 follows immediately after ξ_1, or that ξ_3 follows immediately after ξ_2, etc. There may be intermediate derivatives. As for ζ we make no hypotheses concerning it. It may occur anywhere in the range.

We have assumed our system to be of the type described in the lemma. This means that if in the equation

$$\xi_i-\phi_i=0,$$

ϕ_i contains some derivative, say η anterior to ξ_i, then $\dfrac{\partial\phi_i}{\partial\eta}=0$ at the initial points $x_0, y_0, \ldots, \sigma_0, \ldots, \tau_0, \ldots,$ of the solution.

Now let

$$\zeta=X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}.$$

Suppose $X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}$ to be absent from the group of first members of our system. We shall call it ζ_p. We are going to demonstrate that the system whose first members are

$$\xi_1, \xi_2, \ldots, \xi_{p-1}, \zeta,$$

can, after a linear homogeneous transformation of the independent variables,

$$x_1'=ax_1+bx_2+cx_3,$$
$$x_2'=a'x_1+b'x_2+c'x_3,$$
$$x_3'=a''x_1+b''x_2+c''x_3,$$

---

* *Annales de l'école normale supérieure, loc. cit.*, p. 426.

followed by a resolution with respect to the proper derivatives, be changed into a system whose first members are

$$\xi_1', \xi_2', \ldots, \xi_{p-1}', \xi_p'.$$

It will be sufficient to indicate the points wherein our proof differs from that of Delassus.

Let us return to the system S. It is our object to solve this system with respect to $\xi_1', \xi_2', \ldots, \xi_p'$, starting from the initial values $\xi_1^{(0)}, \xi_2^{(0)}, \ldots, \xi_{(p-1)}^{(0)}$, $\zeta^0, \eta^0, \ldots$, and we must show that the functional determinant does not vanish for these initial values. The functional determinant of our p equations, after they have been transformed with respect to $\xi_1', \xi_2', \ldots, \xi_p'$ is,

$$D=\left|\begin{array}{l} \frac{\partial\xi_1}{\partial\xi_1'}+\Sigma\frac{\partial\phi_1}{\partial\eta}\frac{\partial\eta}{\partial\xi_1'}, \ldots, \frac{\partial\xi_1}{\partial\xi_p'}+\Sigma\frac{\partial\phi_1}{\partial\eta}\frac{\partial\eta}{\partial\xi_p'}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \frac{\partial\xi_{p-1}}{\partial\xi_1'}+\Sigma\frac{\partial\phi_{p-1}}{\partial\eta}\frac{\partial\eta}{\partial\xi_1'}, \ldots, \frac{\partial\xi_{p-1}}{\partial\xi_p'}+\Sigma\frac{\partial\phi_{p-1}}{\partial\eta}\frac{\partial\eta}{\partial\xi_p'}, \\ \frac{\partial\zeta}{\partial\xi_1'}+\Sigma\frac{\partial\phi_p}{\partial\eta}\frac{\partial\eta}{\partial\xi_1'}, \ldots, \frac{\partial\zeta}{\partial\xi_p'}+\Sigma\frac{\partial\phi_p}{\partial\eta}\frac{\partial\eta}{\partial\xi_p'}. \end{array}\right|$$

If $\frac{\partial\phi_\lambda}{\partial\eta}\neq 0$ for the initial values, then by hypothesis $\frac{\partial\xi_\lambda}{\partial\xi_\lambda'}$ is anterior to η, $\{\lambda=1, 2, \ldots, p\}$.

And so following the same course of reasoning as Delassus, we retain only the first members of each element of the determinant. We are left with

$$\Delta=\frac{\partial(\xi_1, \xi_2, \ldots, \xi_{p-1}, \zeta)}{\partial(\xi_1', \xi_2', \ldots, \xi_p')}.$$

Then consider the minors of the above determinant $A_{p,p}, A_{p,p-1}, \ldots, A_{p,1}$. The diagonal of $A_{p,p}$ consists of the elements

$$\frac{\partial\xi_1}{\partial\xi_1'}, \frac{\partial\xi_2}{\partial\xi_2'}, \ldots, \frac{\partial\xi_{p-1}}{\partial\xi_{p-1}'}.$$

Each of these terms is of degree n in a, b', c''. For let $\xi_k'=X_1'^{a_1}X_2'^{a_2}X_3'^{a_3}$. Then, clearly, $\xi_k=X_1^{a_1}X_2^{a_2}X_3^{a_3}$, where $\{k=1, 2, \ldots, p-1\}$. After transforming we obtain

$$X_1^{a_1}X_2^{a_2}X_3^{a_3}=(aX_1'+a'X_2'+a''X_3')^{a_1}(bX_1'+b'X_2'+b''X_3')^{a_2}(cX_1'+c'X_2'+c''X_3')^{a_3}.$$

The coefficient of $X_1'^{a_1}X_2'^{a_2}X_3'^{a_3}$ is $a^{a_1}b'^{a_2}c''^{a_3}$. Hence $\frac{\partial\xi_k}{\partial\xi_k'}=a^{a_1}b'^{a_2}c''^{a_3}$, which contains no other quantities except a, b', c'', and is, consequently, of the n-th degree in the above-mentioned quantities. So $A_{pp} \not\equiv 0$, but is of degree $n(p-1)$ in a, b', c''.

Let us now consider the expression $\frac{\partial \zeta_1}{\partial \xi'_p}$. Let $\zeta_1 = X_1^{a_1} X_2^{a_2} X_3^{a_2}$. Then,

$$\xi'_p = X_1^{a_1} X_2^{a_2-1} X_3^{a_3+1}, \quad \text{or} \quad X_1^{a_1-1} X_2^{a_2+1} X_3^{a_3}, \quad \text{or} \quad X_1^{a_1-1} X_2^{a_2} X_3^{a_3+1}.$$

Let us suppose that the first equality is the one which exists, for the reasoning will be the same in the other cases:

$$X_1^{a_1} X_2^{a_2} X_3^{a_3} = (aX'_1 + a'X'_2 + a''X'_3)^{a_1} (bX'_1 + b'X_2 + b''X_3)^{a_2} (cX'_1 + c'X'_2 + c''X'_3)^{a_3}.$$

The coefficient of $X_1^{a_1} X_2^{a_2-1} X_3^{a_3+1}$ is clearly obtained by expanding

$$a_1^{a_1} b'^{a_2-1} c''^{a_3} X_1'^{a_1} X_2^{a_2-1} X_3^{a_3} (bX'_1 + b'X_2 + b''X_3),$$

and is clearly $a^{a_1} b'^{a_2-1} b'' c''^{a_3}$ and is of degree $n-1$ in a, b', c''. Hence $\frac{\partial \zeta_1}{\partial \xi'_p}$ is of degree $n-1$ in the same quantities.

We can easily convince ourselves that $\frac{\partial \zeta_1}{\partial \xi'_1}, \frac{\partial \zeta_1}{\partial \xi'_2}, \ldots, \frac{\partial \zeta_1}{\partial \xi'_{p-1}}$ are of degree $\not> (n-1)$ in the a, b', c''. Also we easily demonstrate that all the minors $A_{p,\,p-1}, \ldots, A_{p,\,1}$ are of degree less than $n(p-1)$. Therefore, the whole determinant does not vanish identically, but is effectively of degree $n(p-1) + (n-1)$ in a, b', c''.

Thus we see that if $X_1^{a_1} X_2^{a_2} X_3^{a_3}$ is among the group of first members, either $X_1^{a_1} X_2^{a_2+1} X_3^{a_3}$ is a first member, or it can be made a first member by a linear homogeneous transformation followed by a resolution. Similar reasoning will show that we can make the same statement with respect to $X_1^{a_1+1} X_2^{a_2} X_3^{a_3-1}$ and $X_1^{a_1+1} X_2^{a_2-1} X_3^{a_3}$.

We can follow a course of reasoning parallel to that of Delassus in case the various equations of the system are not all of order n, or if there be more unknowns than one. It is easy to see that we can establish the convergence of the series formally satisfying our canonical system of equations by the method of Delassus, *i. e.*, by reducing the given system to a system of systems of the type of Kovalefski.

We should also note that a system can often be made canonical in more than one way. For example:

$$\frac{\partial^2 u}{\partial x_1^2} = f_1(x_1 x_2 x_3), \quad \frac{\partial^2 u}{\partial x_1 \partial x_2} = f_2 \quad \frac{\partial^2 u}{\partial x_1 \partial x_3} = c \frac{\partial^2 u}{\partial x_2^2},$$

and

$$\frac{\partial^2 u}{\partial x_1^3} = f_1(x_1 x_2 x_3), \quad \frac{\partial^2 u}{\partial x_1 \partial x_2} = f_2 \quad \frac{\partial^2 u}{\partial x_2} = \frac{1}{c} \frac{\partial^2 u}{\partial x_1 \partial x_3},$$

are equivalent forms, and both are canonical.

So far the author has shown, that since the canonical form developed is regular, it can be determined whether or not it is passive by a finite number of operations. But the knowledge that the canonical form is regular is not necessary for this purpose. In fact, it can readily be shown, that using the known theorems concerning the invariants λ and μ, we can determine whether or not a given system at certain points not solvable in orthonomic form, is passive throughout the entire space of n dimensions.

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Fourier's Constants of Functions of Several Variables.

By Dr. W. W. Küstermann.

In Vol. LVII of the *Mathematische Annalen*, A. Hurwitz has shown how to express the product of two ordinary Fourier's series in form of another Fourier's series, or, stated differently, how to compute Fourier's constants of the product of two functions from those of the functions themselves. In the following pages we solve the corresponding problem for multiple Fourier's series. In order to simplify matters the work is carried through for double Fourier's series only, but results and proofs admit of immediate and obvious generalization to n-tuple series.

The solution of our problem depends vitally upon a proof of the relation*

$$\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}[f(x,y)]^2\,dx\,dy=\sum_{\mu,\nu=0}^{\infty}\frac{(a^2+b^2+c^2+d^2)_{\mu\nu}}{2^{E\left(\frac{1}{\mu+1}\right)+E\left(\frac{1}{\nu+1}\right)}} \tag{1}$$

where $(a,b,c,d)_{\mu\nu}$ are Fourier's constants of $f(x,y)$ and have the values

$$\left.\begin{matrix} a\\ b\\ c\\ d\end{matrix}\right\}_{\mu\nu}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(\alpha,\beta)\left\{\begin{matrix}\cos\mu\alpha\,\cos\nu\beta\\ \sin\mu\alpha\,\cos\nu\beta\\ \cos\mu\alpha\,\sin\nu\beta\\ \sin\mu\alpha\,\sin\nu\beta\end{matrix}\right\}d\alpha\,d\beta. \tag{2}$$

The analogue of this relation for functions of a single variable is due to Parseval, and was proved by him on the supposition that the function in question is really represented by its Fourier's development and that this series is integrable term by term. In 1893 de la Vallée-Poussin† gave a proof requiring merely that the function and its square be integrable. Hurwitz, in 1903, again called attention to the wide scope of the theorem, gave a new proof, and applied it to the problem mentioned. More recently proofs of varying generality have been added by Fischer, Stekloff, Fatou,[*] and the formula has gained considerably in interest through the researches of Riess and Fischer (Riess-Fischer Theorem).[†]

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* In the following $(a^2+b^2+c^2+d^2)_{\mu\nu}$ stands for $(a^2_{\mu\nu}+b^2_{\mu\nu}+c^2_{\mu\nu}+d^2_{\mu\nu})$ and so on. $E(x)$ denotes the largest positive integer contained in x. Hereafter we will write $\Lambda_{\mu\nu}$ in place of $2^{E\left(\frac{1}{\mu+1}\right)+E\left(\frac{1}{\nu+1}\right)}$,

then $\Lambda_{\mu\nu}=\begin{cases}4 \text{ when } \mu=\nu=0,\\ 2 \quad " \quad \mu=0,\ \nu>0 \text{ or } \mu>0,\ \nu=0,\\ 1 \quad " \quad \mu>0,\ \nu>0.\end{cases}$

† De la Vallée-Poussin, *Ann. de la Soc. Scient. de Bruxelles*, Vol. XVII, p. 18.

The first three paragraphs of our paper are devoted to proving relation (1) by a method which like those of de la Vallée-Poussin, Fischer and Fatou involves the use of Poisson's integral. (For a brief survey see § 3.) We, too, make no assumption regarding the convergence of the Fourier's series of $f(x, y)$.

All we demand of $f(x, y)$ is that it be bounded and possess a Riemann double integral over the fundamental square.[‡] For convenience of language we will suppose $f(x, y)$ periodic 2π, so that

$$f(x \pm 2\pi k,\ y \pm 2\pi l) = f(x, y) \text{ for } \begin{cases} k = 0, 1, 2, \ldots, \\ l = 0, 1, 2, \ldots. \end{cases}$$

These conditions insure the existence of Fourier's constants $(a, b, c, d)_{\mu\nu}$, but do not in any way prejudice the question of convergence[§] of the formal Fourier's development for $f(x, y)$ which we can write:

$$\begin{aligned} f(x, y) \sim \sum_{\mu, \nu = 0}^{\infty} 1/\Lambda_{\mu\nu} [(a_{\mu\nu} \cos \mu x + b_{\mu\nu} \sin \mu x) \cos \nu y \\ + (c_{\mu\nu} \cos \mu x + d_{\mu\nu} \sin \mu x) \sin \nu y]. \end{aligned} \quad (3)$$

How relation (1) can be utilized to solve the problem originally proposed is shown in § 4.

§ 1. *Preliminary Theorems.*

THEOREM I. *If $\phi(x, y)$ is bounded and integrable*[‖] *in the fundamental square and can be represented by a trigonometric double series which is uniformly convergent in this closed region, then the trigonometric series is a Fourier's series* (that is its coefficients are given by formulas of type (2)).

The proof is so closely analogous to the proof of the corresponding theorem for ordinary Fourier's series that we will not reproduce it here.

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[*] Fischer, *Monatshefte*, Vol. XV, p. 69; Stekloff, *Mem. de l'Ac. de St. Petersbourg*, Vol. XV, Ser. 8; Fatou, *Acta Math.*, Vol. XXX.

[†] Riess, "Über orthogonale Funktionensysteme," *Gött. Nachr.*, 1907; Fischer, *Compt. Rend.*, 1907; also Young, *Quart. Journ.*, Vol. XLIV; where this relation is shown to represent the *necessary* and *sufficient* condition that a given trigonometric series be the Fourier's series of a function whose square is integrable.

[‡] By the "fundamental square" we mean the square $(-\pi, -\pi;\ \pi, \pi)$ whose lower left-hand and upper right-hand vertices are $(-\pi, -\pi)$ and (π, π), respectively.

[§] In the Stolz-Pringsheim sense. Cf. *Münchner Sitzungsber.*, Vol. XXVII, p. 101.

[‖] All integrals in this paper are to be taken in the Riemann sense.

THEOREM II. *If $f(x, y)$ and $\phi(x, y)$ are both bounded and integrable in the fundamental square, if furthermore $\phi(x, y)$ can be developed into a double Fourier's series, uniformly convergent in this region, then*

$$\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x, y)\,\phi(x, y)\,dx\,dy = \sum_{\mu,\,\nu=0}^{\infty}(a\alpha+b\beta+c\gamma+d\delta)_{\mu\nu}/\Lambda_{\mu\nu},^{*} \qquad (4)$$

where $(a, b, c, d)_{\mu\nu}$, $(\alpha, \beta, \gamma, \delta)_{\mu\nu}$ are the Fourier's constants of f and ϕ, respectively.

To prove this theorem multiply the Fourier's expansion of $\phi(x, y)$ by $f(x, y)$ and integrate term by term over the fundamental square.† If, in particular, we let $\phi(x, y) = f(x, y)$, equation (4) goes over into (1). The latter relation therefore holds in the special case, where $f(x, y)$ can be developed into a *uniformly* convergent Fourier's series. To prove it for any integrable $f(x, y)$ we need further preparations.

LEMMA: *The double series*

$$S = \sum_{\mu,\,\nu=0}^{\infty}(a^2+b^2+c^2+d^2)_{\mu\nu}/\Lambda_{\mu\nu}, \qquad (5)$$

where $(a, b, c, d)_{\mu\nu}$ are the Fourier's constants of a bounded integrable function $f(x, y)$, is absolutely convergent and can be summed by rows, columns or diagonals.

PROOF: Define $\psi(x, y)$ by the identity

$$\begin{aligned}\psi(x, y) = f(x, y) - \Bigg\{&\frac{a_{00}}{4} + \frac{1}{2}\sum_{\mu=1}^{m}(a_{\mu 0}\cos\mu x + b_{\mu 0}\sin\mu x)\\ &+\frac{1}{2}\sum_{\nu=1}^{n}(a_{0\nu}\cos\nu y + c_{0\nu}\sin\nu y)\\ &+\sum_{\mu'=1}^{m}\sum_{\nu'=1}^{n}\big[(a_{\mu'\nu'}\cos\mu' x + b_{\mu'\nu'}\sin\mu' x)\cos\nu' y\\ &+(c_{\mu'\nu'}\cos\mu' x + d_{\mu'\nu'}\sin\mu' x)\sin\nu' y\big]\Bigg\}.\end{aligned}$$

---

[*] Cf. note * p. 113.

[†] The reduction is effected by means of the well-known formulas;

$$\int_{-\pi}^{\pi}\begin{matrix}\sin\\ \cos\end{matrix}\,\kappa x,\ \begin{matrix}\sin\\ \cos\end{matrix}\,\lambda x\,dx = \begin{cases}0, \text{ if } \kappa \neq \lambda,\\ \pi, \text{ if } \kappa = \lambda \neq 0.\end{cases}$$

$$\int_{-\pi}^{\pi}\sin\kappa x\cdot\cos\lambda x\,dx = 0; \qquad \int_{-\pi}^{\pi}\begin{matrix}\sin\\ \cos\end{matrix}\,\kappa x\,dx = 0, \text{ if } \kappa \neq 0.$$

Squaring both sides of this equation

$$\left.\begin{aligned}
[\psi(x,y)]^2=[f(x,y)]^2-\Big\{&\frac{a_{00}}{2}f(x,y)+\sum_{\mu=1}^{m}(a_{\mu0}f(x,y)\cos\mu x+b_{\mu0}f(x,y)\sin\mu x)\\
&+\sum_{\nu=1}^{n}(a_{0\nu}f(x,y)\cos\nu y+c_{0\nu}f(x,y)\sin\nu y)\\
+2\sum_{\mu'=1}^{m}\sum_{\nu'=1}^{n}&[a_{\mu'\nu'}f(x,y)\cos\mu'x\cos\nu'y+b_{\mu'\nu'}f(x,y)\sin\mu'x\cdot\cos\nu'y\\
&+c_{\mu'\nu'}f(x,y)\cos\mu'x\sin\nu'y+d_{\mu'\nu'}f(x,y)\sin\mu'x\sin\nu'y]\Big\}\\
+\frac{a_{00}^2}{16}+\frac{1}{4}\sum_{\mu=1}^{m}&(a_{\mu0}^2\cos^2\mu x+b_{\mu0}^2\sin^2\mu x)+\frac{1}{4}\sum_{\nu=1}^{n}(a_{0\nu}^2\cos^2\nu y+c_{0\nu}^2\sin^2\nu y)\\
+\sum_{\mu'=1}^{m}\sum_{\nu'=1}^{n}&[a_{\mu'\nu'}^2\cos^2\mu'x\cos^2\nu'y+b_{\mu'\nu'}^2\sin^2\mu'x\cos^2\nu'y\\
&+c_{\mu'\nu'}^2\cos^2\mu'x\sin^2\nu'y+d_{\mu'\nu'}^2\sin^2\mu'x\sin^2\nu'y]+\Sigma\mu\,\Sigma\nu\,\Sigma\mu'\,\Sigma\nu'.
\end{aligned}\right\}\tag{6}$$

Here $\Sigma\mu\,\Sigma\nu\,\Sigma\mu'\,\Sigma\nu'$ contains those product terms (except for constant coefficients) which are of one of the sixteen types $\genfrac{}{}{0pt}{}{\cos}{\sin}\mu x\cdot\genfrac{}{}{0pt}{}{\cos}{\sin}\nu y\cdot\genfrac{}{}{0pt}{}{\cos}{\sin}\mu' x\cdot\genfrac{}{}{0pt}{}{\cos}{\sin}\nu' y$ where some of the numbers μ, ν, μ', ν' may be zero, but in no term of which the equalities $\mu=\mu', \nu=\nu'$ coexist. If we now integrate equation (6) over the fundamental square the terms resulting from $\Sigma\mu\,\Sigma\nu\,\Sigma\mu'\,\Sigma\nu'$ vanish* and we have

$$\begin{aligned}
\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}[\psi(x,y)]^2\,dx\,dy=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}[f(x,y)]^2\,dx\,dy-\pi^2\Big\{&\frac{a_{00}^2}{2}+\sum_{\mu=1}^{m}(a^2+b^2)_{\mu0}\\
&+\sum_{\nu=1}^{n}(a^2+c^2)_{0\nu}+2\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}(a^2+b^2+c^2+d^2)_{\mu\nu}\Big\}\\
+\pi^2\Big\{&\frac{a_{00}^2}{4}+\frac{1}{2}\sum_{\mu=1}^{m}(a^2+b^2)_{\mu0}+\frac{1}{2}\sum_{\nu=1}^{n}(a^2+c^2)_{0\nu}\\
&+\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}(a^2+b^2+c^2+d^2)_{\mu\nu}\Big\}
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}[\psi(x,y)]^2\,dx\,dy\\
&\qquad=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}[f(x,y)]^2\,dx\,dy-\sum_{\mu,\,\nu=0}^{\mu=m,\,\nu=n}(a^2+b^2+c^2+d^2)_{\mu\nu}/\Lambda_{\mu\nu}.
\end{aligned}$$

The integral on the left-hand side can clearly not be negative, hence, setting

$$S_{mn}=\sum^{\mu=m,\,\nu=n}(a^2+b^2+c^2+d^2)_{\mu\nu}/\Lambda_{\mu\nu}$$

we can write

$$S_{mn}\leqq\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}[f(x,y)]^2\,dx\,dy,$$

---

* Cf. note † p. 115.

no matter what positive integral values m and n may have. Since, moreover, all the terms of (5) are positive, it follows from a well-known theorem* that this series converges absolutely and hence can be summed in the various ways mentioned.

COROLLARY: *The Fourier's constants of a bounded and integrable function of two variables approach zero when either one or both subscripts become infinite.*†

§ 2. *Discussion of Poisson's Double Integral.*

If in the double Fourier's series (3) we introduce the convergence factors $r^{\mu+\nu}$, where $0 \leq r < 1$, we obtain a new double series

$$u_r(x,y) = \sum_{\mu,\nu=0}^{\infty} \frac{r^{\mu+\nu}}{\Lambda_{\mu\nu}} \left[(a_{\mu\nu} \cos \mu x + b_{\mu\nu} \sin \mu x) \cos \nu y \right. \\ \left. + (c_{\mu\nu} \cos \mu x + d_{\mu\nu} \sin \mu x) \sin \nu y \right] \qquad (7)$$

which is uniformly (and absolutely) convergent for all values of (x, y) and hence defines a function $u_r(x, y)$ which is finite, continuous, and integrable in the fundamental square. If we replace $(a, b, c, d)_{\mu\nu}$ by their values from (2), interchange summation and integration signs, and effect the summation by means of the formula

$$\frac{1}{2} + \sum_{\mu=1}^{\infty} r^{\mu} \cos \mu\phi = \frac{1}{2} \cdot \frac{1-r^2}{1+r^2-2r\cos\phi} = \frac{1}{2} P_r(\phi),$$

we obtain

$$u_r(x, y) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\alpha, \beta) P_r(\alpha - x) P_r(\beta - y) \, d\alpha \, d\beta,$$

which by the transformation $\alpha - x = \xi$, $\beta - y = \eta$ goes over into

$$u_r(x,y) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+\xi, y+\eta) P_r(\xi) P_r(\eta) \, d\xi \, d\eta$$
$$= \frac{1}{4\pi^2} \int_{0}^{\pi} \int_{0}^{\pi} F(x, y, \xi, \eta) P_r(\xi) P_r(\eta) \, d\xi \, d\eta, \qquad (8)$$

where

$$F(x, y, \xi, \eta) \equiv f(x+\xi, y+\eta) + f(x+\xi, y-\eta) + f(x-\xi, y+\eta) + f(x-\xi, y-\eta). \qquad (9)$$

---

* Cf. Pringsheim, *loc. cit.*, § 2, V and § 3.

† This property of double Fourier's constants was proved from the integrals defining them by W. H. Young, *Lond. Math. Soc. Proc.*, Ser. 2, Vol. XI, p. 133.

We wish to prove

THEOREM III. *If $f(x, y)$ is bounded and integrable in the fundamental square, then*

$$\lim_{r=1-0} u_r(x, y) = f(x, y)$$

at every point of continuity of $f(x, y)$.

Since

$$\int_0^s P_r(\phi)\, d\phi = 2 \arctan\left(\frac{1+r}{1-r} \tan\frac{s}{2}\right) \leqq \pi$$

and

$$\int_0^\pi P_r(\phi)\, d\phi = \pi \text{ for every positive } r<1,$$

we may write

$$\left.\begin{aligned} & u_r(x, y) - f(x, y) \\ & = \frac{1}{4\pi^2} \int_0^\pi \int_0^\pi [F(x, y, \xi, \eta) - 4 f(x, y)]\, P_r(\xi)\, P_r(\eta)\, d\xi\, d\eta \\ & = \frac{1}{4\pi^2} \left\{ \int_0^a \int_0^a + \int_a^\pi \int_0^\pi + \int_0^a \int_a^\pi \right\} \\ & \qquad\qquad [F(x, y, \xi, \eta) - 4 f(x, y)]\, P_r(\xi)\, P_r(\eta)\, d\xi\, d\eta \\ & = \frac{1}{4\pi^2} \left\{ I_1 + I_2 + I_3 \right\}. \end{aligned}\right\} \tag{10}$$

We shall see that the sum of these three integrals can be made as small as we please by choosing r close enough to unity.

$$|I_1| \leqq \int_0^a \int_0^a |F(x, y, \xi, \eta) - 4 f(x, y)| \cdot |P_r(\xi)| \cdot |P_r(\eta)|\, d\xi\, d\eta.$$

Now

$$P_r(\phi) = \frac{1-r^2}{1+r^2-2r\cos\phi} \geqq \frac{1-r^2}{(1+r)^2} = \frac{1-r}{1+r} > 0$$

for $0 \leqq r < 1$; hence, $|P_r(\phi)| = P_r(\phi)$; furthermore,

$$1 + r^2 - 2r\cos\phi = (1-r)^2 + 4r\sin^2\frac{\phi}{2} > 4r\sin^2\frac{a}{2},$$

for $0 \leqq r < 1$ and $0 < a \leqq \phi \leqq \pi$; therefore,

$$P_r(\phi) < (1-r^2)/4r\sin^2\frac{a}{2}.$$

If (x, y) is a point of continuity for $f(x, y)$, then a can be chosen so small that the upper bound of $|F(x, y, \xi, \eta) - 4 f(x, y)|$ in the domain $\begin{cases} 0 \leqq \xi \leqq a \\ 0 \leqq \eta \leqq a \end{cases}$ is less than $\frac{\varepsilon_1}{\pi^2}$. Thus,

$$|I_1| < \varepsilon_1 \text{ for any } 0 \leqq r < 1. \tag{11}$$

The Fourier's constants of ψ', namely $(A', B', C', D')_{\mu\nu}$, can be expressed in terms of those of ϕ. For instance,

$$A'_{\mu\nu}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\psi'\cos\mu x\cos\nu y\,dx\,dy=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\phi\cos mx\cos\mu x\cos ny\cos\nu y\,dx\,dy$$

$$=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\phi\,[\cos(\mu-m)x+\cos(\mu+m)x]\,[\cos(\nu-n)y+\cos(\nu+n)y]\,dx\,dy$$

$$=\frac{1}{4}(\alpha_{\mu-m,\,\nu-n}+\alpha_{\mu-m,\,\nu+n}+\alpha_{\mu+m,\,\nu-n}+\alpha_{\mu+m,\,\nu+n}).$$

Similarly,

$$B'_{\mu\nu}=\tfrac{1}{4}(\beta_{\mu-m,\,\nu-n}+\beta_{\mu-m,\,\nu+n}+\beta_{\mu+m,\,\nu-n}+\beta_{\mu+m,\,\nu+n}),$$

$$C'_{\mu\nu}=\tfrac{1}{4}(\gamma_{\mu-m,\,\nu-n}+\gamma_{\mu-m,\,\nu+n}+\gamma_{\mu+m,\,\nu-n}+\gamma_{\mu+m,\,\nu+n}),$$

$$D'_{\mu\nu}=\tfrac{1}{4}(\delta_{\mu-m,\,\nu-n}+\delta_{\mu-m,\,\nu+n}+\delta_{\mu+m,\,\nu-n}+\delta_{\mu+m,\,\nu+n}).$$

But ψ' satisfies all conditions under which (16) was derived; hence,

$$\mathfrak{A}_{mn}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f\cdot\psi'\,dx\,dy=\sum_{\mu,\,\nu=0}^{\infty}(aA'+bB'+cC'+dD')_{\mu\nu}/\Lambda_{\mu\nu}.$$

If we apply the same line of reasoning to

$$\psi''=\phi\sin mx\cos ny,$$

we find

$$\mathfrak{B}_{mn}=\sum_{\mu,\,\nu=0}^{\infty}(aA''+bB''+cC''+dD'')_{\mu\nu}/\Lambda_{\mu\nu},$$

where

$$A''_{\mu\nu}=\tfrac{1}{4}(-\beta_{\mu-m,\,\nu-n}-\beta_{\mu-m,\,\nu+n}+\beta_{\mu+m,\,\nu-n}+\beta_{\mu+m,\,\nu+n}),$$

$$B''_{\mu\nu}=\tfrac{1}{4}(\alpha_{\mu-m,\,\nu-n}+\alpha_{\mu-m,\,\nu+n}-\alpha_{\mu+m,\,\nu-n}-\alpha_{\mu+m,\,\nu+n}),$$

$$C''_{\mu\nu}=\tfrac{1}{4}(-\delta_{\mu-m,\,\nu-n}-\delta_{\mu-m,\,\nu+n}+\delta_{\mu+m,\,\nu-n}+\delta_{\mu+m,\,\nu+n}),$$

$$D''_{\mu\nu}=\tfrac{1}{4}(\gamma_{\mu-m,\,\nu-n}+\gamma_{\mu-m,\,\nu+n}-\gamma_{\mu+m,\,\nu-n}-\gamma_{\mu+m,\,\nu+n}).$$

Similarly,

$$\mathfrak{C}_{mn}=\sum_{\mu,\,\nu=0}^{\infty}(aA'''+bB'''+cC'''+dD''')_{\mu\nu}/\Lambda_{\mu\nu}$$

with

$$A'''_{\mu\nu}=\tfrac{1}{4}(-\gamma_{\mu-m,\,\nu-n}+\gamma_{\mu-m,\,\nu+n}-\gamma_{\mu+m,\,\nu-n}+\gamma_{\mu+m,\,\nu+n}),$$

$$B'''_{\mu\nu}=\tfrac{1}{4}(-\delta_{\mu-m,\,\nu-n}+\delta_{\mu-m,\,\nu+n}-\delta_{\mu+m,\,\nu-n}+\delta_{\mu+m,\,\nu+n}),$$

$$C'''_{\mu\nu}=\tfrac{1}{4}(\alpha_{\mu-m,\,\nu-n}-\alpha_{\mu-m,\,\nu+n}+\alpha_{\mu+m,\,\nu-n}-\alpha_{\mu+m,\,\nu+n}),$$

$$D'''_{\mu\nu}=\tfrac{1}{4}(\beta_{\mu-m,\,\nu-n}-\beta_{\mu-m,\,\nu+n}+\beta_{\mu+m,\,\nu-n}-\beta_{\mu+m,\,\nu+n}),$$

and

$$\mathfrak{D}_{mn}=\sum_{\mu,\,\nu=0}^{\infty}(aA''''+bB''''+cC''''+dD'''')_{\mu\nu}/\Lambda_{\mu\nu}$$

with

$$\begin{aligned}
A''''_{\mu\nu} &= \tfrac{1}{4}(\delta_{\mu-m,\,\nu-n}-\delta_{\mu-m,\,\nu+n}-\delta_{\mu+m,\,\nu-n}+\delta_{\mu+m,\,\nu+n}),\\
B''''_{\mu\nu} &= \tfrac{1}{4}(-\gamma_{\mu-m,\,\nu-n}+\gamma_{\mu-m,\,\nu+n}+\gamma_{\mu+m,\,\nu-n}-\gamma_{\mu+m,\,\mu+n}),\\
C''''_{\mu\nu} &= \tfrac{1}{4}(-\beta_{\mu-m,\,\nu-n}+\beta_{\mu-m,\,\nu+n}+\beta_{\mu+m,\,\nu-n}-\beta_{\mu+m,\,\nu+n}),\\
D''''_{\mu\nu} &= \tfrac{1}{4}(\alpha_{\mu-m,\,\nu-n}-\alpha_{\mu-m,\,\nu+n}-\alpha_{\mu+m.\,\nu-n}+\alpha_{\mu+m,\,\nu+n}).
\end{aligned}$$

Before using these formulas Fourier's constants with negative subscripts must be transformed into such with positive ones by considering that

$$\begin{aligned}
(\alpha, \beta, \gamma, \delta)_{-\kappa,\,\lambda} &= (\alpha, -\beta, \gamma, -\delta)_{\kappa\lambda},\\
(\alpha, \beta, \gamma, \delta)_{-\kappa,\,-\lambda} &= (\alpha, -\beta, -\gamma, \delta)_{\kappa\lambda},\\
(\alpha, \beta, \gamma, \delta)_{\kappa,\,-\lambda} &= (\alpha, \beta, -\gamma, -\delta)_{\kappa\lambda}.
\end{aligned}$$

Equations Involving the Partial Derivatives of a Function of a Surface.[*]

By Charles A. Fischer.

Introduction.

In a former paper[†] I have defined the derivative of a function of a surface, and proved that if a function $L(S)$ has a derivative $L'(S;x,y)$, which is continuous and approached uniformly, the first variation of $L(S)$ can be given by the equation

$$\frac{dL(S)}{d\lambda}=\int\int L'(S;x,y)\frac{dz(x,y)}{d\lambda}\,dxdy. \tag{1}$$

Conversely, it can be proved that if there is a function $L'(S;x,y)$, continuous in all its arguments, which satisfies this equation for every family of surfaces in a given neighborhood, then $L'(S;x,y)$ is the derivative of $L(S)$. In the present paper, functions depending not only on a surface, but also on the values taken by a function at every point of the surface, are considered. Such a function has two partial functional derivatives.

In the first section these derivatives are defined. In Section 2 the adjoint of a functional is discussed, and the conditions that functionals of a certain type be self-adjoint are found. In Section 3 are found the conditions that two given functionals be the partial derivatives of a function of a surface. The fourth section contains the condition of integrability for an equation involving these derivatives, and the equation is found which is satisfied by the function

$$\Phi=\int\int\int(f_x^2+f_y^2+f_z^2)\,dxdydz,$$

where $f(x,y,z)$ is a potential function. In the last section the characteristics of such an equation are briefly discussed. Similar work for functions of lines has been done by Levy.[‡]

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[*] Read before the American Mathematical Society, Feb. 26, 1916.

[†] American Journal of Mathematics, Vol. XXXVI (1914), No. 3, p. 289.

[‡] "Sur l'integration des equations aux derivees fonctionelle partielles," *Rendiconti del Circolo Matematico di Palermo*, Vol. 37 (1914), p. 113.

§ 1. *The Variation of a Function of a Surface.*

The equations of surfaces discussed in this paper will be given in parametric form; and, consequently, the definition of the derivative of a function of a surface must be modified slightly. The equations of the surface S will be

$$S: \quad x=x(u, v), \quad y=y(u, v), \quad z=z(u, v),$$

defined over a region Ω in the u, v-plane, and those of the varied surface,

$$S_\epsilon: \quad x=x(u, v)+Xn(u, v), \quad y=y(u, v)+Yn(u, v), \quad z=z(u, v)+Zn(u, v),$$

where X, Y and Z are the direction cosines of the normal to S. The function $n(u, v)$ is assumed to be of class $C^{(r)}$, to have a permanent sign, to vanish everywhere excepting in the region $(u_0-\epsilon<u<u_0+\epsilon;\ v_0-\epsilon<v<v_0+\epsilon)$, and the absolute values of its partial derivatives including those of order r are assumed to be less than ϵ. Then the derivative said to be approached with order r, will be defined as

$$L'(S; u, v)=\lim_{\epsilon=0} \frac{L(S_\epsilon)-L(S)}{\sigma},$$

where

$$\sigma=\int_{u_0-\epsilon}^{u_0+\epsilon}\int_{v_0-\epsilon}^{v_0+\epsilon} n(u, v)\,H(u, v)\,dudv, \tag{2}$$

$Hdudv$ being the element of area. That is, $H=\sqrt{EG-F^2}$, where $E=\Sigma x_u^2$, $F=\Sigma x_u x_v$ and $G=\Sigma x_v^2$, the summation sign signifying that the expression is symmetrical in x, y and z. It will always be assumed that $H\neq 0$. If the function $\omega(x, y, \alpha)$ in my paper already referred to,* is replaced by the new function $n(u, v, \lambda)$, equation (9) in that paper can be replaced by the equation,

$$\left.\frac{dL(S_\lambda)}{d\lambda}\right|_{\lambda=0}=\int\int_\Omega L'(S; u, v)\,n_\lambda(u, v, 0)\,Hdudv. \tag{3}$$

In order to make the variation normal to all of the surfaces S_λ, instead of to S_0 alone, these surfaces can be considered as defined by the partial differential equations

$$S_\lambda: \quad x_\lambda=Xn_\lambda, \quad y_\lambda=Yn_\lambda, \quad z_\lambda=Zn_\lambda, \tag{4}$$

with the initial conditions $x(u, v, 0)=x(u, v)$, $y(u, v, 0)=y(u, v)$, $z(u, v, 0)=z(u, v)$. The existence theorems for partial differential equations prove that $x(u, v, \lambda)$, $y(u, v, \lambda)$ and $z(u, v, \lambda)$ are determined uniquely, at least if

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* Fischer, *loc. cit.*, p. 291.

$n(u, v, \lambda)$; $x(u, v)$, $y(u, v)$ and $z(u, v)$ are analytic, and cases where they are not so determined will not be considered here. This change in the equations of S_λ does not affect equation (3).

A function $\Phi(n, f)$ will now be considered which depends on the surface designated by the argument n, and on all of the values taken by another function $f(u, v)$ at points of the surface. Such a function may have two partial functional derivatives,

$$\Phi'_n(0, f; u, v) = \lim_{\epsilon=0} \frac{\Phi(n, f) - \Phi(0, f)}{\sigma}, \tag{5}$$

and

$$\Phi'_f(n, f_0; u, v) = \lim_{\epsilon=0} \frac{\Phi(n, f) - \Phi(n, f_0)}{\sigma'}, \tag{6}$$

where σ is defined by equation (2), and

$$\sigma' = \int_{u_0-\epsilon}^{u_0+\epsilon} \int_{v_0-\epsilon}^{v_0+\epsilon} (f(u, v) - f_0(u, v)) H du dv.$$

It is assumed, of course, that $f - f_0$ has the properties already assumed for n. If these partial derivatives are continuous in all arguments and approached uniformly, the equation

$$\frac{d\Phi(n, f)}{d\lambda} = \iint_\Omega [\Phi'_n(n, f; u, v) n_\lambda + \Phi'_f(n, f; u, v) f_\lambda] H du dv, \tag{7}$$

which is analogous to equation (3), must be satisfied, if $n(u, v, \lambda)$ and $f(u, v, \lambda)$ and all of their partial derivatives considered are continuous everywhere, and independent of λ along the boundary of Ω, and if $n_\lambda(u, v, \lambda)$ and $f_\lambda(u, v, \lambda)$ are approached uniformly. It was also assumed to simplify the proof that the function which corresponds to $n(u, v, \lambda') - n(u, v, \lambda)$ should have a permanent sign,* but this is not essential.

§ 2. *The Adjoint of a Functional.*

The adjoint of a functional of a line has been defined by Levy,† and it was found useful in deriving the condition of integrability for equations involving functional derivatives. The adjoint of a functional of a surface will be defined similarly.

If there are two functionals $L(f)$ and $\overline{L}(f)$ such that the equation

$$\iint_\Omega g(u, v) L(f) H du dv = \iint_\Omega f(u, v) \overline{L}(g) H du dv \tag{8}$$

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* Fischer, *loc, cit.*, p. 291.

† Levy, *loc. cit.*, p. 115.

is satisfied for every pair of functions f and g of class $C^{(r)}$ which vanish together with their partial derivatives along the boundary of Ω, then $\overline{L}$ is said to be the adjoint of L, and vice versa. The proof that a functional can not have two distinct adjoints is essentially the same as for functionals of lines,[*] and will not be repeated. If a functional has an adjoint it must be linear, since if $\overline{L}$ is the adjoint of L the equations

$$\begin{aligned}\int\int_\Omega gL(af_1+bf_2)H\,du\,dv &= a\int\int_\Omega f_1\overline{L}(g)H\,du\,dv+b\int\int_\Omega f_2\overline{L}(g)H\,du\,dv\\ &= \int\int g[aL(f_1)+bL(f_2)]H\,du\,dv;\end{aligned}$$

must be satisfied for every function $g(u, v)$. It follows that

$$L(af_1+bf_2)=aL(f_1)+bL(f_2),$$

which is the condition that L be linear. Every linear functional can be expressed in the form

$$\lim_{\epsilon=0}\int\int_\Omega F\ (u, v, u_1, v_1, \epsilon)f(u_1, v_1)H(u_1, v_1)\,du_1dv_1.\dagger$$

Its adjoint will be

$$\lim_{\epsilon=0}\int\int_\Omega F(u_1, v_1, u, v, \epsilon)f(u_1, v_1)H(u_1, v_1)\,du_1dv_1,$$

provided this limit exists and is approached uniformly. However, there are linear functionals which have no adjoints.

A large class of functionals are expressible in the form

$$L(f)=\int\int_\Omega F(u, v, u_1, v_1)f(u_1, v_1)H(u_1, v_1)\,du_1dv_1 + \sum_{i,j=0}^{i+j=m} A_{ij}(u, v)\,\frac{\partial^{i+j}f(u, v)}{\partial u^i\partial v^j}. \qquad (9)$$

The adjoint is

$$\overline{L}(f)=\int\int_\Omega F(u_1, v_1, u, v)f(u_1, v_1)H(u_1, v_1)\,du_1dv_1 + \frac{1}{H}\sum_{i,j=0}^{i+j=m}(-1)^{i+j}\,\frac{\partial^{i+j}A_{ij}fH}{\partial u^i\partial v^j}. \qquad (10)$$

This can be verified by substituting in equation (8), and applying Green's theorem repeatedly. If the function (9) is self-adjoint, the equation

$$F(u, v, u_1, v_1)=F(u_1, v_1, u, v)$$

will be satisfied, and also the equations obtained by equating the coefficients of $\partial^{k+l}/\partial u^k\partial v^l$ in the equation

$$\sum_{i,j=0}^{i+j=m} A_{ij}\,\frac{\partial^{i+j}f}{\partial u^i\partial v^j}=\frac{1}{H}\sum_{i,j=0}^{i+j=m}(-1)^{i+j}\,\frac{\partial^{i+j}A_{ij}Hf}{\partial u^i\partial v^j}.$$

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[*] Levy, *loc. cit.*, p. 116.

[†] Hadamard, "Leçons sur le calcul des variations," p. 303.

If $m=2$ these equations are seen to be

$$\left.\begin{aligned}
A_{20}&= \quad A_{20}, \quad A_{11}=A_{11}, \quad A_{02}=A_{02},\\
A_{10}&=-A_{10}+\frac{1}{H}\left(2\frac{\partial A_{20}H}{\partial u}+\frac{\partial A_{11}H}{\partial v}\right),\\
A_{01}&=-A_{01}+\frac{1}{H}\left(\frac{\partial A_{11}H}{\partial u}+2\frac{\partial A_{02}H}{\partial v}\right),\\
A_{00}&= \quad A_{00}+\frac{1}{H}\left(\frac{\partial^2 A_{20}H}{\partial u^2}+\frac{\partial^2 A_{11}H}{\partial u\partial v}+\frac{\partial^2 A_{02}H}{\partial v^2}-\frac{\partial A_{10}H}{\partial u}-\frac{\partial A_{01}H}{\partial v}\right).
\end{aligned}\right\} \tag{11}$$

If the values of A_{10} and A_{01} are substituted in the last of these equations it becomes an identity. It follows that if A_{20}, A_{11}, A_{02} and A_{00} are taken arbitrarily, A_{10} and A_{01} can be determined so that the functional will be self-adjoint. It is evident from the equations similar to (11) for higher values of m, that m must be even, and that the functions A_{ij} with $i+j$ odd are determined by the others. It will be proved a little later that the functions A_{ij} with $i+j$ even can always be taken arbitrarily. The following theorems are easily verified.*

If $L(f)$ and $\overline{L}(f)$ are adjoint, then $h(u, v)L(k(uv)f)$ is adjoint to $k(u, v)\overline{L}(h(u, v)f)$, where $h(u, v)$ and $k(u, v)$ are arbitrary. It follows that if $L(f)$ is self-adjoint, $h(u, v)L(h(u, v)f)$ is also.

If $L(f)$ and $M(f)$ have the adjoints $\overline{L}(f)$ and $\overline{M}(f)$, respectively, then $L(M(f))$ is the adjoint of $\overline{M}(\overline{L}(f))$. Similarly, if $L(f)$ is self-adjoint, $\overline{M}(L(M(f)))$ is also.

From the last statement if $L(f)=\phi(u, v)f(u, v)$, where $\phi(u, v)$ is arbitrary, and $M(f)=\partial^{i+j}f(u, v)/\partial u^i\partial v^j$, then the functional

$$\overline{M}(L(M(f)))=\frac{(-1)^{i+j}}{H}\frac{\partial^{i+j}}{\partial u^i\partial v^j}\left(H\phi\frac{\partial^{i+j}f}{\partial u^i\partial v^j}\right) \tag{12}$$

must be self-adjoint. This may be expressed as

$$\sum_{k=i}^{2i}\sum_{l=j}^{2j}A_{kl}\frac{\partial^{k+l}f}{\partial u^k\partial v^l},$$

where $A_{2i2j}=(-1)^{i+j}\phi$. The function

$$\phi\frac{\partial^2 f}{\partial u\partial v}+\frac{1}{2H}\left(\frac{\partial\phi H}{\partial v}\cdot\frac{\partial f}{\partial u}+\frac{\partial\phi H}{\partial u}\cdot\frac{\partial f}{\partial v}\right)$$

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* Compare with Levy, *loc. cit.*, p. 116.

is self-adjoint and can be used instead of $\phi(u, v)f$. It follows that

$$\bar{M}(L(M(f))) = \frac{(-1)^{i+j}}{H}\frac{\partial^{i+j}}{\partial u^i\partial v^j}\left(H\phi\frac{\partial^{i+j+2}f}{\partial u^{i+1}\partial v^{j+1}} + \frac{1}{2}\frac{\partial\phi H}{\partial v}\cdot\frac{\partial^{i+j+1}f}{\partial u^{i+1}\partial v^{j}} + \frac{1}{2}\frac{\partial\phi H}{\partial u}\cdot\frac{\partial^{i+j+1}f}{\partial u^{i}\partial v^{j+1}}\right) \quad (13)$$

is also self-adjoint. This is equal to

$$\sum_{k=i}^{2i+1}\sum_{l=j}^{2j+1} A_{kl}\frac{\partial^{k+l}f}{\partial u^k\partial v^l},$$

where $A_{2i+1\,2j+1} = (-1)^{i+j}\phi$, which is arbitrary. A self-adjoint functional of the type

$$\sum_{i,j=0}^{i+j=m} A_{ij}\frac{\partial^{i+j}f}{\partial u^i\partial v^j} \quad (14)$$

can then be formed by adding those of types (12) and (13), in which the functions A_{ij} with $i+j$ even are arbitrary, as has been stated previously. Since the functions A_{ij} with $i+j$ odd are determined uniquely by the condition that the functional be self-adjoint, every self-adjoint functional of type (14) is expressible as the sum of a finite number of functionals of types (12) and (13).

§ 3. *The Conditions that Given Functionals Be the Derivatives of Functions of Surfaces.*

In finding the variation of a functional depending on the surface S_λ defined by equations (4), the function $H(u, v)$ must be considered a function of λ also. Its derivative will now be calculated. By definition,

$$\frac{\partial H}{\partial\lambda} = \frac{1}{2H}\left(G\frac{\partial E}{\partial\lambda} + E\frac{\partial G}{\partial\lambda} - 2F\frac{\partial F}{\partial\lambda}\right), \quad (15)$$

and

$$\frac{\partial E}{\partial\lambda} = 2\Sigma x_u x_{u\lambda} = 2\Sigma(x_u X_u n_\lambda + x_u X n_{u\lambda}) = -2Dn_\lambda,$$

since

$$\Sigma x_u X = 0, \text{ and } \Sigma x_u X_u = -D.\text{[*]}$$

If $\partial F/\partial\lambda$ and $\partial G/\partial\lambda$ are evaluated in the same way, equation (15) becomes

$$\frac{\partial H}{\partial\lambda} = -\frac{1}{H}(GD + ED'' - 2FD')n_\lambda = -K_m H n_\lambda,$$

where K_m is the mean curvature of the surface.[†]

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[*] The functions D, D' and D'' are called the fundamental coefficients of the second order. See Eisenhart, "Differential Geometry," p. 115.

[†] Eisenhart, *loc. cit.*, p. 123.

The condition that a given functional $\Psi_1(S;\, u, v)$ be the derivative of some function $\Psi(S)$ will now be derived by a method given by Volterra.[*] It will be assumed that Ψ_1 has a differential. That is to say that there is a linear functional $L(n_\lambda)$ such that $\partial\Psi_1/\partial\lambda = L(n_\lambda)$. A two parameter family of surfaces $S_{\lambda,\lambda'}$ will be determined by a function $n(u, v, \lambda, \lambda')$ having the properties assumed for $n(u, v, \lambda)$. Then a closed curve,

$$l: \qquad \lambda = \lambda(s), \quad \lambda' = \lambda'(s),$$

will be chosen bounding a small region ω in the λ, λ'-plane. It follows from Green's theorem that

$$\int_l ds \iint_\Omega \Psi_1(S_s;\, u, v) H \left(n_\lambda \frac{d\lambda}{ds} + n_{\lambda'} \frac{d\lambda'}{ds} \right) dudv$$
$$= \iint_\omega d\lambda d\lambda' \iint_\Omega \left\{ \frac{\partial}{\partial\lambda} (\Psi_1 H n_{\lambda'}) - \frac{\partial}{\partial\lambda'} (\Psi_1 H n_\lambda) \right\} dudv.$$

This is equivalent to the equation

$$\int_l ds \iint_\Omega \Psi_1(S_s;\, u, v) \frac{dn}{ds} H dudv = \iint_\omega d\lambda d\lambda' \iint (L(n_\lambda) n_{\lambda'} - L(n_{\lambda'}) n_\lambda) H dudv. \tag{16}$$

If $\Psi_1(S;\, u, v)$ is the derivative of some function $\Psi(S)$, the left member of this equation is equal to

$$\int_l \frac{d\Psi}{ds} ds,$$

which vanishes for every choice of n_λ and $n_{\lambda'}$. Consequently, the right member must vanish, and $L(n_\lambda)$ is self-adjoint. Conversely, if it is self-adjoint the right member will vanish. The value of Ψ can then be taken arbitrarily for one surface S_0, and for S_λ

$$\Psi(S_\lambda) = \Psi(S_0) + \int_0^\lambda d\lambda \iint_\Omega \Psi_1(S_\lambda;\, u, v) n_\lambda H dudv.$$

There are analogous conditions which must be satisfied if two given functionals $\Phi_n(n, f;\, u, v)$ and $\Phi_f(n, f;\, u, v)$ are to be equal to the partial derivatives of some function $\Phi(n, f)$. These functionals will be assumed to have differentials which satisfy the equations

$$\frac{\partial\Phi_n}{\partial\lambda} = \Phi_{nn}(n_\lambda) + \Phi_{nf}(f_\lambda), \quad \frac{\partial\Phi_f}{\partial\lambda} = \Phi_{fn}(n_\lambda) + \Phi_{ff}(f_\lambda). \tag{17}$$

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[*] Volterra, "Fonctions de lignes," p. 45.

Equation (16) must then be replaced by the analogous equation

$$\begin{aligned}\int_l ds \iint_\Omega \left(\Phi_n \frac{dn}{ds} + \Phi_f \frac{df}{ds}\right) H\,du\,dv\\ =\iint_\omega d\lambda\, d\lambda' \iint_\Omega \{ (\Phi_{nn}(n_\lambda) + \Phi_{nf}(f_\lambda))\, n_{\lambda'} + (\Phi_{fn}(n_\lambda) + \Phi_{ff}(f_\lambda))\, f_{\lambda'}\\ -(\Phi_{nn}(n_{\lambda'}) + \Phi_{nf}(f_{\lambda'}))\, n_\lambda - (\Phi_{fn}(n_{\lambda'}) + \Phi_{ff}(f_{\lambda'}))\, f_\lambda + \Phi_f (f_{\lambda'\lambda} - f_{\lambda\lambda'})\\ -K_m \Phi_f (f_{\lambda'} n_\lambda - f_\lambda n_{\lambda'}) \} H\,du\,dv.\end{aligned} \tag{18}$$

The derivatives $f_{\lambda'\lambda}$ and $f_{\lambda\lambda'}$ will not in general be equal. Their difference will be

$$f_{\lambda'\lambda} - f_{\lambda\lambda'} = \Sigma (f_u u_x + f_v v_x + f_n n_x)\,(x_{\lambda'\lambda} - x_{\lambda\lambda'}). \tag{19}$$

If the equations

$$u_x x_u + u_y y_u + u_z z_u = 1, \quad u_x x_v + u_y y_v + u_z z_v = 0, \quad u_x X + u_y Y + u_z Z = 0,$$

are solved for u_x, u_y and u_z, the equations

$$\Sigma u_x^2 = \frac{1}{H^2} \Sigma [(1-X^2) x_v^2 - 2YZ y_v z_v] = \frac{1}{H^2} [G - (\Sigma X x_v)^2] = \frac{G}{H^2} \tag{20}$$

are easily derived. Similarly, it can be proved that

$$\Sigma u_x v_x = \frac{-F}{H^2}, \quad \Sigma v_x^2 = \frac{E}{H^2}, \quad n_x = X, \quad \Sigma u_x n_x = \Sigma v_x n_x = 0. \tag{21}$$

To evaluate $x_{\lambda'\lambda} - x_{\lambda\lambda'}$ equations (4) must be differentiated with respect to λ', and subtracted from the same equations with λ and λ' interchanged. This gives the equations

$$\begin{aligned}x_{\lambda'\lambda} - x_{\lambda\lambda'} &= \frac{1}{H} \begin{vmatrix} Y & y_v \\ Z & z_v \end{vmatrix} \left(\frac{\partial n_\lambda}{\partial u} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial u} n_\lambda\right) + \frac{1}{H} \begin{vmatrix} y_u & Y \\ z_u & Z \end{vmatrix} \left(\frac{\partial n_\lambda}{\partial v} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial v} n_\lambda\right)\\ &= -u_x \left(\frac{\partial n_\lambda}{\partial u} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial u} n_\lambda\right) - v_x \left(\frac{\partial n_\lambda}{\partial v} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial v} n_\lambda\right).\end{aligned}$$

If these values are substituted in equation (19) it can be reduced to the form

$$f_{\lambda'\lambda} - f_{\lambda\lambda'} = \frac{1}{H^2} \left\{ (-f_u G + f_v F) \left(\frac{\partial n_\lambda}{\partial u} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial u} n_\lambda\right) + (f_u F - f_v E) \left(\frac{\partial n_\lambda}{\partial v} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial v} n_\lambda\right) \right\}.$$

If this expression is substituted in equation (18) it becomes

$$\begin{aligned}\int_l ds \iint_\Omega \left(\Phi_n \frac{dn}{ds} + \Phi_f \frac{df}{ds}\right) H\,du\,dv = \iint_\omega d\lambda\, d\lambda' \iint_\Omega \Bigg\{ \Bigg[\Phi_{nn}(n_\lambda)\\ + \frac{\Phi_f}{H^2} \left((-f_u G + f_v F) \frac{\partial n_\lambda}{\partial u} + (f_u F - f_v E) \frac{\partial n_\lambda}{\partial v} \right) \Bigg] n_{\lambda'}\\ - \left[\Phi_{nn}(n_{\lambda'}) + \frac{\Phi_f}{H^2} \left((-f_u G + f_v F) \frac{\partial n_{\lambda'}}{\partial u} + (f_u F - f_v E) \frac{\partial n_{\lambda'}}{\partial v} \right) \right] n_\lambda\\ + [\Phi_{nf}(f_\lambda) + K_m \Phi_f f_\lambda]\, n_{\lambda'} - \Phi_{fn}(n_{\lambda'}) f_\lambda + \Phi_{fn}(n_\lambda) f_{\lambda'}\\ - [\Phi_{nf}(f_{\lambda'}) + K_m \Phi_f f_{\lambda'}]\, n_\lambda + \Phi_{ff}(f_\lambda) f_{\lambda'} - \Phi_{ff}(f_{\lambda'}) f_\lambda \Bigg\} H\,du\,dv.\end{aligned}$$

If there is a function $\Phi(n, f)$ which has the partial derivatives $\Phi'_n=\Phi_n$ and $\Phi'_f=\Phi_f$ the left member of the last equation, and consequently the right member, must vanish for every choice of the functions n_λ, $n_{\lambda'}$, f_λ and $f_{\lambda'}$. The necessary and sufficient conditions for this are that the functionals

$$\Phi_{nn}(g)+\frac{\Phi_f}{H^2}\left((-f_uG+f_vF)\frac{\partial g}{\partial u}+(f_uF-f_vE)\frac{\partial g}{\partial v}\right)$$

and $\Phi_{ff}(g)$ be self-adjoint, and that $\Phi_{nf}(g)+K_m\Phi_f g$ be the adjoint of $\Phi_{fn}(g)$. The similar conditions for functions of lines are given by Levy.*

§ 4. *Equations Involving Partial Functional Derivatives.*

The condition of integrability of an equation such as

$$\Phi'_n(n, f; u, v)=W(n, f, \Phi'_f; \Phi, u, v) \tag{22}$$

will now be found.† If there is an integral $\Phi(n, f)$ which is equal to an arbitrarily given function $\Psi(f)$ when $n=0$, the equation is said to be completely integrable. It will be assumed that the derivative $\Psi'(f; u, v)$ is continuous and approached uniformly, and that Ψ' and W have differentials, implying the equations

$$\frac{d\Psi'}{d\lambda}=\Psi_{ff}(f_\lambda),\quad \frac{dW}{d\lambda}=Q\left(\frac{d\Phi'_f}{d\lambda}\right)+M(n_\lambda)+L(f_\lambda), \tag{23}$$

for every choice of the functions $n(u, v, \lambda)$ and $f(u, v, \lambda)$ such that f_λ, n_λ and $\partial\Phi'_f/\partial\lambda$ are of class $C^{(r)}$ in u and v.

If equation (22) is differentiated with respect to λ, and values substituted from equations (17) and (23), it becomes

$$\Phi_{nn}(n_\lambda)+\Phi_{nf}(f_\lambda)=Q(\Phi_{fn}(n_\lambda))+Q(\Phi_{ff}(f_\lambda))+M(n_\lambda)+L(f_\lambda).$$

It follows that for any function $g(u, v)$,

$$\Phi_{nn}(g)=Q(\Phi_{fn}(g))+M(g),$$

and

$$\Phi_{nf}(g)=Q(\Phi_{ff}(g))+L(g).$$

On the surface defined by $n=0$, the functional Φ_{ff} is equal to Ψ_{ff} which must be self-adjoint. The functional $\Phi_{fn}(g)$ is the adjoint of $\Phi_{nf}(g)+K_m\Phi_f g$, and, consequently,

$$\Phi_{fn}(g)=\Phi_{ff}(\bar{Q}(g))+\bar{L}(g)+K_m\Phi'_f g. \tag{24}$$

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* Levy, *loc. cit.*, p. 120.

† Compare with Levy, *loc. cit.*, pp. 122–123.

The other functional which was proved to be self-adjoint in the last section is now equal to

$$Q(\Phi_{ff}(\bar{Q}(g)))+Q(\bar{L}(g))+Q(K_m\Phi'_f g)+M(g)$$
$$+\frac{\Phi'_f}{H^2}\left((-f_uG+f_vF)\frac{\partial g}{\partial u}+(f_uF-f_vE)\frac{\partial g}{\partial v}\right).$$

The first term is self-adjoint as was proved in § 2. A necessary condition that equation (22) be completely integrable is, therefore, that the functional

$$Q(\bar{L}(g))+Q(K_m\Phi'_f g)+M(g)+\frac{\Phi'_f}{H^2}\left((-f_uG+f_vF)\frac{\partial g}{\partial u}+(f_uF-f_vE)\frac{\partial g}{\partial v}\right) \tag{25}$$

be self-adjoint.

To illustrate the theory just developed, the equation will be found which must be satisfied by the function

$$\Phi=\iiint_R(f_x^2+f_y^2+f_z^2)\,dxdydz, \tag{26}$$

where $f(x, y, z)$ is a solution of the equation

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=0. \tag{27}$$

The value of f inside the region R is then determined by the values it takes on the surface S which bounds R. If S is considered as fixed, while the boundary values of $f(x, y, z)$ are varied, the equation

$$\frac{d\Phi}{d\lambda}=2\iiint_R\Sigma f_x f_{x\lambda}\,dxdydz$$

will be satisfied. This can be reduced by means of Green's formula and equation (27) to the form

$$\frac{d\Phi}{d\lambda}=2\iint_S f_n f_\lambda H\,dudv.$$

It follows that

$$\Phi'_f(n, f; u, v)=2f_n(x(u, v), y(u, v), z(u, v)). \tag{28}$$

If the value of $f(x, y, z)$ is fixed at each point of R, while S is varied, the boundary values of $f(x, y, z)$ will vary according to the law $f_\lambda=f_n n_\lambda$. In this case it follows from equations (7) and (26) that

$$\iint(\Phi'_n+f_n\Phi'_f)\,n_\lambda H\,dudv=\iint(f_x^2+f_x^2+f_z^2)\,n_\lambda H\,dudv,$$

and, consequently,

$$\Phi'_n+f_n\Phi'_f=\Sigma f_x^2.$$

If f_x is replaced by $f_u u_x+f_v v_x+f_n n_x$ and equations (20) and (21) applied, the right member of this equation becomes

$$\frac{1}{H^2}(f_u^2G-2f_uf_vF+f_v^2E)+f_n^2.$$

Substituting the value of f_n from equation (28),

$$\Phi'_n = -\tfrac{1}{4}\Phi_f'^2 + \frac{1}{H^2}(f_u^2 G - 2f_u f_v F + f_v^2 E).$$

This is the desired equation.

It can easily be proved that the condition of integrability is satisfied. The functionals Q, M and L are, evidently,

$$Q(g) = -\tfrac{1}{2}\Phi'_f g, \quad M(g) = 0,$$

$$L(g) = \frac{2}{H^2}\left[(f_u G - f_v F)\frac{\partial g}{\partial u} + (-f_u F + f_v E)\frac{\partial g}{\partial v}\right].$$

If these values are substituted in the expression (25) the terms involving $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial v}$ cancel and the other terms are self-adjoint.

§ 5. *Characteristics.*

There is a set of functionals which have the same relation to a solution of equation (22) that the characteristics have to a solution of a partial differential equation. The characteristics of equations involving partial derivatives of functions of lines have been discussed by Levy.* There are characteristics of various orders, but the most interesting ones are the "caractéristiques de première espèce," and these are the only ones which will be considered here. After they are defined it will be proved that if an integral contains an element of a characteristic, it contains the whole characteristic. An element and an integral will be defined first.

An element is any set of functions $n(u, v)$, $f(u, v)$, $\Phi(n, f)$, $\Phi'_n(n, f; u, v)$ and $\Phi'_f(n, f; u, v)$ which satisfy equation (22).

An integral is a function $\Phi(n, f)$ whose partial derivatives satisfy equation (22) for every pair of admissible functions $n(u, v)$ and $f(u, v)$.

A characteristic is a set of functions $g(u, v)$, $\Psi(n)$, $\Psi'_n(n; u, v)$ and $\Psi'_f(n; u, v)$ which satisfy the equations

$$\frac{dg}{d\lambda} = -\bar{Q}(n_\lambda),$$

$$\frac{d\Psi}{d\lambda} = \iint_\Omega (\Psi'_n - Q(\Psi'_f))\, n_\lambda H\, du\, dv, \tag{29}$$

$$\frac{d\Psi'_n}{d\lambda} = Q(\bar{L}(n_\lambda)) + Q(K_m \Psi'_f n_\lambda) - L(\bar{Q}(n_\lambda)) + M(n_\lambda), \tag{30}$$

$$\frac{d\Psi'_f}{d\lambda} = \bar{L}(n_\lambda) + K_m \Psi'_f n_\lambda, \tag{31}$$

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* Levy, *loc. cit.*, Ch. II.

where the function Ψ is used instead of Φ in deriving Q, L and M. If $\Phi(n, f)$ is an integral the equations

$$\frac{d\Phi}{d\lambda} = \int\int_{\Omega} (\Phi'_n n_\lambda + \Phi'_f f_\lambda) H\,du\,dv, \tag{32}$$

$$\frac{d\Phi'_n}{d\lambda} = Q\left(\frac{d\Phi'_f}{d\lambda}\right) + L(f_\lambda) + M(n_\lambda), \tag{33}$$

and

$$\frac{d\Phi'_f}{d\lambda} = \Phi_{ff}(f_\lambda) + \Phi_{ff}(\bar{Q}(n_\lambda)) + \bar{L}(n_\lambda) + K_m \Phi'_f n_\lambda, \tag{34}$$

will be satisfied, on account of equations (7), (17), (22), (23) and (24). If the value given for g_λ is substituted for f_λ in equations (32), (33) and (34), they become equivalent to (29), (30) and (31). It follows that if the integral $\Phi(n, g)$ and its derivatives are equal to $\Psi(n)$, $\Psi'_n(n; u, v)$ and $\Psi'_f(n; u, v)$, respectively, for $\lambda = \lambda_0$, they are equal for all values of λ for which the equations of the characteristics have unique solutions.

COLUMBIA UNIVERSITY.

Invariants and Covariants of the Cremona Cubic Surface.

By C. P. Sousley.

Introduction.

If $a, b, \ldots, f$ are properly chosen cubic curves on P_6^2, *i. e.*, on six points in a plane, the cubic surface mapped on the plane by these curves may be given by the equation

$$a^3+b^3+\ldots+f^3=0,$$

where the variables are subject to the relations

$$a+b+\ldots+f=0,$$
$$\bar{a}a+\bar{b}b+\ldots+\bar{f}f=0.$$

This form is known as the hexahedral cubic surface of Cremona.*

As to the invariants and covariants of the Sylvester pentahedral form of the surface much is known and given in explicit form in Salmon's geometry of three dimensions. Nothing, however, is known as to the invariants and covariants of this Cremona form.

The purpose of this paper is to obtain some of these invariants and covariants and to outline the further steps necessary for the determination of the invariants and linear covariants.

The results are important in determining the lines on a general cubic surface. In particular for the Cremona form, the equations of the lines are known. If, then, one can find a typical representation for the general cubic surface in terms of the Cremona form and can determine the irrational invariants for the Cremona form, then the required lines of the general surface can be determined from the known lines of the Cremona cubic surface.

This requires the calculation of the invariants and linear covariants for the Cremona form and their identification with the corresponding invariants and covariants of the general cubic surface.

The results obtained are valuable also for any study that involves the behavior of the lines of a cubic surface with reference to the covariants of the surface.

Section 1.

Given in S_3 the cubic surface $(\alpha x)^3\equiv(\beta x)^3\equiv\ldots$ a known comitant $|\alpha\beta\gamma\xi|\ldots(\delta x)\ldots$ becomes $|\alpha\beta\gamma\eta\zeta\xi|\ldots(\delta x)$, when the surface is taken in S_5, and the variables are subject to the conditions $\overset{6}{\Sigma}\eta_1 x_1=0$, $\overset{6}{\Sigma}\zeta_1 x_1=0$.

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* *Mathematische Annalen,* Vol. XIII.

This process is known as the transference principle of Clebsch. The straightforward method for obtaining the corresponding comitant in S_8 would be to eliminate two variables, thus getting the surface in S_3, and for this form to calculate the comitant, but this would give an unsymmetrical result; whereas, by the Clebsch transference principle we obtain a symmetrical result.

The process of contravariant differentiation is not affected by going up to S_5 from S_3 in this way. This fact is noted immediately on performing corresponding operations in both dimensions.

Let us examine the degree of the invariants and covariants of the Cremona form. A covariant of a cubic surface of degree d, weight w, class c, order σ, has $3d$ symbols α; distributed in $w-c$ symbols $|\alpha\beta\gamma\delta|$, c symbols $|\alpha\beta\gamma\xi|$ and σ symbols (αx), whence $3d=4(w-c)+3c+\sigma$, or $3d=4w-c+\sigma$. These symbols become by Clebsch's principle of transference $|\alpha\beta\gamma\delta\eta\zeta|$, $|\alpha\beta\gamma\xi\eta\zeta|$, (αx), where for the Cremona cubic surface the coefficients α are now numerical and the coefficients η all are 1 and the coefficients ζ are $\bar{a}, \bar{b}, \ldots, \bar{f}$. Hence, if d' is the degree in $\bar{a}, \bar{b}, \ldots, \bar{f}$ of the covariant for the Cremona form, we have

$$d'=w=\tfrac{1}{4}(3d-\sigma+c). \tag{1}$$

Hence for this canonical form, the invariants have the degree $d'=6, 12, 18, 24, 30, 75$; the linear covariants have the degree $d'=8, 14, 20, 32$.

The invariants of the surface being invariants of P_6^2, must be expressible in terms of the rational and symmetric functions of the irrational system $\bar{a}, \ldots, \bar{f}$ of P_6^2. The linear covariants of the surface can be expressed in terms of the functions

$$K_i=\bar{a}^i a+\bar{b}^i b+\ldots+\bar{f}^i f, \qquad (i=2, \ldots, 5). \tag{2}$$

Let us now calculate for P_6^2, referred to a special coordinate system, the irrational system $\bar{a}, \ldots, \bar{f}$, and from these form the rational system $a_2, \ldots, a_6$,* which are the elementary symmetric functions of $\bar{a}, \ldots, \bar{f}$.

Section 2.

If we take P_6^2 in the prepared form

$$\left.\begin{array}{llll@{\qquad\qquad}llll} (1) & 1 & 0 & 0, & (4) & 1 & 1 & 1, \\ (2) & 0 & 1 & 0, & (5) & x & y & u, \\ (3) & 0 & 0 & 1, & (6) & z & t & u, \end{array}\right\} \tag{3}$$

we have (*loc. cit.*, p. 170)

$$\left.\begin{array}{ll} 3\bar{a}=\rho-3(ux+ut), & 3\bar{d}=\rho-3(uy+uz), \\ 3\bar{b}=\rho-3(ux+yz), & 3\bar{e}=\rho-3(uy+xt), \\ 3\bar{c}=\rho-3(ut+yz), & 3\bar{f}=\rho-3(uz+xt), \\ \multicolumn{2}{c}{\rho=u(x+y+z+t)+xt+yz.} \end{array}\right\} \tag{4}$$

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* Coble, "Point Sets and Allied Cremona Groups," *Trans. American Math. Soc.*, Vol. XVI (1915), p. 155; in particular § 5.

Let us introduce the following notation:

$$\left.\begin{aligned}
&\gamma_1=x+y+z+t,\\
&\gamma_2=(x+t)(y+z),\quad \bar{\gamma}_2=yz+xt,\\
&\gamma_3=xyz+yzt+xyt+xzt,\\
&\gamma_4=xyzt.
\end{aligned}\right\}\qquad(5)$$

For purposes of calculation it is more convenient to use the sums of the powers of $\bar{a},\ldots,\bar{f}$ for the rational invariants of P_6^2, rather than their symmetric functions, *i. e.*, if $p_2=\Sigma\bar{a}^2$, $p_3=\Sigma\bar{a}^3$, $\ldots$, $p_5=\Sigma\bar{a}^5$, and if we substitute the values of $\bar{a},\ldots,\bar{f}$, given by (4), we find

$$\left.\begin{aligned}
9p_2={}&6[(2\gamma_1^2-3\bar{\gamma}_2-6\gamma_2)u^2-(2\gamma_1\bar{\gamma}_2-3\gamma_3)u+2(\bar{\gamma}_2^2-3\gamma_4)],\\
27p_3={}&3[(9\gamma_1\bar{\gamma}_2+18\gamma_1\gamma_2-4\gamma_1^3-27\gamma_3)u^3+3(2\gamma_1^2\bar{\gamma}_2-6\bar{\gamma}_2^2\\
&-3\gamma_1\gamma_3-3\bar{\gamma}_2\gamma_2+36\gamma_4)u^2+3(2\gamma_1\bar{\gamma}_2^2-3\gamma_1\gamma_4-3\bar{\gamma}_2\gamma_3)u\\
&\quad+2\bar{\gamma}_2(9\gamma_4-2\bar{\gamma}_2^2)],\\
81p_4={}&18[(2\gamma_1^4-12\gamma_1^2\gamma_2-6\gamma_1^2\bar{\gamma}_2+18\gamma_2^2+9\bar{\gamma}_2^2+18\bar{\gamma}_2\gamma_2-18\gamma_4)u^4\\
&+2(3\gamma_1\bar{\gamma}_2^2-2\gamma_1^3\bar{\gamma}_2+6\gamma_1\bar{\gamma}_2\gamma_2-9\bar{\gamma}_2\gamma_3-9\gamma_2\gamma_3+3\gamma_1^2\gamma_3+9\gamma_1\gamma_4)u^3\\
&+3(2\gamma_1^2\bar{\gamma}_2^2-2\bar{\gamma}_2^3-6\gamma_1\bar{\gamma}_2\gamma_3+6\bar{\gamma}_2\gamma_4+7\gamma_3^2-10\gamma_2\gamma_4)u^2\\
&+2(3\bar{\gamma}_2^2\gamma_3-2\gamma_1\bar{\gamma}_2^3+6\gamma_1\bar{\gamma}_2\gamma_4-9\gamma_3\gamma_4)u+2(\bar{\gamma}_2^4-6\bar{\gamma}_2^2\gamma_4+9\gamma_4^2)],\\
243p_5={}&5[3(-4\gamma_1^5+30\gamma_1^3\gamma_2+15\gamma_1^3\bar{\gamma}_2-27\gamma_1^2\gamma_3-54\gamma_1\gamma_2^2\\
&-27\gamma_1\bar{\gamma}_2^2-54\gamma_1\bar{\gamma}_2\gamma_2-27\gamma_1\gamma_4+81\bar{\gamma}_2\gamma_3+81\gamma_2\gamma_3)u^5\\
&+3(10\gamma_1^4\bar{\gamma}_2-45\gamma_1^2\bar{\gamma}_2\gamma_2-36\gamma_1^2\bar{\gamma}_2^2+54\bar{\gamma}_2^3-270\bar{\gamma}_2\gamma_4-15\gamma_1^3\gamma_3\\
&\quad+108\gamma_1^2\gamma_4+108\gamma_1\bar{\gamma}_2\gamma_3+54\gamma_1\gamma_2\gamma_3+27\bar{\gamma}_2\gamma_2^2+216\gamma_2\gamma_4-108\gamma_3^2)u^4\\
&+6(9\gamma_1^2\bar{\gamma}_2\gamma_3-27\bar{\gamma}_2^2\gamma_3+9\gamma_1\bar{\gamma}_2^3-108\gamma_1\bar{\gamma}_2\gamma_4-3\gamma_1^3\gamma_4-2\gamma_1^3\bar{\gamma}_2^2\\
&\quad+243\gamma_3\gamma_4-18\gamma_1\gamma_3^2+27\bar{\gamma}_2\gamma_2\gamma_3-9\gamma_1\gamma_2\gamma_4)u^3\\
&+6(3\gamma_1^2\bar{\gamma}_2\gamma_4-6\bar{\gamma}_2^4-2\gamma_1^2\bar{\gamma}_2^3+6\gamma_1\bar{\gamma}_2^2\gamma_3+108\bar{\gamma}_2^2\gamma_4\\
&\quad-21\bar{\gamma}_2\gamma_2\gamma_4+27\gamma_1\gamma_3\gamma_4-15\bar{\gamma}_2\gamma_3^2-324\gamma_4^2)u^2\\
&+3(10\gamma_1\bar{\gamma}_2^4-45\gamma_1\bar{\gamma}_2^2\gamma_4+27\gamma_1\gamma_4^2-15\bar{\gamma}_2^3\gamma_3+54\bar{\gamma}_2\gamma_3\gamma_4)u\\
&+2(45\bar{\gamma}_2^3\gamma_4-6\bar{\gamma}_2^5-81\bar{\gamma}_2\gamma_4^2)],\\
729a_6={}&(4\gamma_1^6-36\gamma_1^4\gamma_2-18\gamma_1^4\bar{\gamma}_2+54\gamma_1^3\gamma_3-162\gamma_1^2\gamma_4+81\gamma_1^2\gamma_2^2\\
&\quad+81\gamma_1^2\bar{\gamma}_2\gamma_2-243\gamma_1\gamma_2\gamma_3+729\gamma_2\gamma_4)u^6\\
&+3(6\gamma_1^4\gamma_3-4\gamma_1^5\bar{\gamma}_2+24\gamma_1^3\bar{\gamma}_2\gamma_2+21\gamma_1^3\bar{\gamma}_2^2+90\gamma_1^3\gamma_4-135\gamma_1^2\bar{\gamma}_2\gamma_3\\
&\quad-27\gamma_1^2\gamma_2\gamma_3-27\gamma_1\bar{\gamma}_2\gamma_4-27\gamma_1\bar{\gamma}_2\gamma_2^2+108\gamma_1\gamma_3^2-351\gamma_1\gamma_2\gamma_4\\
&\quad+162\bar{\gamma}_2\gamma_2\gamma_3)u^5\\
&+3(-\gamma_1^4\bar{\gamma}_2^2-3\gamma_1^4\gamma_4+6\gamma_1^3\bar{\gamma}_2\gamma_3-9\gamma_1^2\gamma_3^2-9\gamma_1^2\bar{\gamma}_2^3-27\gamma_1^2\bar{\gamma}_2\gamma_4\\
&\quad+9\gamma_1^2\gamma_2\gamma_4+81\gamma_1\bar{\gamma}_2^2\gamma_3-27\gamma_1\bar{\gamma}_2\gamma_2\gamma_3-108\bar{\gamma}_2\gamma_3^2+27\bar{\gamma}_2^2\gamma_4\\
&\quad+54\gamma_2\gamma_3^2+27\bar{\gamma}_2\gamma_2\gamma_4+27\gamma_2^2\gamma_4)u^4\\
&+(26\gamma_1^3\bar{\gamma}_2^3-117\gamma_1^2\bar{\gamma}_2^2\gamma_3-72\gamma_1\bar{\gamma}_2^4+162\gamma_1\bar{\gamma}_2^2\gamma_4-9\gamma_1\bar{\gamma}_2\gamma_2\gamma_4\\
&\quad+225\gamma_1\bar{\gamma}_2\gamma_3^2+54\bar{\gamma}_2^3\gamma_3-81\gamma_2\gamma_3\gamma_4-162\gamma_3^3)u^3\\
&+3(3\gamma_1^2\bar{\gamma}_2^2\gamma_4-\gamma_1^2\bar{\gamma}_2^4+3\gamma_1^2\gamma_4^2+3\gamma_1\bar{\gamma}_2^3\gamma_3-30\gamma_1\bar{\gamma}_2\gamma_3\gamma_4\\
&\quad-6\bar{\gamma}_2^2\gamma_3^2+57\gamma_3^2\gamma_4+15\gamma_2\gamma_4^2-45\bar{\gamma}_2^3\gamma_4+12\bar{\gamma}_2^5)u^2\\
&+3(24\gamma_1\bar{\gamma}_2^3\gamma_4-4\gamma_1\bar{\gamma}_2^5-27\gamma_1\bar{\gamma}_2\gamma_4^2+6\bar{\gamma}_2^4\gamma_3-27\bar{\gamma}_2^2\gamma_3\gamma_4)u\\
&+(4\bar{\gamma}_2^6-36\bar{\gamma}_2^4\gamma_4+81\bar{\gamma}_2^2\gamma_4^2).
\end{aligned}\right\}\qquad(6)$$

We can change to symmetric functions at any time by means of the relations

$$\left.\begin{aligned}p_2&=-2a_2,\\ p_3&=-3a_3,\\ p_4&=-4a_4+2a_2^2,\\ p_5&=-5a_2a_3+5a_5,\\ p_6&=-6a_6+6a_2a_4+3a_3^2-2a_2^3.\end{aligned}\right\}\qquad(7)$$

To determine the cubic curves $a, \ldots, f$ on the prepared set (3), solve the five relations (*loc. cit.*, p. 168) with $\Sigma a=0$, thus getting the following values, where we write in order the coefficients of $x_0x_1^2$, $x_0^2x_1$, $x_1^2x_2$, $x_1x_2^2$, $x_0^2x_2$, $x_0x_2^2$, $x_0x_1x_2$:

$$\left.\begin{aligned}\frac{a}{2}&=u^2-ux-uz,\ -u^2+uy+ut,\ xz+uz-ux,\ xt-xz-yz,\\ &\qquad\qquad ut-uy-ty,\ yz-tx+ty,\ 2(ux-ut)\\ \frac{b}{2}&=-u^2+ux-uz,\ u^2+ut-uy,\ ux+uz-xz,\ xz-yz-xt,\\ &\qquad\qquad uy-ut-ty,\ xt+ty-yz,\ 2(yz-ux),\\ \frac{c}{2}&=-u^2-ux+uz,\ u^2+uy-ut,\ xz+ux-uz,\ yz-xz-xt,\\ &\qquad\qquad ty-uy-ut,\ yz+tx-ty,\ 2(ut-yz),\\ \frac{d}{2}&=-u^2+ux+uz,\ u^2-uy-ut,\ uz-ux-xz,\ xz-yz+xt,\\ &\qquad\qquad ty-uy+ut,\ yz-xt-yt,\ 2(uy-uz)\\ \frac{e}{2}&=u^2-ux+uz,\ -u^2+uy-ut,\ ux-uz-xz,\ xz+yz-xt,\\ &\qquad\qquad uy+ut-ty,\ ty-yz-tx,\ 2(xt-uy),\\ \frac{f}{2}&=u^2+ux-uz,\ -u^2+ut-uy,\ xz-ux-uz,\ yz+tx-xz,\\ &\qquad\qquad ty+uy-ut,\ xt-yz-ty,\ 2(uz-xt).\end{aligned}\right\}\qquad(8)$$

Combining these with the values of $\bar{a}, \ldots, \bar{f}$, given by (4), we find immediately the functions $K_2, \ldots, K_5$.

From the values of $p_2, \ldots, p_5$, given by (6), we find that $p_2^2-4p_4$ or $a_2^2-4a_4$, and $12p_5-5p_2p_3$ or $a_2a_3-2a_5$ contain u^2 as a factor. If, then,

$$q_4=p_2^2-4p_4,\qquad q_5=12p_5-5p_2p_3,$$

we find

$$\left.\begin{aligned}q_4&=u^2[(\bar\gamma_2^2-4\gamma_4)u^2+2(2\gamma_1\gamma_4-\bar\gamma_2\gamma_3)u+\gamma_3^2-4\gamma_2\gamma_4],\\ q_5&=2430u^2[(3\bar\gamma_2\gamma_3-2\gamma_1\gamma_4-\gamma_1\bar\gamma_2^2)u^3\\ &\quad+(2\bar\gamma_2^3-8\bar\gamma_2\gamma_4-\gamma_1\bar\gamma_2\gamma_3+5\gamma_1^2\gamma_4-12\gamma_2\gamma_4)u^2\\ &\quad+(3\bar\gamma_3\gamma_2\gamma_3-\bar\gamma_2^2\gamma_3-7\gamma_1\bar\gamma_2\gamma_4-2\gamma_1\gamma_2\gamma_4-\gamma_1\gamma_3^2+18\gamma_3\gamma_4)u\\ &\quad+(6\bar\gamma_2^2\gamma_4-\bar\gamma_2\gamma_3^2-2\bar\gamma_2\gamma_2\gamma_4+3\gamma_1\gamma_3\gamma_4-24\gamma_4^2)].\end{aligned}\right\}\qquad(9)$$

Calculating the terms free of u in the functions $K_2, \ldots, K_5$, I find

$$\left.\begin{aligned} &K_2=36\gamma_4(xt-yz)(x_1x_2^2-x_0x_2^2),\\ &K_3=0,\\ &K_4=108\gamma_4(\bar{\gamma}_2^2-3\gamma_4)(xt-yz)(x_1x_2^2-x_0x_2^2),\\ &K_5=36\gamma_4(-2\bar{\gamma}_2^3-81\bar{\gamma}_2\gamma_4)(xt-yz)(x_1x_2^2-x_0x_2^2);\end{aligned}\right\} \qquad (10$$

hence, the combinations $K_4'=9p_2K_2-4K_4$, $K_5'=9(p_3-20\bar{\gamma}_2\gamma_4)K_2-2K_5$, contai u as a factor. It follows from the value of K_5' that it could not be used t advantage in expressing the linear covariants.

An interesting particular case occurs when the last two points of (3) ar apolar to the pencil of conics on the first four, $x_2x_3=x_3x_1=x_1x_2$; *i. e.*, whe two of the points are a corresponding pair in the quadratic Cremona involutio with fixed points at the other four. This requires that $xt+yz=u(y+t$ $=u(x+z)$ whence from (4) $\bar{a}+\bar{d}=\bar{b}+\bar{f}=\bar{c}+\bar{e}=\lambda$. Since $\Sigma\bar{a}=0$, $\lambda=0$ and therefore, $p_3=\Sigma\bar{a}^3=0$ and $p_5=\Sigma\bar{a}^5=0$. If, conversely, $p_3=p_5=0$ and if fiv points of P_6^2 be fixed there are fifteen possible positions for the sixth. Thes all are accounted for by choosing any four of the five as fixed points or an three of the five as fixed points. Hence,

The necessary and sufficient condition that the invariants of P_6^2 of od degree (other than the skew invariant $d_2\sqrt{d}$) vanish is that two of the point be apolar to the pencil of conics on the other four.

Section 3.

Let the cubic surface be given by

$$U\equiv(\alpha x)^3\equiv(\beta x)^3\equiv\ldots\equiv(\delta x)^3,\ \text{or}\ \equiv x_1^3+x_2^3+\ldots+x_6^3=0, \qquad (11$$

for the Cremona form, where the variables are subject to the relations

$$\left.\begin{aligned} x_1+x_2+\ldots+x_6=0,\\ \bar{a}x_1+\bar{b}x_2+\ldots+\bar{f}x_6=0.\end{aligned}\right\} \qquad (12$$

The Hessian is the locus of points whose polar quadrics are cones and therefore, obtained by writing the discriminant of a polar quadric.

According to the principle of transference of Clebsch it is given by

$$H=\frac{1}{24}\,|\alpha\beta\gamma\delta\eta\zeta|^2(\alpha x)(\beta x)(\gamma x)(\delta x)\ \text{ or }\ =\sum_{(56)}^{15}x_1x_2x_3x_4\overline{56}^2 \qquad (13$$

for the Cremona form, where $\overline{ij}=\begin{vmatrix}\eta_i & \eta_j\\ \zeta_i & \zeta_j\end{vmatrix}$, and since $\eta_1, \ldots, \eta_6=1, \ldots, 1$ $\zeta_1, \ldots, \zeta_6=\bar{a}, \ldots, \bar{f}\equiv b_1, \ldots, b_6$, we have $\overline{ij}=b_i-b_j$. The super- an subscript of Σ refer to the fifteen terms obtained by choosing the pair 5 from 1, ..., 6.

The contravariant S, as given by the Clebsch transference principle, is

$$S=\frac{1}{6}|\alpha\beta\gamma\eta\zeta\xi|\,|\alpha\beta\delta\eta\zeta\xi|\,|\alpha\gamma\delta\eta\zeta\xi|\,|\beta\gamma\delta\eta\zeta\xi| \text{ or } =4\sum_{(56)}^{15}[\overline{156}\,\overline{256}\,\overline{356}\,\overline{456}] \quad (14)$$

for the Cremona form, where

$$\overline{ijk}=\begin{vmatrix} \xi_i & \xi_j & \xi_k \\ \eta_i & \eta_j & \eta_k \\ \zeta_i & \zeta_j & \zeta_k \end{vmatrix}.$$

By expanding and collecting the coefficients of ξ we have

$$\begin{aligned}\frac{1}{4}S=&\sum_{(1)}^{6}\xi_1^4\sum_{(2)}^{5}\overline{23}\,\overline{24}\,\overline{25}\,\overline{26}+\sum_{(1)(2)}^{30}\xi_1^3\xi_2[\sum_{(3)}^{4}\overline{31}\,\overline{24}\,\overline{25}\,\overline{26}+\sum_{(3)}^{4}\overline{13}\,\overline{34}\,\overline{35}\,\overline{36}]\\ &+\sum_{(56)}^{15}\xi_5^2\xi_6^2\sum_{(12)}^{6}\overline{16}\,\overline{26}\,\overline{35}\,\overline{45}+\sum_{(12),(3)}^{60}\xi_1\xi_2\xi_3^2[\sum_{(4)}^{3}\overline{34}^2\,\overline{45}\,\overline{46}+\sum_{(1)}^{2}\overline{31}\sum_{(4)}^{3}\overline{34}\,\overline{15}\,\overline{16}]\\ &+\sum_{(1234)}^{15}\xi_1\xi_2\xi_3\xi_4[\overline{56}^4+\sum_{(1)}^{4}\sum_{(5)}^{2}\overline{15}^3\,\overline{56}-\sum_{(12)}^{6}\sum_{(5)}^{2}\overline{12}^2\,\overline{15}\,\overline{26}].\end{aligned}$$

Let us indicate the coefficients of S as follows:

$$\frac{1}{4}S=\sum_{(1)}^{6}S_1\xi_1^4+\sum_{(1)}^{6}\sum_{(2)}^{5}S_{1,2}\xi_1^3\xi_2+\sum_{(12)}^{15}S_{12}\xi_1^2\xi_2^2+\sum_{(1)}^{6}\sum_{(23)}^{10}S_{1,23}\xi_1^2\xi_2\xi_3+\sum_{(1234)}^{15}S_{1234}\xi_1\xi_2\xi_3\xi_4.$$

We find that these coefficients expressed in terms of $a_2, \ldots, a_6$, the elementary symmetric functions of $b_1, \ldots, b_6$ are:

$$\left.\begin{aligned} S_1 &=-15a_4+4a_2^2+9a_3b_1-6a_2b_1^2-9b_1^4,\\ S_{1,2} &=12a_4-4a_2^2+3a_3\{1\}-2a_2\{1,2\}-6\{0,1,0\}-6b_2[a_3+2b_1b_2^2],\\ S_{12} &=6a_4-9a_3\{1\}+2a_2\{5,-2\}+10\{1,-2,1\},\\ S_{1,23} &=-8a_4+4a_2^2-a_3[10b_1+\{1\}]+2a_2[b_1\{4\}+\{3,-2\}]\\ &\quad+2[2b_1\{4,-9\}+\{2,-3,0\}],\\ S_{1234} &=8a_2^2+18a_3\{1\}+12a_2\{1,-2\}-4\{2,-7,-4\}.\end{aligned}\right\} \quad (15)$$

Here $\{\ldots\}$ refers to the coefficients of polynomials in ρ_1, ρ_2 the symmetric combinations of the two isolated letters.

Identities among these which are sometimes useful are:

$$\left.\begin{aligned} &\{1,-4,3\}+a_2\{1,-2\}-a_3\{1,-1\}+a_4\{1\}-a_5=0,\\ &[b_1^4+b_1^3\{1\}+b_1^2\{1,-1\}+b_1\{1,-2\}+\{1,-3,1\}]\\ &\quad+a_2[b_1^2+b_1\{1\}+\{1,-1\}]-a_3[b_1+\{1\}]+a_4=0.\end{aligned}\right\} \quad (16)$$

The coefficients of S are connected by the relations

$$4S_1\eta_1+\sum_{(2)}^{5}S_{1,2}\eta_2=0,$$

$$3S_{1,2}\eta_1+2S_{12}\eta_2+\sum_{(3)}^{4}S_{1,23}\eta_3=0,$$

$$2(S_{1,23}\eta_1+S_{2,31}\eta_2+S_{3,12}\eta_3)+S_{1234}\eta_4+S_{1235}\eta_5+S_{1236}\eta_6=0.$$

Hence, $\eta_1, \ldots, \eta_6 = 1, \ldots, 1$, or $b_1, \ldots, b_6$, also $\sum\limits_{(2)}^{5} S_{2,1} = \sum\limits_{(2)}^{5} S_{1,2} = -4S_1$.

A numerical check is furnished by taking $\left\{\begin{matrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ 1 & 1 & 1 & 1 & -2 & -2 \end{matrix}\right\}$, then $a_2 = -6$, $a_3 = -4$, $a_4 = 9$, $a_5 = 12$, $a_6 = 4$, while $S_5 = S_6 = 81$, $S_{56} = 18 \cdot 27$, $S_{5,6} = S_{6,5} = -18 \cdot 18$, and all the other coefficients S vanish.

By operating with S on H (in operating with a form of order n on a form of class $m > n$ we remove the factor $m(m-1) \ldots (m-n+1)$ after differentiation) we get the first invariant

$$\begin{aligned} I_1 &= \sum_{(56)}^{15} \overline{56}^6 - 2 \sum_{(1)}^{6} \sum_{(23)}^{10} \overline{12}^3\, \overline{13}^3 - 2 \sum_{(12)\,(34)}^{45} \overline{12}^2\, \overline{34}^2 [\overline{13}\,\overline{24} + \overline{14}\,\overline{23}] \\ &= -15 \Sigma b_1^6 + 18 \Sigma b_1^5 b_2 - 9 \Sigma b_1^4 b_2^2 - 18 \Sigma b_1^4 b_2 b_3 \\ &\quad + 12 \Sigma b_1^3 b_2^3 + 24 \Sigma b_1^3 b_2 b_3 b_4 + 18 \Sigma b_1^2 b_2^2 b_3^2 - 12 \Sigma b_1^2 b_2^2 b_3 b_4, \end{aligned}$$

and in terms of the symmetric functions $a_2, \ldots, a_6$ we find

$$I_1 = 24[4a_2^3 - 3a_3^2 - 16a_2 a_4 + 12a_6]. \tag{17}$$

By operating with S on U^2 we get the first covariant quadric, *i. e.*, a $C_{6,2,0}$. Denoting this quadric by Q, we find

$$5Q = \sum_{(1)}^{6} (20S_1 + \sum_{(2)}^{5} 2S_{2,1}) x_1^2 + 4 \sum_{(12)}^{15} S_{12} x_1 x_2 \equiv \sum_{(1)}^{6} Q_{11} x_1^2 + \sum_{(12)}^{15} Q_{12} x_1 x_2, \tag{18}$$

where

$$Q_{11} = 12 S_1, \qquad Q_{12} = 4 S_{12}, \tag{19}$$

(S_1 and S_{12} are given by (15)).

By operating with Q on S we get the contravariant quadric $C_{10,0,2}$ Denoting this quadric by q we find

$$5 \cdot 3\, q \equiv \sum_{(1)}^{6} q_{11} \xi_1^2 + \sum_{(12)}^{15} q_{12} \xi_1 \xi_2, \tag{20}$$

where if $q_4 = a_2^2 - 4a_4$ and $q_5 = a_2 a_3 - 2a_5$,

$$\left.\begin{aligned} \frac{1}{72} q_{11} &= -3 \cdot 32\, q_4 b_1^4 + 3 \cdot 8\, q_5 b_1^3 - 3 \cdot 2 [I_1 + 8a_2 q_4] b_1^2 \\ &\quad + 3 \cdot 2 [5a_2 q_5 + 17 a_3 q_4] b_1 + [5a_2 I_1 + 15 a_3 q_5 + 27 q_4^2 - 15 a_2^2 q_4], \\ \frac{1}{32} q_{12} &= -32 a_2 a_6 - 135 a_3 a_5 - 216 a_4^2 + 198 a_2^2 a_4 - 36 a_2^4 \\ &\quad + 135 a_2 a_5 \{1\} - 108 a_3 a_4 \{1\} + 27 a_2^2 a_3 \{1\} + 12 a_6 \{-43, 33\} \\ &\quad + 2a_2 a_4 \{144, -47\} + 81 a_3^2 \{0, 1\} - 36 a_2^3 \{2, 1\} + 6a_5 \{45, -4\} \\ &\quad - 248 a_2 a_3 \{0, 1\} + 36 a_4 \{6, -9, -14\} + 2a_2^2 \{-27, -11, 173\} \\ &\quad + 6a_3 \{0, -27, 61\} + 2a_2 \{0, 16, 93, -83\} + 12 \{0, 0, 10, 13, -6\}. \end{aligned}\right\} \tag{21}$$

By operating with q on U we get the first linear covariant $C_{11,1,0}$. Denoting this covariant by L_1 we find $L_1 = q_{11}x_1 + q_{22}x_2 + \dots + q_{66}x_6$, or in terms of the functions (2),

$$L_1 = -16\,q_4K_4 + 4\,q_5K_3 + [-I_1 - 8\,a_2q_4]K_2. \tag{22}$$

By operating with q on Q we get the second invariant I_2, where

$$5^2 \cdot 6\,I_2 = 2\sum_{(1)}^{6} q_{11}Q_{11} + \sum_{(12)}^{15} q_{12}Q_{12}.$$

By another method I have found a second invariant to be

$$I_2' = 2^8\,q_5^2q_2 + 3 \cdot 5 \cdot 2^4\,q_5q_4q_3 - 2 \cdot 3^2 \cdot 5^2\,q_4^3 - 2 \cdot 3 \cdot 5^2\,q_4^2q_2^2 + 3 \cdot 5^2\,I_1q_4q_2; \tag{23}$$

hence, we must find $I_2 = c_0I_1^2 + c_1I_2'$, where c_0 and c_1 are numerical constants.

If we denote by M the mixed concomitant obtained by writing the plane equation of the polar quadric of any point with respect to the cubic surface, we have, according to the principle of transference of Clebsch,

$$\left.\begin{aligned} M &= \frac{1}{6}\,|\alpha\beta\gamma\eta\zeta\xi|\,(\alpha x)\,(\beta x)\,(\gamma x), \text{ or } = \sum^{20} x_1x_2x_3\overline{456}^2, \\ &= \sum^{20} x_1x_2x_3\Big(\sum_{(4)}^{3} u_4^2\,\overline{56}^2 - 2\sum_{(56)}^{3} u_5u_6\,\overline{45}\,\overline{46} \end{aligned}\right\} \tag{24}$$

for the Cremona form.

By operating on this with $\frac{5}{4}Q = \sum_{(1)}^{6} 3\,S_1x_1^2 + \sum_{(12)}^{15} S_{12}x_1x_2$, we get a $C_{9,3,0}$. Denoting this by N we have

$$N = \sum_{(123)}^{20} x_1x_2x_3\Big[\sum_{(4)}^{3}(3\,S_4\overline{56}^2 - S_{56}\overline{45}\,\overline{46}\Big] \equiv \sum_{(123)}^{20} C_{123}x_1x_2x_3, \tag{25}$$

where we find

$$\begin{aligned} C_{123} = {} & 24\,q_4(\sigma_1^2 - 3\,\sigma_2) + 36\,a_3(\sigma_1\sigma_2 - 9\,\sigma_3) + 72\,a_2(3\,\sigma_1\sigma_3 - \sigma_2^2) \\ & + 3\,[19\,(4\,\sigma_1^3\sigma_3 - \sigma_1^2\sigma_2^2) + 28\,\sigma_2^3 - 150\,\sigma_1\sigma_2\sigma_3 + 81\,\sigma_3^2]. \end{aligned} \tag{26}$$

Here $\sigma_1, \sigma_2, \sigma_3$ are the elementary symmetric functions of b_4, b_5, b_6.

The result of operating with N on S is a $C_{18,0,1}$. Denoting this by C, we have

$$C = \sum_{(1)}^{6} \xi_1\Big[\sum_{(56)}^{10}(S_{1234}C_{234} + 2\,S_{1,56}C_{156})\Big] \equiv C_1\xi_1 + \dots + C_6\xi_6. \tag{27}$$

Multiplying $S_{1,56}$ and C_{156} together, summing for $\sum^{10}$ and indicating by $(ijkl)$ the symmetric sum for five things of $\sum b_2^ib_3^jb_4^kb_5^l$, we get

$$\left.\begin{aligned}
\sum^{10} 2S_{1,56}C_{156} = 6\{&24q_4[4a_2^2-8a_4-10a_3b_1][2(2)-(11)] \\
&+\{24q_4[-a_3+8a_2b_1]+36a_3[4a_2^2-8a_4-10a_3b_1]\}[(21)-2(111)] \\
&+48a_3[-a_3+8a_2b_1][(211)-6(1111)] \\
&\qquad\qquad+24a_2[4a_2^2-8a_4-10a_3b_1][(211)-3(22)] \\
&+32a_2q_4[9(22)-(211)-12(1111)] \\
&+[4a_2^2-8a_4-10a_3b_1][38(411)-57(42) \\
&\qquad\qquad+10(321)-30(33)-30(222)] \\
&+32q_4b_1[12(32)-8(311)+6(221)-9(2111)] \\
&+144a_2a_3[2(221)-(2111)-40(11111)] \\
&\qquad\qquad+24a_2[-a_3+8a_2b_1][3(2111)-2(221)] \\
&+96a_2^2[(2211)-9(222)+12(21111)] \\
&\qquad\qquad+16\cdot 12a_3b_1[2(321)-6(3111)+3(2211)-18(21111)] \\
&+16q_4[6(42)-4(411)+5(321)-15(3111) \\
&\qquad\qquad+18(222)-6(2211)] \\
&+48a_3[2(421)-6(4111)+5(3211)-60(31111) \\
&\qquad\qquad+9(2221)-18(22111)] \\
&+96a_2b_1[4(3211)-8(322)+18(22111)-9(2221)] \\
&+2[-a_3+8a_2b_1][19\cdot 12(31111)-19(421) \\
&\qquad\qquad+20(3211)-20(331)-30(2221)] \\
&+48a_2[2(4211)-4(422)+15(32111)-5(3221) \\
&\qquad\qquad+18(22211)-36(2222)] \\
&+4a_2[19\{(4211)-6(422)+24(41111)\}+10\{3(3221)-3(332) \\
&\qquad\qquad-18(2222)+6(32111)-2(3311)-6(22211)\}] \\
&+38[8(4411)-8(442)+30(43111)-5(4321) \\
&\qquad\qquad+12(42211)-18(4222)] \\
&+20[2(4321)-4(433)-6(4222)+20(33211)-15(3331) \\
&\qquad\qquad+3(32221)-6(3322)-180(22222)]\}.
\end{aligned}\right\}\quad(28)$$

In order to get $\sum^{10} S_{1234}C_{234}$, we write

$$\begin{aligned}
S_{1234} &= 2[4a_2^2+9a_3(b_5+b_6)+6a_2(b_5^2+b_6^2)-4(b_5^4+b_6^4)-2b_5b_6(b_5^2+b_6^2)+12b_5^2b_6^2], \\
C_{234} &= 3[8q_4b_1^2+(-8q_4b_1+12a_3b_1^2)(b_5+b_6)+(8q_4+12a_3b_1-24a_2b_1^2 \\
&\quad -19b_1^4)(b_5^2+b_6^2)+(-8q_4-72a_3b_1+24a_2b_1^2+38b_1^4)b_5b_6-10b_1^3(b_5^3+b_6^3) \\
&\quad +(12a_3+24a_2b_1+10b_1^3)(b_5+b_6)b_5b_6-19b_1^2(b_5^4+b_6^4)+10b_1^2b_5b_6(b_5^2+b_6^2) \\
&\quad +(-24a_2-30b_1^2)b_5^2b_6^2+38b_1b_5b_6(b_5^3+b_6^3)+10b_1b_5^2b_6^2(b_5+b_6) \\
&\quad -19b_5^2b_6^2(b_5^2+b_6^2)-10b_5^3b_6^3].
\end{aligned}$$

$$\left.\begin{aligned}
\therefore \sum^{10} S_{1234} C_{234} = 6\{ & 320\, a_2^2 q_4 b_1^2 + [72 a_3 q_4 b_1^2 + 48 a_3 a_2^2 b_1^2 - 32 a_2^2 q_4 b_1] 4(1) \\
& + [-76 a_2^2 b_1^4 + 12(-8a_2^3 + 9a_3^2 + 4a_2 q_4) b_1^2 + 24(2a_2^2 a_3 - 3a_3 q_4) b_1 \\
& \qquad + 32 a_2^2 q_4] 4(2) \\
& + [8 \cdot 19 a_2^2 b_1^4 + 24(9a_3^2 + 4a_2^3) b_1^2 - 144(a_3 q_4 + 2a_2^2 a_3) b_1 - 32 a_2^2 q_4] (11) \\
& + [-9 \cdot 19 a_3 b_1^4 - 40 a_2^2 b_1^3 - 144 a_2 a_3 b_1^2 + 12(9a_3^2 - 4a_2 q_4) b_1 \\
& \qquad + 72 a_3 q_4] 4(3) \\
& + [9 \cdot 19 a_3 b_1^4 + 40 a_2^2 b_1^3 + 72 a_2 a_3 b_1^2 + 12(8a_2^3 + 9a_3^2 - 4a_2 q_4) b_1 \\
& \qquad + 48 a_2^2 a_3] (21) \\
& + [-6 \cdot 19 a_2 b_1^4 - 90 a_3 b_1^3 - 4(55 a_2^2 + 8q_4) b_1^2 + 72 a_2 a_3 b_1 + 48 a_2 q_4] 4(4) \\
& + [12 \cdot 19 a_2 b_1^4 + 8(23a_2^2 - 2q_4) b_1^2 - 24 \cdot 9 a_2 a_3 b_1 + 12(9a_3^2 - 4a_2 q_4)] (31) \\
& + [-12 \cdot 19 a_2 b_1^4 + 180 a_3 b_1^3 + 24(-17a_2^2 + 4q_4) b_1^2 + 24 \cdot 24 a_2 a_3 b_1 \\
& \qquad + 24(9a_3^2 - 4a_2^3 + 4a_2 q_4)] (22) \\
& + [-60 a_2 b_1^3 - 3 \cdot 73 a_3 b_1^2 + 32 q_4 b_1] 4(5) \\
& + [60 a_2 b_1^3 - 9 \cdot 17 a_3 b_1^2 + (8 \cdot 37 a_2^2 + 48 q_4) b_1 + 72 a_2 a_3] (41) \\
& + [-60 a_3 b_1^2 + 8(23a_2^2 - 10q_4) b_1 - 144 a_2 a_3] (32) \\
& + [4 \cdot 19 b_1^4 - 18 a_2 b_1^2 - 48 a_3 b_1 - 32 q_4] 4(6) \\
& + [-6 \cdot 19 b_1^4 + 12 a_2 b_1^2 + 606 a_3 b_1 + 16 q_4] (51) \\
& + [-12 \cdot 19 b_1^4 - 6 \cdot 89 a_2 b_1^2 + 24 \cdot 28 a_3 b_1 + 20(4q_4 - 11a_2^2)] (42) \\
& + [28 \cdot 19 b_1^4 + 12 \cdot 42 a_2 b_1^2 - 6 \cdot 122 a_3 b_1 - 8(16q_4 + 5a_2^2)] (33) \\
& + [40 b_1^3] 4(7) + [-20 b_1^3 + 12 \cdot 11 a_2 b_1 - 48 a_3] (61) \\
& + [-180 b_1^3 - 84 a_2 b_1 - 9 \cdot 27 a_3] (52) \\
& + [160 b_1^3 + 48 \cdot 11 a_2 b_1 - 3 \cdot 47 a_3] (43) \\
& + [4 \cdot 19 b_1^2] 4(8) + [-2 b_1^2] (71) + [-16 \cdot 8 b_1^2 - 18 a_2] (62) \\
& + [2 \cdot 89 b_1^2 - 12 a_2] (53) + [-4 \cdot 62 b_1^2 - 129 a_2] (44) \\
& - 8 \cdot 19 b_1 (81) - 4 \cdot 29 b_1 (72) + 12 \cdot 33 b_1 (63) - 4 \cdot 32 b_1 (54) \\
& + 4 \cdot 19 (82) + 78 (73) - 4 \cdot 33 (64) - 4 \cdot 11 (55).
\end{aligned}\right\} \quad (29)$$

Using brackets for symmetric functions of six things, we have

$$\left.\begin{aligned}
(i) &= [i] - b_1^i, \\
(ii) &= [ii] - b_1^i [i] + b_1^{2i}, \\
(ij) &= [ij] - b_1^i [j] - b_1^j [i] + 2 b_1^{i+j}, \text{ etc.};
\end{aligned}\right\} \quad (30)$$

thus, we get C_1 in final form by making these substitutions in (28) and (32) and collecting.

The polar form of (18) is $\frac{1}{2}[\sum\limits_{(1)}^{6} x_1 (2Q_{11} y_1 + \sum\limits_{(2)}^{5} Q_{12} y_2)]$, and by operating with this on (20) we get a collineation $C_{16,1,1}$. Denoting this by K, we have

$$2K = \sum_{(1)}^{6} \xi_1 [x_1 (4 q_{11} Q_{11} + \sum_{(2)}^{5} q_{12} Q_{12}) + \sum_{(2)}^{5} x_2 (2 q_{12} Q_{23} + 2 q_{11} Q_{13} + \sum_{(3)}^{4} q_{13} Q_{23})]. \quad (31)$$

The Lines of Electric Force Due to a Moving Electron.

BY FRANCIS D. MURNAGHAN.

INTRODUCTION.

The values of the electric and magnetic forces in the electromagnetic field due to a moving point charge, or electron, have been given by Liénard,* Wiechert,† Heaviside,‡ and, in detail, by Abraham.§ The expressions are, however, so complicated in the case of perfectly general motion that the form of the lines of force was not obtained except for the simplest case—that of uniform motion in a straight line. The field of force for this case was known ‖ long before that for the general case was worked out, and it was shown that the lines of electric force at any time t were straight lines through the position of the electron at that time t; a result which may be expressed by saying that an electron in uniform motion along a straight line convects its field along with it.

The form of the lines of electric force is of interest in connection with a theory, advanced by Sir J. J. Thomson,¶ of the structure of the ether and the nature of Röntgen rays and light. Following Faraday,** Thomson regards the lines of force not merely as an abstract mathematical concept but as physical realities. He makes, however, one important modification in Faraday's theory. If we have two charges of opposite sign they will be joined by lines of force, the direction of a line at any point being the direction of the electric force at that point. These are the lines of force which Faraday imagined as physical realities endowed with such properties as tension along their length and mutual transverse repulsion. In Thomson's modification of the theory each charge is supposed to carry its own lines of force with it independently of the presence of other charges. In a field containing many charges there will be many lines of force crossing at any point and these give a resultant electric force whose direction is that of the Faraday line at the point. Accordingly, the lines of force attached to a point charge are conceived by Thomson as more fundamental than the idea of the point charge itself and may, in fact, be used to define it.

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* *L'éclairage électrique*, Vol. XVI (1898), pp. 5, 53, 106.

† *Arch. néerlandaises* (2), Vol. V (1900), p. 549.

‡ *Nature*, Vol. LXVII (1902), p. 6.

§ *Ann. d. Phys.*, Vol. XIV (1904), p. 236; "Theorie der Elektrizität," Band 2, §§ 13–15, Leipzig (1914).

‖ O. Heaviside, *Phil. Mag.*, Vol. XXVII (1889), p. 324.

¶ Thomson, *Proc. Camb. Phil. Soc.*, Vol. XV (1909), p. 65.

** A full account of Faraday's theory is given by Thomson in his "Recent Researches in Electricity and Magnetism," Chapter I.

To account for results obtained in experiments on the ionization of gases by Röntgen rays and ultra-violet light Thomson[*] has modified his theory by supposing that the lines of force are not distributed in a continuous manner round the point charge. (He had previously[†] made this modification in Faraday's concept.)

SECTION 1.

Recently a method was given by Bateman[‡] which reduces the problem of finding the equations, at any time t, of the lines of electric force due to a point charge moving in any way, to the solution of a differential equation of Riccati's type. It will be shown that this equation can always be integrated if the path of the electron lies wholly in a plane. The following cases, which are those of most interest, of motion in a plane have been worked out in detail:

(a) Rectilinear motion with any velocity and acceleration;
(b) Uniform motion in a circle.

It has been found possible to integrate the equation in one case of motion in three dimensions—that of uniform motion in a circular helix. Further, it has been found that if one solution of the Riccatian equation is known a second can be derived from it and accordingly the general solution is reduced to the evaluation of a single quadrature.

It will be convenient to give here a brief account of the method by which Bateman obtained the Riccatian equation referred to. Let C be the curve along which the electron is moving and let $x=\xi$, $y=\eta$, $z=\zeta$ be the coordinates of the point of C, occupied by the electron at time τ, so that ξ, η, ζ are functions of τ. Denote differentiations with regard to τ by primes so that the velocity of the electron is $\bar{v}=(\xi', \eta', \zeta')$ and its acceleration is $\bar{v}'=(\xi'', \eta'', \zeta'')$; where bars are used to distinguish vector quantities from scalars. Let P be any point in space. Then if x, y, z are the coordinates of P, a disturbance emanating from the electron at time τ will reach P at time t if $c(t-\tau)=r$ where c is the velocity of light and $r^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2$. Denote by $\bar{r}$ the vector

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[*] *Phil. Mag.*, Vol. XIX (1910), p. 301.

[†] *Proc. Camb. Phil. Soc.*, Vol. XIV, p. 417. (For this and other theories as to the structure of the ether see Bateman "Electrical and Optical Wave Motion," Camb., 1915, Chapter 8.

[‡] *Bulletin Amer. Math. Soc.*, 2d Series, Vol. XXI (1915), No. 6, p. 308; *American Journal*, April (1915).

OP (where O is the position of the electron at time τ) and let (l, m, n) be the direction cosines of $\bar{r}$. Then the electric force at P at time t is known* to be

$$E=\frac{e}{r^2}\left\{\frac{\bar{r}}{r}-\frac{\bar{v}}{c}\right\}\left\{1-\frac{v^2}{c^2}+\frac{(\bar{v}'\bar{r})}{c^2}\right\}\left(\frac{\partial\tau}{\partial t}\right)^3-\frac{e\,\bar{v}'}{r\,c^2}\left(\frac{\partial\tau}{\partial t}\right)^2,$$

where e is the charge on the electron, round brackets denoting scalar products.

If we differentiate the equation $r^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2$ we get $\frac{\partial r}{\partial\tau}=\lambda-c$ where $\lambda=c-\frac{1}{r}(\bar{r}\bar{v})$ and since $c(t-\tau)=r$ we have $c\frac{\partial t}{\partial\tau}=\lambda$. We shall be interested in the direction only of E at P and an easy transformation of the expression for E shows that it has the direction of the vector

$$(c^2-v^2)\left\{\bar{v}-c\frac{\bar{r}}{r}\right\}+r\left\{\lambda\bar{v}'+\left(\bar{v}-c\frac{\bar{r}}{r}\right)p\right\},$$

where $p=l\xi''+m\eta''+n\zeta''=\frac{1}{r}(\bar{r}\bar{v}')$.

Let P' be a point consecutive to P and let its coordinates be $x+dx$, $y+dy$, $z+dz$. P' will be reached at time t by a disturbance which has emanated from the electron at time $\tau+d\tau$; at this time let O' be the position of the electron so that $\xi+d\xi$, $\eta+d\eta$, $\zeta+d\zeta$ are the coordinates of O'. The x component of the vector equation $\overline{PP'}=\overline{PO}+\overline{OO'}+\overline{O'P'}$ may be written

$$dx=-rl+d\xi+c(l+dl)(t-\tau-d\tau)\quad\text{or}\quad\frac{dx}{d\tau}=r\frac{dl}{d\tau}+\frac{d\xi}{d\tau}-cl.$$

Thus, if $\bar{s}$ is the vector $\overline{PP'}$ we have $\bar{s}'=\left(\bar{v}-c\frac{\bar{r}}{r}\right)+r(l', m', n')$ where, as previously, (l', m', n') is used to denote the vector whose components are l', m', n'. Comparing this with the expression giving the direction of the electric force at P, we see that PP' is the direction of the electric force at P if the three equations

$$(c^2-v^2)\frac{d}{d\tau}(l, m, n)=\lambda(\xi'', \eta'', \zeta'')+p(\xi'-cl, \eta'-cm, \zeta'-cn)\ldots\quad\text{(A)}$$

are satisfied. From the manner in which these equations were derived we see that, if (l, m, n) is a solution of these equations satisfying also the condition $l^2+m^2+n^2=1$; the equations

$$x=\xi+c(t-\tau)l,\quad y=\eta+c(t-\tau)m,\quad z=\zeta+c(t-\tau)n$$

(where τ is a parameter satisfying the inequality $\tau<t$) give a line of electric force at time t. It will be noticed that, if at each position of the electron we imagine a particle projected with velocity c in the direction (l, m, n), the

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* Abraham, "Theorie der Elektrizität."

aggregate of these particles will at any time t form a line of electric force. All the lines of force at any time t pass through the position of the electron at that time.

In order to obtain the solutions (l, m, n) of (A) which satisfy $l^2+m^2+n^2=1$ put $\sigma=\dfrac{l+im}{1+n}=\dfrac{1-n}{l-im}$ where $i^2=-1$. After an easy reduction we obtain the Riccatian equation

$$2(c^2-v^2)\frac{d\sigma}{d\tau}=\phi''(c-\zeta')+\phi'\zeta''-2\sigma\{c\zeta''+i(\xi'\eta''-\xi''\eta')\}+\sigma^2\{\psi'\zeta''-(c+\zeta')\psi''\}$$

where $\phi=\xi+i\eta$ and ψ is its conjugate $\xi-i\eta$.

If σ is a solution of this equation and if its conjugate is s, $l=\dfrac{\sigma+s}{1+\sigma s}$, $m=-i\dfrac{\sigma-s}{1+\sigma s}$, $n=\dfrac{1-\sigma s}{1+\sigma s}$ is a solution of the equations (A) which satisfies $l^2+m^2+n^2=1$.

It is interesting to note that the equations (A) are a generalization of the equations

$$\frac{d}{d\tau}(l, m, n)=(mr-nq,\ np-lr,\ lq-mp)\ldots. \qquad \text{(B)}$$

to which they reduce on making $c=0$ and $v^2(p, q, r)=[\bar{v}'\bar{v}]$ the square brackets being used to denote a vector product. These equations occur in the kinematics of a rigid body and have been considered at length by Darboux.* Several writers† have extended them to the case of four or more variables and have obtained results similar to those found by Darboux.

§ 2. ASSOCIATED DIRECTIONS OF PROJECTION.

A solution (l, m, n) of the equations

$$(c^2-v^2)\frac{d}{d\tau}(l, m, n)=\lambda(\xi'', \eta'', \zeta'')+p(\xi'-cl,\ \eta'-cm,\ \zeta'-cn),$$

which have been mentioned in the preceding paragraph, will be called a *direction of projection* provided $l^2+m^2+n^2=1$. It will now be shown that to every direction of projection there corresponds, in an involutory manner, a second direction which shall be called the *associated direction of projection.*

Let O be the position at time τ of the moving electron so that the Cartesian coordinates of O are $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$. Let OP be a direction of projection at O so that (l, m, n), the direction cosines of OP, satisfy the fundamental equations (A) of § 1. Let Q be the position which would have been occupied by the electron at a time $t>\tau$ if it had continued to move with velocity $v=(\xi', \eta', \zeta')$ along the tangent to its path at O. Thus, the coordinates of Q are

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* "Théorie des Surfaces," Tome 1, Chapters 2 and 3.

† Craig, Cole, Eiesland, *Amer. Journal*, Vol. XX (1898); Hatzidakis, *Amer. Journal*, Vol. XXIII (1901); Eiesland, *Amer. Journal*, Vol. XXVIII (1906); Matsumoto, *Memoirs of Coll. of Science*, Kyoto Imp. Univ., Vol. I, No. 7, January (1916).

$$(x_1, y_1, z_1)=(\xi+\xi'\overline{t-\tau},\ \eta+\eta'\overline{t-\tau},\ \zeta+\zeta'\overline{t-\tau}).$$

If now we make $OP=c(t-\tau)$ and draw a sphere with centre O and radius OP, then PQ will meet the sphere again in a point P'. *OP' is the associated direction of projection at O.* To prove this write $c(t-\tau)=r$ so that the coordinates of P are $(x_0, y_0, z_0)=(\xi+rl, \eta+rm, \zeta+rn)$ and PQ has the equations $x-x_1/x_1-x_0=y-y_1/y_1-y_0=z-z_1/z_1-z_0=\theta$, say. But $x_1-x_0=(t-\tau)(\xi'-cl)$ and so we have three equations of the type $x-\xi=\xi'(t-\tau)+\theta(\xi'-cl)(t-\tau)$. Let (x, y, z) be the coordinates of P'. P' being on PQ we have $x-\xi=(t-\tau)\{\xi'+\theta_1(\xi'-cl)\}$ where θ_1 is the value of the parameter θ giving P'. Since P' is on the sphere we have the equation

$$(x-\xi)^2+(y-\eta)^2(z-\zeta)^2=c^2(t-\tau)^2$$

and from these two equations we get

$$\theta_1^2\{c^2+v^2-2\frac{c}{r}(\bar{r}\,\bar{v})\}+2\theta_1\{v^2-\frac{c}{r}(\bar{r}\,\bar{v})\}+v^2-c^2=0.$$

The roots of this equation are $\theta_1=-1$ and $\theta_1=(c^2-v^2)/\left[c^2+v^2-2\frac{c}{r}(\bar{r}\,\bar{v})\right]$. The root $\theta_1=-1$ is the parameter of the point P itself. The second root is the parameter of P' and so if (L, M, N) are the direction cosines of OP', so that $x-\xi=c(t-\tau)L$, etc., we obtain the following three equations for L, M, N:

$$(\xi'-cL, \eta'-cM, \zeta'-cN)=h(v^2-c^2)(\xi'-cl, \eta'-cm, \zeta'-cn),$$

where $h^{-1}=c^2-2\frac{c}{r}(\bar{r}\,\bar{v})+v^2$. If as in § 1 we use the abbreviations

$$\lambda=c-\frac{1}{r}(\bar{r}\,\bar{v}),\ \mu=c^2-v^2,\ \text{then}\ h^{-1}=2c\lambda-\mu.$$

Write $\bar{R}$ to denote the vector OP', and Γ to denote the expression $c-\frac{1}{r}(\bar{R}\bar{v})$; then from the values obtained for (L, M, N) it is easy to deduce the equation $\Gamma=h\lambda\mu$.

In order to prove that OP' is a direction of projection we have merely to show that (L, M, N) satisfy the equations (A) of § 1. In other words, it must be shown that the three equations

$$\mu\frac{d}{d\tau}(L, M, N)=\Gamma(\xi'', \eta'', \zeta'')+P(\xi'-cL, \eta'-cM, \zeta'-cN)$$

are true, where $P=L\xi''+M\eta''+N\zeta''=\frac{1}{r}(\bar{R}\bar{v}')$ and, as before, $\bar{v}'$ is the vector acceleration of the electron at O. If now we multiply the equations (A) by ξ', η', ζ', respectively, and add they give $\mu(p+q)=\lambda\{cp+(\bar{v}\bar{v}')\}$ where $q=l'\xi'+m'\eta'+n'\zeta'$. This equation is equivalent to

$$-\mu\lambda'=\lambda\{cp+(\bar{v}\bar{v}')\}.$$

Again $\xi'-cL=-\mu h(\xi'-cl)$. Our object is to obtain L' so we differentiate this equation and it gives

$$cL'-\xi''=\mu h(\xi''-cl')-2h(\bar{v}\bar{v}')(\xi'-cl)-2\mu h^2(\xi'-cl)[c\lambda'+(\bar{v}\bar{v}')].$$

On combining the first two terms on the right-hand side of this equation with the term $-\xi''$ on the left-hand side we obtain, after substituting for l' its value from the equation (A) of § 1 and for λ' its value just given

$$cL'=h\{c\lambda\xi''-(\xi'-cl)[cp+2(\bar{v}\bar{v}')]\}+2h^2(\xi'-cl)\{c^2\lambda p+(c\lambda-\mu)(\bar{v}\bar{v}')\}$$

or

$$L'=h\{\lambda\xi'-(\xi'-cl)p\}+2h^2\lambda(\xi'-cl)\{cp-(\bar{v}\bar{v}')\}.$$

Again from the values already given for (L, M, N) we derive

$$cP=(\bar{v}\bar{v}')+\mu h\{(\bar{v}\bar{v}')-cp\} \quad \text{or} \quad P=h[2\lambda(\bar{v}\bar{v}')-\mu p] \quad \text{since } h^{-1}=2c\lambda-\mu.$$

Substituting these values of L', P in the equation $\mu L'=\Gamma\xi''+(\xi'-cL)P$ we see on dividing across by $\mu hp(\xi'-cl)$ that this equation is true if $1-2hc\lambda+\mu h=0$ which is so since $h^{-1}=2c\lambda-\mu$.

This proves the theorem stated. If we know a particular solution $OP=(l, m, n)$ of the equations (A) we can obtain a second solution $OP'=(L, M, N)$ in the manner described at the beginning of this paragraph. Darboux[*] has given a similar result for the equations (B) of § 1.

There is one particular case, when the associated direction coincides with the original direction of projection, in which we can not obtain by this method a new solution of the differential equations. In this case PP' is a tangent line to the sphere and so $l\xi'+m\eta'+n\zeta'=v\frac{c}{v}$ or $\lambda=0$. It is easy, as a matter of fact, to verify that if we put $\lambda=0$ in the equations for L, M, N that $L=l$, $M=m$, $N=n$. In this case the velocity v of the electron is greater than c, the velocity of light. From the equation for λ' we see that when $\lambda=0$ λ' also $=0$. In the cases of motion for which the solution has been worked out it will be seen that there are always a single infinity of solutions compatible with $\lambda=0$.

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[*] "Théorie des Surfaces," Tome 1 (1887), p. 23.

§ 3. GENERAL SOLUTION WHEN MOTION IS IN ONE PLANE.

Let θ be the angle between OP and OQ so that $v\cos\theta = l\xi' + m\eta' + n\zeta'$. It will now be seen that if the associated directions OP, OP' are distinct and equally inclined to OQ, the motion of the electron is in one plane and the solution can be found. OP and OP' are equally inclined to OQ if $\cos\theta = \dfrac{v}{c}$ for then PP' is perpendicular to OQ. Also since $\lambda = c - v\cos\theta$ we have $\mu = c\lambda$, and so $-c\lambda' = 2(\bar{v}\bar{v}')$. On substitution in the equation $-\mu\lambda' = \lambda\{cp + (\bar{v}\bar{v}')\}$ of § 2 we get $-c\lambda' = cp + (\bar{v}\bar{v}')$ or $cp = (\bar{v}\bar{v}')$. Substituting in the first of the equations (A) this gives $\mu(\xi'' - cl') = -(\xi' - cl)(\bar{v}\bar{v}')$ which is at once integrable, giving $\xi' - cl = a(c^2 - v^2)^{\frac{1}{2}}$ where a is a constant of integration. Similarly, we obtain the equations $\eta' - cm = b(c^2 - v^2)^{\frac{1}{2}}$, $\zeta' - cn = d(c^2 - v^2)^{\frac{1}{2}}$ where b and d are constants of integration. On multiplication by ξ', η', ζ', respectively, and addition, these three integrals give $a\xi' + b\eta' + d\zeta' = 0$ (remembering $v^2 = c^2 - c\lambda$).

Similarly, on using ξ'', η'', ζ'' as multipliers, we get, since $(\bar{v}\bar{v}') = cp$,

$$a\xi'' + b\eta'' + d\zeta'' = 0.$$

Thus, the velocity and the acceleration of the electron are in the same plane; hence, if a solution of the fundamental equations can be found which satisfies the equation $c\cos\theta = v$, the motion of the electron is in one plane. Conversely, if the motion is plane it is easy to pick out a solution for which $c\cos\theta = v$. The theorem of associated directions, then, gives a second solution which also satisfies $c\cos\theta = v$. Knowing these two solutions the integration of the Riccatian equation of § 1 is effected by means of a single quadrature.

Take the plane of motion to be $z = 0$ so that $\zeta' = 0$, $\zeta'' = 0$. It is evident that $cl = \xi'$, $cm = \eta'$ satisfy the first two equations. These values of l, m satisfy $c\cos\theta = v$. To determine n we have the condition $l^2 + m^2 + n^2 = 1$, giving $cn = \pm(c^2 - v^2)^{\frac{1}{2}}$. Either of these values of n satisfies the third equation which is, in this case, $\mu n' = -cn(l\xi'' + m\eta'')$. Hence, the solutions which give $c\cos\theta = v$ are $cl = \xi'$, $cm = \eta'$, $cn = \pm\mu^{\frac{1}{2}}$.

To obtain the general direction of projection it is necessary to solve the Riccatian equation. In the case of plane motion this is somewhat simplified and is $2(c^2 - v^2)\dfrac{d\sigma}{d\tau} = c\phi'' - 2i\sigma(\xi'\eta'' - \xi''\eta') - c\sigma^2\psi''$ where $\sigma(1+n) = l + im$. [illegible]tituting for l, m, n the values just given we have the two particular solu- [illegible]en by $c(1+\omega)\sigma_1 = \phi'$, $c(1-\omega)\sigma_2 = \phi'$ where $c\omega = (c^2 - v^2)^{\frac{1}{2}}$. It is [illegible] σ_0 satisfy the Riccatian equation.

The general solution of the Riccatian equation is, therefore,*

$$\sigma = \sigma_2 + \frac{\sigma_1 - \sigma_2}{1 + A e^{\int(\sigma_1 - \sigma_2) R d\tau}},$$

where A is a constant of integration and $2(c^2 - v^2)R = -c\psi''$. A is, in general, complex since σ is complex.

Thus, when the motion is plane, the problem is reduced to the integration

$$\int(\sigma_1 - \sigma_2) R d\tau \text{ or } c\int \frac{(\xi'' - i\eta'')\, d\tau}{(\xi' - i\eta')\sqrt{c^2 - \xi'^2 - \eta'^2}}.$$

The integration can always be effected if the scalar quantity v is constant, and in this case the integral is $[c/(c^2 - v^2)^{\frac{1}{2}}] \log \psi'$, so that

$$\sigma = \sigma_2 + \frac{\sigma_1 - \sigma_2}{1 + A\psi'^{\frac{1}{\omega}}}.$$

If $\eta = 0$, $\eta' = 0$, $\eta'' = 0$, it is evident that the integration can be carried out. This is the case of rectilinear motion which will be treated in detail. The remaining case of most interest is that of uniform motion in a circle and the solution is in this case given in terms of the trigonometric functions. In general, the explicit determination of the equations of the lines of force depends on the evaluation of the integral given above.

§ 4. PARTICULAR CASES OF PLANE MOTION.

(a) *Rectilinear Motion.*

Taking the x-axis as the line of motion we see from § 3 that the solution depends on the integration of $c\int \frac{\xi'' d\tau}{\xi'\sqrt{c^2 - \xi'^2}}$. In this particular case, however, the general solution can be obtained at once directly from the Riccatian equation which becomes $2(c^2 - \xi'^2)\sigma' = c\xi''(1 - \sigma^2)$.

This gives $\log \frac{1+\sigma}{1-\sigma} = \frac{1}{2} \log \frac{c+v}{c-v} + \log A$ (for $\xi' = v$). Write $A = \alpha + i\beta$ and we have $\frac{1+\sigma}{1-\sigma} = (\alpha + i\beta)\left(\frac{c+v}{c-v}\right)^{\frac{1}{2}}$. Whence,

$$\sigma = \{(\alpha + i\beta)(c^2 - v^2)^{\frac{1}{2}} - (c - v)\} \div \{(\alpha + i\beta)(c^2 - v^2)^{\frac{1}{2}} + (c - v)\}.$$

From this point we must distinguish between the cases $v \lessgtr c$.

Linear Motion When $v < c$.

$(c^2 - v^2)^{\frac{1}{2}}$ is real and so if s denotes the conjugate of σ we have

$$s\{(\alpha - i\beta)(c^2 - v^2)^{\frac{1}{2}} + (c - v)\} = (\alpha - i\beta)(c^2 - v^2)^{\frac{1}{2}} - (c$$ [illegible]

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* Cohen, "Differential Equations," p. 175.

Forming the product σs we find

$$\sigma s\{(\alpha^2+\beta^2)(c^2-v^2)+(c-v)^2+2(c-v)(c^2-v^2)^{\frac{1}{2}}\alpha\}$$
$$=(\alpha^2+\beta^2)(c^2-v^2)+(c-v)^2-2(c-v)(c^2-v^2)^{\frac{1}{2}}\alpha,$$

and since $(1+\sigma s)l=\sigma+s$, $(1+\sigma s)im=\sigma-s$, $(1+\sigma s)n=1-\sigma s$ we have

$$Rl=(c+v)(\alpha^2+\beta^2)-(c-v),\ Rm=2\beta(c^2-v^2)^{\frac{1}{2}},\ Rn=2\alpha(c^2-v^2)^{\frac{1}{2}},$$

where $R=(c+v)(\alpha^2+\beta^2)+(c-v)$.

The lines of force at time t are, with these values for l, m, n,

$$x=\xi+c(t-\tau)l,\ y=c(t-\tau)m,\ z=c(t-\tau)n,$$

and α, β are two arbitrary real constants.

It is seen that $\alpha y=\beta z$ or the lines of force lie in planes through the direction of motion a result evident, à priori, from symmetry. If v is constant, so also are l, m, n, and we get the well-known result* that in the case of uniform motion in a right line the line of electric force at time t through any point x, y, z is got by joining x, y, z to the position of the electron at that time t. Since the lines of force are symmetrical round the direction of motion it is sufficient to consider the lines in the plane $z=0$, so that $\alpha=0$. For instance, in the case of simple harmonic motion we may write $\xi=a\cos p\tau$ and the lines of electric force in the xy-plane are $x=a\cos p\tau+c(t-\tau)l$, $y=c(t-\tau)m$, where

$$l\{\beta^2(c-ap\sin p\tau)+(c+ap\sin p\tau)\}=\beta^2(c-ap\sin p\tau)-(c+ap\sin p\tau),$$
$$m\{\beta^2(c-ap\sin p\tau)+(c+ap\sin p\tau)\}=2\beta(c^2-a^2p^2\sin^2 p\tau)^{\frac{1}{2}}.$$

The single arbitrary constant β gives the single infinity of lines in the plane. To get an idea of the shape of the lines take the particular line $\beta=1$. Then, $cl=-ap\sin p\tau$, $cm=(c^2-a^2p^2\sin^2 p\tau)^{\frac{1}{2}}$; put $t=0$ and $\tau=-\tau'$ so that τ' takes positive values. Then, $x=a\cos p\tau'+c\tau' l$, $y=c\tau' m$, where $cl=ap\sin p\tau'$. Thus, $x=a(\cos p\tau'+p\tau'\sin p\tau')$, $y=\tau'(c^2-a^2p^2\sin^2 p\tau')^{\frac{1}{2}}$, where τ' takes positive values. If a is small we have at distances far from the origin

$$x^2+y^2=c^2\tau'^2,\quad x=a(\cos p\tau'+p\tau'\sin p\tau').$$

The line of force is of an oscillatory character with increasing amplitude about the line $x=0$.

Case of Uniformly Accelerated Motion in a Straight Line.

Let g be the uniform acceleration and suppose the electron to start from rest at the origin at time $\tau=0$. Then, $\xi=\frac{1}{2}g\tau^2$, $v=g\tau$ so that

$Rl=\beta^2(c+g\tau)-(c-g\tau)$, $Rm=2\beta(c^2-g^2\tau^2)^{\frac{1}{2}}$, where $R=\beta^2(c+g\tau)+(c-g\tau)$. Consider the particular line $\beta=1$. Then, $R=2c$ and $cl=g\tau$ and the equations to the line of force are $x=\frac{1}{2}g\tau^2+(t-\tau)g\tau$, $y=(t-\tau)(c^2-g^2\tau^2)^{\frac{1}{2}}$.

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* J. J. Thomson, *Phil. Mag.*, Vol. XI (1881), p. 229; O. Heaviside, *Phil. Mag.*, Vol. XXVII (1889), p. 324.

These equations take a somewhat simpler form if we transfer the origin to the position of the electron at time t. Then, $x=-\frac{1}{2}g(t-\tau)^2$ and taking $t-\tau$ as the parameter s we have $x=-\frac{1}{2}gs^2$; $y^2=s^2\{c-g(t-s)\}\{c+g(t-s)\}$, where s takes values from t to zero. On eliminating s this is a quartic curve

$$\left\{y^2+4x^2+\frac{2x}{g}(c^2-g^2t^2)\right\}^2+8gt^2x^3=0.$$

It is easy to see how the lines of force due to a point charge in rectilinear motion are altered in shape when the moving charge is accelerated or retarded. For example, imagine the charge to have been moving with uniform velocity for a long time and, then, to be suddenly affected with a retardation g, and consider, as above, the line $\beta=1$ for which $l=\frac{v}{c}$; the electron, being at $\tau=0$ at $x=0$, is reduced to rest at time $\tau=\frac{v}{g}$. Consider the line of force at this time. The part given by the aggregate of particles imagined projected at times $\tau<0$ is a straight line which would pass if produced through the point $x=\frac{v^2}{g}$. At the point where this straight line meets the line $cl=v$ through the origin, the line of force bends and passes through the point $x=\frac{v^2}{2g}$ in a direction perpendicular to the line of motion. At any later time t the line of force is made up of three parts: (1) A straight line part which goes, if produced, through $x=vt$, (2) a bent part beginning where this line meets the sphere $x^2+y^2+z^2=c^2t^2$ and ending at $x=\frac{v^2}{2g}$, $y=c\left(t-\frac{v}{g}\right)$, and (3) the part of the line $x=\frac{v^2}{2g}$ from this point to $y=0$. A similar result holds for the lines got from other values of β. It is evident that if the electron is stopped instantaneously we can get the lines of force in the following way: Draw the pencil of lines through $x=vt$, $y=0$. The sphere with centre at origin and radius ct cuts these lines in points where the lines of force bend suddenly, the remaining portions being the radii of this sphere. This explains how the discontinuity due to the sudden stoppage of an electron spreads out in a spherical wave. The commonly accepted theory of Röntgen rays is that they are pulses of this type due to the sudden stoppage of cathode particles. It is important to notice that in all cases the direction of projection is determined by the velocity at the point of projection and is independent of its acceleration.

If we give particular values to v, g, c, it is easy to draw the bent portion of the line of force due to the retarded part of the motion. For instance, take $v=4$, $g=c=8$; then the electron is reduced to rest in time $\tau=\frac{1}{2}$, its position at that time being $x=1$. Taking $\beta=1$ we have $l=\frac{v}{c}$. First take $t=\frac{1}{2}$, so that we are drawing the line for the instant when the electron is reduced to rest. Then, if $0<\tau<\frac{1}{2}$, $l=\frac{4-8\tau}{8}=\frac{1}{2}-\tau$, $\xi=4(\tau-\tau^2)$. The distance from ξ to the corresponding point on the line of force is $c(t-\tau)=4(1-2\tau)=8l$. It is found that the bent portion of the line of force resembles closely the part in the positive quadrant of an ellipse whose axes are in the ratio of $10:34$. Having drawn the line of force at time $t=\frac{1}{2}$ we get the form of the same line at any subsequent time by producing the radii vectores from ξ to the corresponding point on the line of force at time $t=\frac{1}{2}$, a constant distance. This method of deducing the successive positions of a given line of force from the position at any one time follows from the equations of the lines of force and is applicable in the general case.

Motion in a Straight Line When $v>c$.

In the equation

$$\{(\alpha+i\beta)(c^2-v^2)^{\frac{1}{2}}+(c-v)\}\sigma=\{(\alpha+i\beta)(c^2-v^2)^{\frac{1}{2}}-(c-v)\}$$

$(c^2-v^2)^{\frac{1}{2}}$ is now a pure imaginary. Writing it in form $i(v^2-c^2)^{\frac{1}{2}}$ we see that s, the conjugate of σ, is given by the equation

$$\{(\alpha-i\beta)(v^2-c^2)^{\frac{1}{2}}+i(c-v)\}s=(\alpha-i\beta)(v^2-c^2)^{\frac{1}{2}}-i(c-v).$$

From these we obtain as before

$$Rl=(\alpha^2+\beta^2)(v+c)-(v-c),\quad Rm=2\alpha(v^2-c^2)^{\frac{1}{2}},\quad Rn=2\beta(v^2-c^2),$$

where $R=(\alpha^2+\beta^2)(v+c)+(v-c)$.

The lines of force lie in planes through the axis of x; putting $\beta=0$ we obtain the lines in the plane xy. It will be noticed that, for these, if $\alpha=1$, $l=\frac{c}{v}$, and so $\lambda=0$. Hence, in the case of rectilinear motion with constant velocity $v>c$ the generators of the cone $l=\frac{c}{v}$ are directions of projection. The cone with vertex at $x=\xi(t)$ and semi-vertical angle $\alpha=\sin^{-1}\frac{c}{v}$ accordingly separates the regions where there are lines of electric force from those regions, where there are no lines of force. The generators of the cone are lines of electric force. This is a known result.* If v is not constant, these successive cones envelope a surface of revolution, the meridians on which are lines of electric force, and which enclose all the lines of force.

Motion in a Circle with Uniform Velocity.

Let p be the constant angular velocity and d the radius of the circle. Then, the plane of the circle being $z=0$, and the centre of the circle being the origin of coordinates, we have $\xi=a\cos p\tau$, $\eta=a\sin p\tau$, $\zeta=0$; whence, $\phi=ae^{ip\tau}$ and the particular solutions of the Riccatian equation are σ_1, σ_2, where

$$c(1+\omega)\sigma_1=iape^{ip\tau},\ c(1-\omega)=iape^{ip\tau} \text{ and } c^2\omega^2=c^2-v^2=c^2-a^2p^2.$$

On substituting these values in the general solution for plane motion it is found that

$$\sigma=ie^{ip\tau}\left[\frac{c(1+\omega)}{ap}+\frac{1}{(\alpha+i\beta)e^{\frac{-ip\tau}{\omega}}-ap/2c\omega}\right]^{\dagger}.$$

If $v<c$, ω is real; if $v>c$, ω is a pure imaginary, so there will be two distinct solutions according as $v\gtrless c$.

(a) *Case When $v<c$.*

Since ω is real, s is got by merely changing the sign of i wherever it appears in the expression for σ. Adding we have

$$\sigma+s=-\frac{2c(1+\omega)}{ap}\sin p\tau-2\frac{P\sin p\tau+Q\cos p\tau}{R},$$

where P, Q, R are defined by the equations

$$\begin{aligned}
P&=\alpha\cos p\tau/\omega+\beta\sin p\tau/\omega-ap/2c\omega,\\
Q&=\alpha\sin p\tau/\omega-\beta\cos p\tau/\omega,\\
R&=(\alpha^2+\beta^2)+a^2p^2/4c^2\omega^2-ap/c\omega\,(\alpha\cos p\tau/\omega+\beta\sin p\tau/\omega).
\end{aligned}$$

Similarly, on subtracting we find

$$-i(\sigma-s)=2c(1+\omega)/ap\cos p\tau+2\frac{P\cos p\tau-Q\sin p\tau}{R},$$

and on multiplication

$$\sigma s=\frac{c^2(1+\omega)^2}{a^2p^2}+\frac{\frac{2c(1+\omega)}{ap}P+1}{R};$$

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* O. Heaviside, "Electrical Papers."

† The integral $\int\frac{\psi''}{\psi'\omega}d\tau$ of § 3 is, in this case, $\frac{1}{\omega}\log\psi'$ and $\psi=ae^{-ip\tau}$.

from these equations we obtain $R(1+\sigma s)=S$, $R(1-\sigma s)=T$, where S and T are defined by the equations

$$S=(\alpha^2+\beta^2)\frac{2}{1-\omega}+\frac{1-\omega}{2\omega^2}-\frac{2ap}{c\omega}\left(\alpha\cos\frac{p\tau}{\omega}+\beta\sin\frac{p\tau}{\omega}\right),$$

$$T=(\alpha^2+\beta^2)\frac{2\omega}{\omega-1}+\frac{1-\omega}{2\omega},$$

and hence l, m, n are given by

$$l\equiv\frac{\sigma+s}{1+\sigma s}=-\frac{ap}{c}\sin p\tau+\frac{2}{S}\left\{\left(P+\frac{ap}{2c\omega}\right)\omega\sin p\tau-Q\cos p\tau\right\},$$

$$m\equiv-i\frac{\sigma-s}{1+\sigma s}=\frac{ap}{c}\cos p\tau-\frac{2}{S}\left\{\left(P+\frac{ap}{2c\omega}\right)\omega\cos p\tau+Q\sin p\tau\right\},$$

$$n=\frac{1-\sigma s}{1+\sigma s}=\frac{T}{S},$$

and with these values of l, m, n the parametric equations to the lines of force are $x=a\cos p\tau+c(t-\tau)l$, $y=a\sin p\tau+c(t-\tau)m$, $z=c(t-\tau)n$.

By a suitable choice of the constants α, β so as to get particular lines of force these expressions may be considerably simplified. Thus, if we put $\alpha=0$, $\beta=0$, we have $l=-(ap/c)\sin p\tau$, $m=(ap/c)\cos p\tau$, $n=\omega$, and the line of force is given by the equations

$$x=a\cos p\tau-ap(t-\tau)\sin p\tau,\quad y=a\sin p\tau+ap(t-\tau)\cos p\tau,\quad z=c\omega(t-\tau).$$

It is obvious that it lies on the hyperboloid of revolution

$$(x^2+y^2)/a^2-p^2z^2/c^2\omega^2=1$$

which contains the circle of motion. Writing this equation in the form

$$(x/a-pz/c\omega)(x/a+pz/c\omega)=(1+y/a)(1-y/a),$$

we see that the directions of projection are generators of the hyperboloid. The form of the line of force is evident if we consider its projection on the plane of motion. The equations of the projection are

$$x=a\cos p\tau-ap(t-\tau)\sin p\tau,\quad y=a\sin p\tau+ap(t-\tau)\cos p\tau,\quad z=0;$$

giving

$$(x-a\cos p\tau)+(y-a\sin p\tau)\tan p\tau=0$$

$$\text{and }(x-a\cos p\tau)^2+(y-a\sin p\tau)^2=a^2p^2(t-\tau)^2.$$

These are the equations to an involute of the circle $x^2+y^2=a^2$; the involute meeting the circle at the position of the electron at time $\tau=t$. To obtain, then, a line of electric force due to an electron moving with velocity $v<c$ in a circle we unwrap a string which is wound round the circle, the tracing point at the end of the string leaving the circle at the position of the electron at time t. Having obtained this involute we project it by lines parallel to the axis of z on the hyperboloid $(x^2+y^2)/a^2-p^2z^2/c^2\omega^2=1$.

Lines of Force in the Plane of Motion.

To obtain these put $n=0$. Then, $T=0$ or $(\alpha^2+\beta^2)=(\overline{1-\omega}/2\omega)^2$, and so we may write $2\omega\alpha=(1-\omega)\cos\theta$, $2\omega\beta=(1-\omega)\sin\theta$ and these give $\omega^2 S=(1-\omega)k$ where $k=1-(ap/c)\cos(p\tau/\omega-\theta)$, and, finally,

$$kl=\sin p\tau\cos(p\tau/\omega-\theta)-(ap/c)\sin p\tau-\omega\cos p\tau\sin(p\tau/\omega-\theta),$$
$$km=-\cos p\tau\cos(p\tau/\omega-\theta)+(ap/c)\cos p\tau-\omega\sin p\tau\sin(p\tau/\omega-\theta).$$

The single constant θ gives the single infinity of lines in the plane of motion; from symmetry we need only consider a particular value of θ, say $\theta=0$; then,

$$kl=\quad\sin p\tau\cos p\tau/\omega-(ap/c)\sin p\tau-\omega\cos p\tau\sin p\tau/\omega,$$
$$km=-\cos p\tau\cos p\tau/\omega+(ap/c)\cos p\tau-\omega\sin p\tau\sin p\tau/\omega,$$

where the Doppler factor k is now $1-(ap/c)\cos p\tau/\omega$.

Circular Motion When $v>c$.

ω is now a pure imaginary and so it may be written $=i\theta$ where θ is real. The functions σ and s are given by the equations

$$\sigma=\quad ie^{ip\tau}\left\{\frac{c(1+i\theta)}{ap}+\frac{1}{(\alpha+i\beta)e^{-p\tau/\theta}+iap/2c\theta}\right\},$$
$$s=-ie^{-ip\tau}\left\{\frac{c(1-i\theta)}{ap}+\frac{1}{(\alpha-i\beta)e^{-p\tau/\theta}-iap/2c\theta}\right\},$$

whence

$$D(\sigma+s)=-D\left(\frac{2c}{ap}\sin p\tau+\frac{2c\theta}{ap}\cos p\tau\right)+2e^{-p\tau/\theta}\{\beta\cos p\tau-\alpha\sin p\tau\}+\frac{ap}{c\theta}\cos p\tau,$$

where

$$D=\alpha^2e^{-2p\tau/\theta}+\left\{\beta e^{-p\tau/\theta}+\frac{ap}{2c\theta}\right\}^2.$$

Similarly,

$$-iD(\sigma-s)=\frac{2c}{ap}D(\cos p\tau-\theta\sin p\tau)+2e^{-p\tau/\theta}\{\alpha\cos p\tau+\beta\sin p\tau\}+\frac{ap}{c\theta}\sin p\tau$$
$$\text{and }\sigma s=1+\frac{2c}{apD}e^{-p\tau/\theta}(\alpha-\theta\beta).$$

Using these values we obtain, after some reduction, and on writing

$$F=(\alpha^2+\beta^2)e^{-2p\tau/\theta}-a^2p^2/4c^2\theta^2,$$
$$G=(\alpha^2+\beta^2)e^{-2p\tau/\theta}+a^2p^2/4c^2\theta^2+\frac{c}{ap}e^{-p\tau/\theta}(\alpha+\beta/\theta),$$

the equations

$$l=-\frac{c}{ap}\left[\sin p\tau+\frac{\theta\{F\cos p\tau+\frac{c}{ap}e^{-p\tau/\theta}(\theta\alpha+\beta)\sin p\tau\}}{G}\right],$$
$$m=\quad\frac{c}{ap}\left[\cos p\tau-\frac{\theta\{F\sin p\tau-\frac{c}{ap}e^{-p\tau/\theta}(\theta\alpha+\beta)\cos p\tau\}}{G}\right],$$
$$n=\quad\frac{c}{ap}e^{-p\tau/\theta}\frac{(\theta\beta-\alpha)}{G}.$$

On Inequalities of Certain Types in General Linear Integral Equation Theory.

By Mary Evelyn Wells.

Part I.

§ 1. *Introduction.*

In the theory of the classical linear integral equation

$$\xi(s)=\eta(s)-\lambda\int_a^b\varkappa(s,t)\eta(t)\,dt$$

we find the inequality of Schwarz*

$$\int_a^b[\xi(s)]^2ds\int_a^b[\eta(s)]^2ds-\left[\int_a^b\xi(s)\eta(s)\,ds\right]^2\geqq 0.$$

In the theory of the general linear integral equation

$$\xi(s)=\eta(s)-\lambda J\varkappa(s,t)\eta(t), \tag{1}$$

in which appear more general functions and operator, explained later, E. H. Moore has found, among other inequalities, the analogue of the Schwarz inequality given above, namely,

$$J\xi\bar{\xi}J\eta\bar{\eta}-J\xi\bar{\eta}J\eta\bar{\xi}\geqq 0, \tag{2}$$

where — denotes the conjugate imaginary. We proceed to the discussion of such inequalities.

In a memoir "On the Foundations of the Theory of Linear Integral Equations",† E. H. Moore has given the basis Σ_5, and system of postulates, of a theory of the general linear integral equation (1) which we shall write

$$\xi=\eta-\lambda J\varkappa\eta. \tag{1}$$

E. H. Moore has defined the properties of class $\mathfrak{M}\equiv[\mu]$ of functions μ on a general range $\mathfrak{P}\equiv[p]$ to the class $\mathfrak{A}\equiv[a]$ of all real or complex numbers, properties of class $(\mathfrak{M}\mathfrak{M})_*$ to which the kernel function $\varkappa$ belongs, and also the properties of the functional operation J, necessary for the theory of the general linear integral equation. For the convenience of the reader some of the definitions are given here.

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* Heywood-Fréchet, "L'Equation de Fredholm."

† *Bulletin of the American Mathematical Society*, Ser. 2, Vol. XVIII.

$\mathfrak{M}$ is linear (L), in notation $\mathfrak{M}^L$, in case

$$\mathfrak{M}=\mathfrak{M}_L\equiv[\text{all } \mu=a_1\mu_1+a_2\mu_2+\dots\dots+a_n\mu_n].$$

$\mathfrak{M}$ is real (R), in notation $\mathfrak{M}^R$, in case the class $\mathfrak{M}$ is the same as the class of conjugate elements,

$$\mathfrak{M}=\overline{\mathfrak{M}}.$$

J is an operator of binary quality eliminating both the arguments s and t when it operates on such a function as $\mu_1(s)\mu_2(t)$ or $\varkappa(s,t)$ giving a number of class $\mathfrak{A}$. That J operates on a function $\varkappa$ to give a number a, we indicate by the notation $J^{\text{on } \mathfrak{K} \text{ to } \mathfrak{A}}$. The notation $J_{(s,t)}\varkappa(s,t)$ will be replaced, throughout, by $J\varkappa$, and $J_{(s,t)}\mu_1(s)\mu_2(t)$ by $J\mu_1\mu_2$. That $J\mu_1\mu_2$ is in general different from $J\mu_2\mu_1$ is seen by the examples of J used in § 4.

The operator J is linear (L), in notation J^L, in case

$$a_1\varkappa_1+a_2\varkappa_2=\varkappa \quad \text{implies} \quad a_1J\varkappa_1+a_2J\varkappa_2=J\varkappa.$$

The operator J is hermitian (H), in notation J^H, in case

$$\overline{J\mu_1\mu_2}=J\bar{\mu}_2\bar{\mu}_1.$$

The operator J is positive (P), in notation J^P, in case for every function μ of $\mathfrak{M}$

$$J\bar{\mu}\mu\geqq 0.$$

Using the foundation

$$\Sigma=(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M}^{LR};\ \mathfrak{K}\equiv(\mathfrak{M}\mathfrak{M})_*;\ J^{\text{on } \mathfrak{K} \text{ to } \mathfrak{A}\cdot LHP}),$$

it is the purpose of the first part of this paper to show the existence of certain inequalities of the type

$$\sum_{ijkl}^{(12)} a_{ijkl}J_{ij}J_{kl}\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi^{\mathfrak{M}};\ \eta^{\mathfrak{M}};\ J^{LHP}); \tag{3}$$

where $ijkl$ is an arrangement of the digits 1234 which refer to the functions in the first, second, third and fourth positions in the product $\xi\bar{\xi}\eta\bar{\eta}$ of which the arguments are omitted for convenience. For instance the term $a_{1824}J_{18}J_{24}\xi\bar{\xi}\eta\bar{\eta}$ indicates $a_{1824}J\xi\eta J\bar{\xi}\bar{\eta}$. Of the twenty-four terms to correspond with the twenty-four arrangements of the subscripts 1234 in $J_{ij}J_{kl}$ only twelve are distinct since $J_{ij}J_{kl}=J_{kl}J_{ij}$. Thus, the inequality is one of twelve terms as indicated by the notation in (3). For definiteness the digit 1 will be kept in the first place or the last place in the arrangements of 1234, and the other digits will be in dictionary order. With this plan the coefficients of the twelve distinct terms are as indicated in table (7). In further explanation of (3) it should be said that the parenthesis indicates that the inequalities are valid for all functions ξ and η of the class $\mathfrak{M}$ and all operators which have the properties *LHP*. Since throughout this paper ξ and η will always be considered of the class $\mathfrak{M}$, and each J will have the properties *LHP*, these superscripts will often be omitted, and the superscripts used will usually indicate still more special properties of ξ, η, and J.

Further, it will be shown that the inequalities exhibited form a fundamental set for the two cases, (1°) where $J = \breve{J}$, *i. e.*, $J_{(s,t)}\varkappa(s,t) = J_{(s,t)}\varkappa(t,s)$[*], (2°) where ξ and η are real. By saying that certain inequalities form a fundamental set, it is meant that any inequality of type (3) which satisfies conditions shown to be necessary, can be expressed as a sum of positive or zero multiples of the inequalities forming the fundamental set.

The three following inequalities, containing a numerical parameter u, together with those obtainable from these three by transformations on ξ and η, form the fundamental set (1°) when $J = \breve{J}$, (2°) when ξ and η are real.

$$\{(1 + u\bar{u})J_{12}J_{34} + (u + \bar{u})J_{14}J_{32}\}\xi\bar{\xi}\eta\bar{\eta} \geqq 0 \qquad (\xi^{\mathfrak{M}};\ \eta^{\mathfrak{M}};\ u^{\mathfrak{A}};\ J^{LHP}). \qquad (4)$$

$$\{J_{12}J_{43} + \bar{u}J_{14}J_{23} + uJ_{32}J_{41} + u\bar{u}J_{34}J_{21}\}\xi\bar{\xi}\eta\bar{\eta} \geqq 0 \qquad (\xi^{\mathfrak{M}};\ \eta^{\mathfrak{M}};\ u^{\mathfrak{A}};\ J^{LHP}). \qquad (5)$$

$$\{J_{13}J_{42} + \bar{u}J_{13}J_{24} + uJ_{42}J_{31} + u\bar{u}J_{24}J_{31}\}\xi\bar{\xi}\eta\bar{\eta} \geqq 0 \qquad (\xi^{\mathfrak{M}};\ \eta^{\mathfrak{M}};\ u^{\mathfrak{A}};\ J^{LHP}). \qquad (6)$$

§ 2. *Proofs of the Inequalities.*

A special case of the first inequality (4) is the generalized Schwarz inequality (2) stated and proved by E. H. Moore. The proof of inequality (4) was exhibited by E. H. Moore in class lectures given in January, 1914. For the derivation of (4) and (5) E. H. Moore proved that when a set of classes of functions $\mathfrak{M}', \mathfrak{M}'', \ldots\ldots, \mathfrak{M}^{(n)}$, each of which is linear (L) and real (R), is used to form the class $\mathfrak{M} \equiv (\mathfrak{M}'\mathfrak{M}'' \ldots\ldots \mathfrak{M}^{(n)})_L$,[†] the resulting class is linear (L) and real (R); and that the corresponding functional operation $J \equiv J'J'' \ldots\ldots J^{(n)}$, in which each $J^{(i)}$ has the properties *LP*, itself has the properties *LP*. Accordingly, when $m = 2$, we have

$$J'_{13}J''_{24}(\xi\eta + u\eta\xi)\overline{(\xi\eta + u\eta\xi)} \geqq 0 \qquad (\xi^{\mathfrak{M}};\ \eta^{\mathfrak{M}};\ u^{\mathfrak{A}};\ J'^{LHP};\ J''^{LHP}).$$

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[*] E. H. Moore, *Bulletin of the American Mathematical Society*, April, 1912.

[†] "On the Fundamental Functional Operation of a General Theory of Linear Integral Equations" published in the "Proceedings of the Fifth International Congress of Mathematicians, Cambridge, August, 1912."

Since the two operators are not necessarily the same, two inequalities are obtained, (4) by using JJ and (5) by using $J\breve{J}$.

To prove (6) it is only necessary to write the inequality as a product of a number and its conjugate:

$$(J\xi\eta+uJ\eta\xi)\,\overline{(J\xi\eta+uJ\eta\xi)}\geqq 0 \qquad (\xi^{\mathfrak{M}};\ \eta^{\mathfrak{M}};\ u^{\mathfrak{A}};\ J^{LHP}).$$

Table (7), in which the inequalities are tabulated by means of their coefficients a_{ijkl} and labeled $A_1, A_2, \ldots, A_{12}$ for convenience in reference, shows the complete set of inequalities (4), (5), (6) and those obtainable from (4), (5) and (6) by transformations on ξ and η. The transformations used may be indicated by the usual transformation notation:* $(\xi\bar{\xi})$, $(\xi\bar{\xi})(\eta\bar{\eta})$, $(\eta\bar{\eta})$.

	a_{1234}	a_{1243}	a_{1324}	a_{1342}	a_{1423}	a_{1432}	a_{2341}	a_{2431}	a_{3241}	a_{3421}	a_{4231}	a_{4321}		
$A_{1,u}$	$1+u\bar{u}$	0	0	0	0	$u+\bar{u}$	0	0	0	0	0	0	(4′)	(7)
$A_{2,u}$	0	$1+u\bar{u}$	0	$u+\bar{u}$	0	0	0	0	0	0	0	0		
$A_{3,u}$	0	0	0	0	0	0	0	$u+\bar{u}$	0	$1+u\bar{u}$	0	0		
$A_{4,u}$	0	0	0	0	0	0	$u+\bar{u}$	0	0	0	0	$1+u\bar{u}$		
$A_{5,u}$	0	1	0	0	$\bar{u}$	0	0	0	u	$u\bar{u}$	0	0	(5′)	
$A_{6,u}$	1	0	$\bar{u}$	0	0	0	0	0	0	0	u	$u\bar{u}$		
$A_{7,u}$	0	$u\bar{u}$	0	0	$\bar{u}$	0	0	0	u	1	0	0		
$A_{8,u}$	$u\bar{u}$	0	$\bar{u}$	0	0	0	0	0	0	0	u	1		
$A_{9,u}$	0	0	$\bar{u}$	1	0	0	0	$u\bar{u}$	0	0	u	0	(6′)	
$A_{10,u}$	0	0	$\bar{u}$	$u\bar{u}$	0	0	0	1	0	0	u	0		
$A_{11,u}$	0	0	0	0	$\bar{u}$	$u\bar{u}$	1	0	u	0	0	0		
$A_{12,u}$	0	0	0	0	$\bar{u}$	1	$u\bar{u}$	0	u	0	0	0		

§ 3. *The Twelve Inequalities* (7) *Form a Fundamental Set When* $J=\breve{J}$.

PROOF. Since the operator J is self-transpose $(J=\breve{J})$, $J\mu_1\mu_2=J\mu_2\mu_1$. As an example of such an operator may be mentioned the classical unary J of the classical instances II_n, III, IV,† which, in the respective instances, is

$$\sum_{p=1}^{n},\quad \sum_{p=1}^{\infty},\quad \int_a^b dp.$$

When the self-transpose J operates upon $\xi\bar{\xi}\eta\bar{\eta}$, we have

$$\left.\begin{aligned} J_{12}J_{34}&=J_{12}J_{43}=J_{34}J_{21}=J_{43}J_{21},\\ J_{13}J_{24}&=J_{13}J_{42}=J_{24}J_{31}=J_{42}J_{31},\\ J_{14}J_{23}&=J_{14}J_{32}=J_{23}J_{41}=J_{32}J_{41}.\end{aligned}\right\} \qquad (8)$$

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* Cf. Cajori, "Theory of Equations."

† E. H. Moore, "On the Fundamental Functional Operation of a General Theory of Linear Integral Equations."

Consequently, we have a grouping of coefficients a_{ijkl} which we designate

$$\left.\begin{aligned} b_1 &= a_{1234}+a_{1243}+a_{3421}+a_{4321}, \\ b_2 &= a_{1324}+a_{1342}+a_{2431}+a_{4231}, \\ b_3 &= a_{1423}+a_{1432}+a_{2341}+a_{3241}. \end{aligned}\right\} \tag{9}$$

In this notation the general inequality which we wish to build from the given inequalities (7) is

$$(b_1J_{12}J_{34}+b_2J_{13}J_{24}+b_3J_{14}J_{23})\xi\bar{\xi}\eta\bar{\eta} \geqq 0 \qquad (\xi;\ \eta;\ J=\check{J}). \tag{10}$$

In this field, $J=\check{J}$, the given inequalities (7), with grouping of coefficients in accordance with (9), reduce to four distinct inequalities B_1, B_2, B_3, B_4, tabulated by means of the coefficients b_1, b_2, b_3, as follows:

	b_1	b_2	b_3
$B_{1,u}$	$1+u\bar{u}$	0	$u+\bar{u}$
$B_{2,u}$	$1+u\bar{u}$	$u+\bar{u}$	0
$B_{3,u}$	0	$1+u\bar{u}+u+\bar{u}$	0
$B_{4,u}$	0	0	$1+u\bar{u}+u+\bar{u}$

(11)

To build the general inequality (10), as the sum of positive or zero multiples of the fundamental inequalities (11), it is evident that the multipliers must be expressed in terms of the coefficients b_1, b_2, and b_3. First, then, we determine certain necessary conditions involving b_1, b_2 and b_3, which are certain expressions in b_1, b_2, b_3 found to be necessarily positive or zero numbers. By suitable choice of certain of these expressions we build (10) as a sum of positive or zero multiples of the inequalities (11).

For the determination of necessary conditions on b_1, b_2, b_3 we need only to use a binary operator, and use for ξ and η the vectors (x_1, x_2) and (y_1, y_2). In this binary algebraic case, the linearity of J demands that $J\xi\eta$ have the form

$$j_{11}x_1y_1+j_{12}x_1y_2+j_{21}x_2y_1+j_{22}x_2y_2.$$

Therefore, the binary operator may be written

$$J=\begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix},$$

which effects the ordinary matricial combination with the product $\xi\eta$ which is

$$\begin{pmatrix} x_1y_1, & x_1y_2 \\ x_2y_1, & x_2y_2 \end{pmatrix}.$$

And, conversely, this form of J implies the linearity of J.

The fact that this operator is hermitian (H)* is expressed in the equality

$$\overline{j_{11}x_1y_1+j_{12}x_1y_2+j_{21}x_2y_1+j_{22}x_2y_2}=j_{11}\overline{x_1y_1}+j_{21}\overline{x_1y_2}+j_{12}\overline{x_2y_1}+j_{22}\overline{x_2y_2} \qquad (x_1,\, x_2,\, y_1,\, y_2),$$

for which it is necessary and sufficient that the matrix (j_{rs}) be hermitian, viz., $\bar{j}_{rs}=j_{sr}$, $(r=1,\,2;\; s=1,\,2)$.

The property P of J^* demands

$$j_{11}x_1\bar{x}_1+j_{12}x_1\bar{x}_2+j_{21}x_2\bar{x}_1+j_{22}x_2\bar{x}_2 \geqq 0 \qquad (x_1,\, x_2).$$

For the positiveness of this hermitian form the necessary and sufficient conditions are

$$j_{11}\geqq 0, \qquad j_{22}\geqq 0, \qquad j_{11}j_{22}-j_{12}j_{21}\geqq 0.$$

Hence the matrix (j_{rs}) is both hermitian and positive. In general, it is true that such an operator, binary or n-ary, is hermitian (H) if, and only if, the matrix (j_{rs}) is hermitian; and such an operator is positive (P) if, and only if, each principal minor is positive or zero.

The binary operator $\begin{pmatrix} j_{11} & 0 \\ 0 & j_{22} \end{pmatrix}$ affords, as a special instance of (10),

$$\left.\begin{aligned} &b_1(j_{11}x_1\bar{x}_1+j_{22}x_2\bar{x}_2)\,(j_{11}y_1\bar{y}_1+j_{22}y_2\bar{y}_2) \\ +&b_2(j_{11}x_1y_1+j_{22}x_2y_2)\,(j_{11}\bar{x}_1\bar{y}_1+j_{22}\bar{x}_2\bar{y}_2) \\ +&b_3(j_{11}x_1\bar{y}_1+j_{22}x_2\bar{y}_2)\,(j_{11}\bar{x}_1y_1+j_{22}\bar{x}_2y_2) \end{aligned}\right\}\geqq 0 \quad \begin{pmatrix} x_1,\; y_1,\; j_{11}\geqq 0 \\ x_2,\; y_2,\; j_{22}\geqq 0 \end{pmatrix}. \tag{12}$$

The cases

$$\begin{aligned} (\xi;\ \eta;\ J) &= (x_1,\, x_2;\ y_1,\, y_2;\ j_{11},\, j_{22}) \\ &= (1,\,0;\,1,\,0;\,1,\,0),\ (0,\,-1;\ 1,\,0;\ 1,\,1),\ (1,\,i;\,1,\,-i;\,1,\,1), \\ &\qquad (1,\,i;\,1,\,i;\,1,\,1), \end{aligned}$$

where $i=\sqrt{-1}$, show that we have as necessary conditions:

$$b_1+b_2+b_3\geqq 0, \qquad b_1\geqq 0, \qquad b_1+b_2\geqq 0, \qquad b_1+b_3\geqq 0. \tag{13}$$

With these conditions on the coefficients we are able at once to build the general inequality (10) in the field indicated by $J=\check{J}$ from the fundamental inequalities (11).

If $b_3\geqq 0$, we secure the desired inequality by using

$$\tfrac{1}{2}\,b_1B_{2,-1}+(b_1+b_2)B_{3,0}+b_3B_{4,0},$$

which is the sum of positive or zero multiples of positive or zero forms, hence is positive or zero. If $b_3<0$, we use

$$-\tfrac{1}{2}\,b_3B_{1,-1}+\tfrac{1}{2}(b_1+b_3)B_{2,-1}+(b_1+b_2+b_3)B_{3,0},$$

which is the sum of positive or zero multiples of positive or zero forms, hence is positive or zero.

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* Cf. Section 1.

§ 4. *The Twelve Inequalities (7) Form a Fundamental Set When ξ and η Are Real.*

PROOF. In case ξ and η are real we have:

$$\left.\begin{aligned} &J_{12}J_{34}=J_{12}J_{43}=J_{34}J_{21}=J_{43}J_{21}, \\ &J_{13}J_{42}=J_{14}J_{32}=J_{23}J_{41}=J_{24}J_{31}, \\ &J_{13}J_{24}=J_{14}J_{23}, \\ &J_{32}J_{41}=J_{42}J_{31}. \end{aligned}\right\} \tag{14}$$

We shall designate the consequent grouping of coefficients a_{ijkl},

$$\left.\begin{aligned} &c_1=a_{1234}+a_{1243}+a_{3421}+a_{4321}, \\ &c_2=a_{1342}+a_{1432}+a_{2341}+a_{2431}, \\ &c_3=a_{1324}+a_{1423}, \\ &c_4=a_{3241}+a_{4231}. \end{aligned}\right\} \tag{15}$$

The corresponding general inequality which is to be expressed as the sum of positive or zero multiples of the fundamental inequalities is

$$(c_1J_{12}J_{34}+c_2J_{13}J_{42}+c_3J_{13}J_{24}+c_4J_{32}J_{41})\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi^R;\ \eta^R;\ J). \tag{16}$$

With ξ and η real, the given inequalities (7), with grouping of coefficients in accordance with (15), reduce to three distinct inequalities C_1, C_2, C_3, tabulated by means of their coefficients c_1, c_2, c_3, c_4 as follows:

	c_1	c_2	c_3	c_4
$C_{1,u}$	$1+u\bar{u}$	$u+\bar{u}$	0	0
$C_{2,u}$	$1+u\bar{u}$	0	$\bar{u}$	u
$C_{3,u}$	0	$1+u\bar{u}$	$\bar{u}$	u

(17)

The multipliers in terms of c_1, c_2, c_3, c_4 with which to build the general inequality (16), as a sum of positive or zero multiples of known inequalities (17) are to be chosen from necessary conditions involving the coefficients c_1, c_2, c_3, c_4. Again, it is possible to determine the necessary conditions by use of a binary operator; but in order to secure sufficient conditions, a more general binary operator must be used than served for the situation $J=\breve{J}$. We use j_{12} pure imaginary (I) and have

$$J=\begin{pmatrix} j_{11} & j_{12} \\ -j_{12} & j_{22} \end{pmatrix}.$$

The inequality (16) becomes, for this special case,

$$\left.\begin{aligned} &c_1(j_{11}x_1^2+j_{22}x_2^2)(j_{11}y_1^2+j_{22}y_2^2)\\ &+c_2\{j_{11}x_1y_1+j_{12}(x_1y_2-x_2y_1)+j_{22}x_2y_2\}\\ &\qquad\qquad\{j_{11}x_1y_1+j_{12}(x_2y_1-x_1y_2)+j_{22}x_2y_2\}\\ &+c_3\{j_{11}x_1y_1+j_{12}(x_1y_2-x_2y_1)+j_{22}x_2y_2\}^2\\ &+c_4\{j_{11}x_1y_1+j_{12}(x_2y_1-x_1y_2)+j_{22}x_2y_2\}^2 \end{aligned}\right\}\geqq 0 \begin{pmatrix} x_1^R & y_1^R\\ x_2^R & y_2^R\\ j_{11}\geqq 0 & \\ j_{22}\geqq 0 & j_{12}^I\\ j_{11}j_{22}+j_{12}^2\geqq 0 & \end{pmatrix}. \tag{18}$$

By making suitable choice of the variables and elements of the matrix J, we secure the necessary conditions. In this instance it is convenient to use the most general matrix whose determinant is zero. Such a matrix is

$$\begin{pmatrix} j_1\bar{j}_1 & j_1\bar{j}_2\\ j_2\bar{j}_1 & j_2\bar{j}_2 \end{pmatrix} \text{ where } j_2=eij_1 \text{ and } e^R. \tag{19}$$

In this work we need frequently to remember that

$$me^2+2ne+p\geqq 0 \quad (e^R) \tag{20}$$

implies

$$m^R,\ n^R,\ p^R,\quad m\geqq 0,\ p\geqq 0,\quad mp-n^2\geqq 0. \tag{21}$$

The cases $(x_1, x_2;\ y_1, y_2;\ j_{11}, j_{12}, j_{22})$
$=(1, 0;\ 1, 0;\ 1, 0, 1),\ (1, 0;\ 0, 1;\ 1, i, 1),\ (1, 1; 1, -1; 1, 0, 1)$

give as necessary conditions

$$\text{hence,}\quad \left.\begin{aligned} &c_1+c_2+c_3+c_4\geqq 0,\quad c_1+c_2-c_3-c_4\geqq 0,\quad c_1\geqq 0,\\ &(c_1+c_2)^R\geqq 0,\quad (c_3+c_4)^R,\quad c_1^R,\ c_2^R. \end{aligned}\right\} \tag{22}$$

The case $(x_1, x_2;\ y_1, y_2;\ j_{11}, j_{12}, j_{22})=(1, 0;\ 1, 1;\ 1, -e'i, e'^2)$ gives the inequality

$$(c_1+c_2-c_3-c_4)e'^2-2i(c_3-c_4)e'+(c_1+c_2+c_3+c_4)\geqq 0 \quad (e'^R)$$

which, by (21), ensures the conditions

$$(c_3-c_4)^I,\quad \therefore c_3=\bar{c}_4,\quad \text{and}\quad (c_1+c_2)^2-4c_3c_4\geqq 0. \tag{23}$$

With the conditions expressed in (22) and (23) it is possible to build the general inequality (16) in this field $(\xi^R;\eta^R)$, from the fundamental inequalities (17).

If $c_2>0$, we may build first for the case $c_1=0$, using the consequent condition:

$$c_2^2-4c_3c_4\geqq 0.$$

The desired inequality is given by

$$\tfrac{1}{2}\{c_2+(c_2^2-4c_3c_4)^{\frac{1}{2}}\}C_{3,\,2c_4/[c_2+(c_2^2-4c_3c_4)^{\frac{1}{2}}]}. \tag{24}$$

If $c_1 \neq 0$ it follows from (22) that $c_1 > 0$, and we may obtain the desired inequality by adding $c_1 C_{2,0}$ to the inequality (24) already built.

If $c_2 = 0$ the necessary conditions (22) and (23) reduce to

$$c_1 \geqq 0, \quad c_1^2 - 4c_3c_4 \geqq 0.$$

Hence, the desired inequality may be expressed as

$$\tfrac{1}{2}\{c_1 + (c_1^2 - 4c_3c_4)^{\frac{1}{2}}\} C_{2,\, 2c_4/[c_1+(c_1^2-4c_3c_4)^{\frac{1}{2}}]}.$$

If $c_2 < 0$ and $c_1 + c_2 \neq 0$, we use

$$-\tfrac{1}{2}c_2 C_{1,-1} + dC_{2,\, c_4/d}, \quad \text{where} \quad d = \tfrac{1}{2}[c_1 + c_2 + \{(c_1+c_2)^2 - 4c_3c_4\}^{\frac{1}{2}}].$$

If $c_1 + c_2 = 0$, it follows that $c_3 = c_4 = 0$. Hence, we secure the desired inequality by using $\frac{1}{2} c_1 C_{1,-1}$.

Part II.

§ 5. *Definition of Polarizable Inequality.*

The inequality

$$\sum_{ijkl}^{(12)} a_{ijkl} J_{ij} J_{kl} \xi\bar{\xi}\eta\bar{\eta} \geqq 0 \qquad (\xi^{\mathfrak{M}};\ \eta^{\mathfrak{M}};\ J^{LHP})$$

is said to be polarizable if it is true that

$$\sum_{ijkl}^{(12)} a_{ijkl}(J'_{ij}J''_{kl} + J''_{ij}J'_{kl})\xi\bar{\xi}\eta\bar{\eta} \geqq 0 \qquad (\xi^{\mathfrak{M}};\ \eta^{\mathfrak{M}};\ J'^{LHP};\ J''^{LHP}). \tag{25}$$

It will be shown in this part of the paper that the first eight inequalities of (7) are polarizable, while the last four are not polarizable, even for the case $J = \breve{J}$, or for the case ξ and η real. Also, it will be shown that these eight polarizable inequalities (4′) and (5′) form a fundamental set of polarizable inequalities (1°) when $J = \breve{J}$, and (2°) when ξ and η are real.

§ 6. *Proof of the Polarized Forms.*

In § 2 we have

$$J'_{13}J''_{24}(\xi\eta + u\eta\xi)\overline{(\xi\eta + u\eta\xi)} \geqq 0 \qquad (\xi;\ \eta;\ u^{\mathfrak{A}};\ J';\ J'').$$

We may equally well have

$$J''_{13}J'_{24}(\xi\eta + u\eta\xi)\overline{(\xi\eta + u\eta\xi)} \geqq 0 \qquad (\xi;\ \eta;\ u^{\mathfrak{A}};\ J';\ J'').$$

The sum of these two inequalities is

$$\{(1+u\bar{u})(J'_{12}J''_{34} + J''_{12}J'_{34}) + (u+\bar{u})(J'_{14}J''_{32} + J''_{14}J'_{32})\}\xi\bar{\xi}\eta\bar{\eta} \geqq 0$$
$$(\xi;\ \eta;\ u^{\mathfrak{A}};\ J';\ J''). \tag{26}$$

This is the polarized form of (4). Similarly, by using $J'\breve{J}''$ and $J''\breve{J}'$ we obtain the polarized form of (5). Transformations* on ξ and η afford the polarized forms of the remaining six inequalities of (4′) and (5′). It will be seen from conditions (30) and (34) that the inequalities (6′) are not polarizable even when J is self-transpose or ξ and η are real.

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* Cf. Section 2.

§ 7. *The Eight Polarizable Inequalities* (4′) *and* (5′) *Form a Fundamental Set of Polarizable Inequalities When* $J=\breve{J}$.

PROOF. Since $J'=\breve{J}'$ and $J''=\breve{J}''$, and the operand is $\xi\bar{\xi}\eta\bar{\eta}$, it follows that

$$\left.\begin{array}{l} J'_{12}J''_{34}=J'_{12}J''_{43}=J'_{21}J''_{34}=J'_{21}J''_{43}, \\ J'_{34}J''_{12}=J'_{34}J''_{21}=J'_{43}J''_{12}=J'_{43}J''_{21}, \\ J'_{13}J''_{24}=J'_{13}J''_{42}=J'_{31}J''_{24}=J'_{31}J''_{42}, \\ J'_{24}J''_{13}=J'_{24}J''_{31}=J'_{42}J''_{13}=J'_{42}J''_{31}, \\ J'_{14}J''_{23}=J'_{14}J''_{32}=J'_{41}J''_{23}=J'_{41}J''_{32}, \\ J'_{23}J''_{14}=J'_{23}J''_{41}=J'_{32}J''_{14}=J'_{32}J''_{41}. \end{array}\right\} \tag{27}$$

The corresponding grouping of coefficients a_{ijkl} is

$$a_{1234}+a_{1243}+a_{3421}+a_{4321}, \quad a_{1324}+a_{1342}+a_{2431}+a_{4231}, \quad a_{1423}+a_{1432}+a_{2341}+a_{3241},$$

which have already been defined (9) as b_1, b_2, b_3, respectively.

The general inequality which we wish to build from the eight given inequalities (4′) and (5′) is

$$(b_1J_{12}J_{34}+b_2J_{13}J_{24}+b_3J_{14}J_{23})\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi;\eta;J=\breve{J}). \tag{10}$$

However, since this inequality is polarizable, it must be true that

$$\{b_1(J'_{12}J''_{34}+J'_{34}J''_{12})+b_2(J'_{13}J''_{24}+J'_{24}J''_{13})+b_3(J'_{14}J''_{23}+J'_{23}J''_{14})\}\xi\bar{\xi}\eta\bar{\eta}\geqq 0$$
$$(\xi;\eta;J'=\breve{J}';J''=\breve{J}''). \tag{28}$$

Using the operators

$$J'=\begin{pmatrix} j'_{11} & 0 \\ 0 & j'_{22}\end{pmatrix} \text{ and } J''=\begin{pmatrix} j''_{11} & 0 \\ 0 & j''_{22}\end{pmatrix},$$

we have as a special instance of (28),

$$\left.\begin{array}{l} b_1\left\{\begin{array}{l} \quad(j'_{11}x_1\bar{x}_1+j'_{22}x_2\bar{x}_2)(j''_{11}y_1\bar{y}_1+j''_{22}y_2\bar{y}_2) \\ +(j'_{11}y_1\bar{y}_1+j'_{22}y_2\bar{y}_2)(j''_{11}x_1\bar{x}_1+j''_{22}x_2\bar{x}_2)\end{array}\right\} \\ +b_2\left\{\begin{array}{l} \quad(j'_{11}x_1y_1+j'_{22}x_2y_2)(j''_{11}\bar{x}_1\bar{y}_1+j''_{22}\bar{x}_2\bar{y}_2) \\ +(j'_{11}\bar{x}_1\bar{y}_1+j'_{22}\bar{x}_2\bar{y}_2)(j''_{11}x_1y_1+j''_{22}x_2y_2)\end{array}\right\} \\ +b_3\left\{\begin{array}{l} \quad(j'_{11}x_1\bar{y}_1+j'_{22}x_2\bar{y}_2)(j''_{11}\bar{x}_1y_1+j''_{22}\bar{x}_2y_2) \\ +(j'_{11}\bar{x}_1y_1+j'_{22}\bar{x}_2y_2)(j''_{11}x_1\bar{y}_1+j''_{22}x_2\bar{y}_2)\end{array}\right\}\end{array}\right\}\geqq 0 \quad \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ j'_{11}\geqq 0 & j'_{22}\geqq 0 \\ j''_{11}\geqq 0 & j''_{22}\geqq 0\end{pmatrix}. \tag{29}$$

The cases $(x_1, x_2;\ y_1, y_2;\ j'_{11}, j'_{22}, j''_{11}, j''_{22})$

$=(1, 0;\ 1, 0;\ 1, 0;\ 1, 0)$, $(1, i;\ 1, -i;\ 0, 1;\ 1, 0)$,

$(1, i;\ 1, i;\ 0, 1;\ 1, 0)$, $(-1, 1;\ 1, 1;\ 0, 1;\ 1, 0)$, show that we have as necessary conditions

$$b_1+b_2+b_3 \geqq 0, \quad b_1+b_2-b_3 \geqq 0, \quad b_1-b_2+b_3 \geqq 0, \quad b_1-b_2-b_3 \geqq 0. \tag{30}$$

Since all polarizable inequalities for the field $J=\breve{J}$ must satisfy condition (30), and since neither $B_{3,u}$ nor $B_{4,u}$ of (11) satisfies the last condition of (30), it is now seen, as was suggested in § 5, that neither $B_{3,u}$ nor $B_{4,u}$ is polarizable. It follows that the inequalities (6′) from which $B_{3,u}$ and $B_{4,u}$ were obtained do not satisfy the definition of polarizable inequality, as that definition demands that the polarized form be positive or zero for all J^{LHP}, of which we have $J=\breve{J}^{LHP}$ as an instance.

The eight inequalities (4′) and (5′) which were proved polarizable, reduce to two when $J=\breve{J}$, designated as $B_{1,u}$ and $B_{2,u}$ under (11).

The desired inequality (10) is shown to be the sum of positive or zero multiples of the fundamental polarizable inequalities $B_{1,u}$ and $B_{2,u}$ as follows:

When $b_3 \geqq 0$ and $b_1 \neq b_3$, the desired inequality has the form

$$\tfrac{1}{2} b_3 B_{1,1} + \tfrac{1}{2} d_1 B_{2,\, b_2/d_1},$$

where

$$d_1 = b_1 - b_3 + \{(b_1-b_3)^2 - b_2^2\}^{\frac{1}{2}}.$$

When $b_3<0$ and $b_1+b_3 \neq 0$, the desired inequality has the form

$$-\tfrac{1}{2} b_3 B_{1,-1} + \tfrac{1}{2} d_2 B_{2,\, b_2/d_2},$$

where

$$d_2 = b_1 + b_3 + \{(b_1+b_3)^2 - b_2^2\}^{\frac{1}{2}}.$$

When $b_1=b_3$, we use $\tfrac{1}{2} b_1 B_{1,1}$, since $b_2=0$. When $b_1=-b_3$, we use $\tfrac{1}{2} b_1 B_{1,-1}$, since $b_2=0$.

§ 8. *The Eight Polarizable Inequalities* (4′) *and* (5′) *Form a Fundamental Set When* ξ *and* η *Are Real.*

PROOF. In this instance (ξ^R, η^R)

$$\left.\begin{aligned}
&J'_{12}J''_{34}=J'_{12}J''_{43}=J'_{21}J''_{34}=J'_{21}J''_{43},\\
&J'_{34}J''_{21}=J'_{43}J''_{21}=J'_{34}J''_{12}=J'_{43}J''_{12},\\
&J'_{13}J''_{24}=J'_{14}J''_{23}=J'_{24}J''_{13}=J'_{23}J''_{14},\\
&J'_{13}J''_{42}=J'_{14}J''_{32}=J'_{23}J''_{41}=J'_{24}J''_{31},\\
&J'_{42}J''_{13}=J'_{32}J''_{14}=J'_{41}J''_{23}=J'_{31}J''_{24},\\
&J'_{32}J''_{41}=J'_{42}J''_{31}=J'_{41}J''_{32}=J'_{31}J''_{42}.
\end{aligned}\right\} \tag{31}$$

The corresponding grouping of coefficients is that indicated in (15). Hence the inequality (16) is the general inequality which is to be expressed as the sum of positive or zero multiples of the known polarizable inequalities, $C_{1,\kappa}$ and $C_{2,\kappa}$, to which (4′) and (5′) reduce when ξ and η are real. The positive or zero multipliers will be chosen from necessary conditions on c_1, c_2, c_3, c_4. The conditions (22) and (23), which must be satisfied by the coefficients of all inequalities (16) must necessarily be satisfied by the coefficients of the polarizable inequality (16). In addition to these conditions (22) and (23), necessary conditions are obtained by using the fact that the inequality (16) is polarizable; *i. e.*

$$\begin{aligned}\{c_1(J'_{12}J''_{34}+J'_{34}J''_{12})+c_2(J'_{13}J''_{42}+J'_{42}J''_{13})+c_3(J'_{13}J''_{24}+J'_{24}J''_{13})\\ +c_4(J'_{32}J''_{41}+J'_{41}J''_{32})\}\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi^R;\ \eta^R;\ J';\ J'').\end{aligned} \tag{32}$$

The use of

$$J'=\begin{pmatrix} j'_{11} & j'_{12}\\ -j'_{12} & j'_{22}\end{pmatrix},\quad J''=\begin{pmatrix} j''_{11} & j''_{12}\\ -j''_{12} & j''_{22}\end{pmatrix},$$

where j'_{12} and j''_{12} are pure imaginary, gives as a special case of (32):

$$\left.\begin{array}{l} c_1\{(j'_{11}x_1^2+j'_{22}x_2^2)(j''_{11}y_1^2+j''_{22}y_2^2)\\ \qquad\qquad +(j'_{11}y_1^2+j'_{22}y_2^2)(j''_{11}x_1^2+j''_{22}x_2^2)\}\\ +c_2\left[\begin{array}{l}\{j'_{11}x_1y_1+j'_{12}(x_1y_2-x_2y_1)+j'_{22}x_2y_2\}\\ \qquad\{j''_{11}x_1y_1+j''_{12}(x_2y_1-x_1y_2)+j''_{22}x_2y_2\}\\ +\{j'_{11}x_1y_1+j'_{12}(x_2y_1-x_1y_2)+j'_{22}x_2y_2\}\\ \qquad\{j''_{11}x_1y_1+j''_{12}(x_1y_2-x_2y_1)+j''_{22}x_2y_2\}\end{array}\right]\\ +2c_3\{j'_{11}x_1y_1+j'_{12}(x_1y_2-x_2y_1)+j'_{22}x_2y_2\}\\ \qquad\{j''_{11}x_1y_1+j''_{12}(x_1y_2-x_2y_1)+j''_{22}x_2y_2\}\\ +2c_4\{j'_{11}x_1y_1+j'_{12}(x_2y_1-x_1y_2)+j'_{22}x_2y_2\}\\ \qquad\{j''_{11}x_1y_1+j''_{12}(x_2y_1-x_1y_2)+j''_{22}x_2y_2\}\end{array}\right\}\geqq 0\quad \begin{pmatrix} x_1^R & x_2^R\\ y_1^R & y_2^R\\ j'_{11}\geqq 0 & j'_{22}\geqq 0\\ j''_{11}\geqq 0 & j''_{22}\geqq 0\\ j'^{I}_{12} & j''^{I}_{12}\\ \multicolumn{2}{l}{j'_{11}j'_{22}+j'^2_{12}\geqq 0}\\ \multicolumn{2}{l}{j''_{11}j''_{22}+j''^2_{12}\geqq 0}\end{pmatrix}. \tag{33}$$

The special values

$$\begin{aligned}&(x_1, x_2;\ y_1, y_2;\ j'_{11}, j'_{12}, j'_{22};\ j''_{11}, j''_{12}, j''_{22})\\ &\quad=(1, 0;\ 0, 1;\ 1, i, 1;\ 1, -i, 1),\ (1, 1;\ 1, -1;\ 0, 0, 1;\ 1, 0, 0)\end{aligned}$$

give the necessary conditions

$$\left.\begin{array}{c} c_1-c_2+c_3+c_4\geqq 0,\quad c_1-c_2-c_3-c_4\geqq 0,\\ c_1-c_2\geqq 0.\end{array}\right\} \tag{34}$$

whence

Again, it is convenient to use, for J, a matrix whose determinant is zero. We choose

$$J'=\begin{pmatrix} j'_1\overline{j}'_1 & j'_1\overline{j}'_2\\ j'_2\overline{j}'_1 & j'_2\overline{j}'_2\end{pmatrix},\quad J''=\begin{pmatrix} j''_1\overline{j}''_1 & j''_1\overline{j}''_2\\ j''_2\overline{j}''_1 & j''_2\overline{j}''_2\end{pmatrix},$$

where $j_2'=e'ij_1'$ and $j_2''=e''ij_1''$ with e' and e'' real. Then (33) becomes a quadratic in e' and also in e''. As an instance we set

$$(x_1, x_2; y_1, y_2; j_{11}', j_{12}', j_{22}'; j_{11}'', j_{12}'', j_{22}'') = (1, 1; 1, -1; 1, -e'i, e'^2; 1, -e''i, e''^2),$$

and (33) gives the quadratic expression $P(e', e'')$ in e' and e'' which must be positive or zero for every real e' and e'', and for which, therefore, the discriminant must be positive or zero.

$$\left.\begin{aligned} P(e', e'') \equiv {} & 2c_1(1+e'^2)(1+e''^2) \\ & +c_2\{(1+2e'i-e'^2)(1-2e''i-e''^2) \\ & \qquad +(1-2e'i-e'^2)(1+2e''i-e''^2)\} \\ & +2c_3(1+2e'i-e'^2)(1+2e''i-e''^2) \\ & +2c_4(1-2e'i-e'^2)(1-2e''i-e''^2) \end{aligned}\right\} \geqq 0 \quad (e'^R, e''^R).$$

The discriminant of $P(e', 0)$ gives the condition

$$c_1^2-(c_2+c_3+c_4)^2+(c_3-c_4)^2 \geqq 0;$$

the discriminant of $P(e', 1)$ gives

$$c_1^2-(c_2-c_3-c_4)^2+(c_3-c_4)^2 \geqq 0,$$

whence

$$c_1^2-c_2^2-4c_3c_4 \geqq 0,$$

and, thus,

$$c_1^2+c_2^2-4c_3c_4 \geqq 0. \tag{35}$$

The discriminant of $P(e' e'')$ as to e'' is a homogeneous quadratic expression in $(1-e'^2, 2e')$, and since for e' real, even between the values -1 and $+1$, $2e'/(1-e'^2)$ takes every real value, the discriminant of this homogeneous expression must be positive or zero. Hence, we have the condition

$$(c_1^2+c_2^2-4c_3c_4)^2-4c_1^2c_2^2 \geqq 0,$$

and, therefore, by virtue of (35)

$$c_1^2+c_2^2-4c_3c_4 \geqq \pm 2c_1c_2$$

or

$$(c_1 \pm c_2)^2-4c_3c_4 \geqq 0. \tag{36}$$

It can be seen from (34) that $C_{3,u}$ of (17) is not polarizable. By use of (22), (23), (34) and (36) we build all polarizable inequalities in this field $(\xi^R; \eta^R)$ as the sum of positive or zero multiples of the polarizable inequalities $C_{1,u}$ and $C_{2,u}$ of (17) in the following ways:

If $c_2 \geqq 0$ and $c_1 \neq c_2$, the desired inequality may be expressed as

$$\tfrac{1}{2} c_2 C_{1,1} + d_1 C_{2,\, c_4/d_1},$$

where

$$d_1 = \tfrac{1}{2}[c_1-c_2+\{(c_1-c_2)^2-4c_3c_4\}^{\frac{1}{2}}].$$

If $c_2<0$ and $c_1+c_2 \neq 0$, the desired inequality has the form

$$-\tfrac{1}{2}c_2C_{1,-1}+d_2C_{2,c_4/d_2},$$

where

$$d_2=\tfrac{1}{2}[c_1+c_2+\{(c_1+c_2)^2-4c_3c_4\}^{\frac{1}{2}}].$$

If $c_2\geqq 0$ and $c_1=c_2$, it follows from (36) that $c_3=c_4=0$, and we may express the inequality as

$$\tfrac{1}{2}c_1C_{1,1}.$$

If $c_2<0$ and $c_1+c_2=0$, it follows from (36) that $c_3=c_4=0$, and we may use as the desired inequality

$$\tfrac{1}{2}c_1C_{1,-1}.$$

PART III.

§ 9. *Bilinear Inequalities.*

Somewhat related to the preceding problem is that of determining all bilinear inequalities of the form

$$\sum_{ijkl}^{(24)} z_{ijkl}J'_{ij}J''_{kl}\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi;\eta;J';J'').$$

It is the purpose of this portion of the paper to exhibit sixteen such inequalities and prove that they form a fundamental set of bilinear inequalities for the cases (1°) $J=\check{J}$ and (2°) ξ and η real functions.

We have, in § 2, as a special instance of the theorem proved by E. H. Moore, the inequality

$$J'_{13}J''_{24}(\xi\eta+u\eta\xi)\,\overline{(\xi\eta+u\eta\xi)}\geqq 0 \qquad (\xi;\eta;u^{\mathfrak{A}};J';J''),$$

which is also written

$$(J'_{12}J''_{34}+\bar{u}J'_{14}J''_{32}+uJ'_{32}J''_{14}+u\bar{u}J'_{34}J''_{12})\,\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi;\eta;u^{\mathfrak{A}};J';J'').$$

By using J' instead of J', J'' instead of J'', and J' and J'' instead of J' and J'' we have also

$$(J'_{21}J''_{34}+\bar{u}J'_{41}J''_{32}+uJ'_{23}J''_{14}+u\bar{u}J'_{43}J''_{12})\,\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi;\eta;u^{\mathfrak{A}};J';J''),$$

$$(J'_{12}J''_{43}+\bar{u}J'_{14}J''_{23}+uJ'_{32}J''_{41}+u\bar{u}J'_{34}J''_{21})\,\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi;\eta;u^{\mathfrak{A}};J';J''),$$

$$(J'_{21}J''_{43}+\bar{u}J'_{41}J''_{23}+uJ'_{23}J''_{41}+u\bar{u}J'_{43}J''_{21})\,\xi\bar{\xi}\eta\bar{\eta}\geqq 0 \qquad (\xi;\eta;u^{\mathfrak{A}};J';J'').$$

Four more inequalities may be written by interchange of J' and J'', the eight being $Z_{1,u}$ to $Z_{8,u}$ inclusive, tabulated according to their coefficients z_{ijkl} in (37). By replacing ξ by $\bar{\xi}$ the inequalities $Z_{9,u},\ldots,Z_{16,u}$ of (37) are obtained from inequalities $Z_{1,u},\ldots,Z_{8,u}$.

	z_{1234}	z_{1243}	z_{1324}	z_{1342}	z_{1423}	z_{1432}	z_{2341}	z_{2431}	z_{3241}	z_{3421}	z_{4231}	z_{4321}	z_{3412}	z_{4312}	z_{2413}	z_{4213}	z_{2314}	z_{3214}	z_{4123}	z_{3124}	z_{4132}	z_{2134}	z_{3142}	z_2
$Z_{1,u}$	1	0	0	0	0	$\bar{u}$	0	0	0	0	0	0	$u\bar{u}$	0	0	0	0	u	0	0	0	0	0	0
$Z_{2,u}$	0	0	0	0	0	0	0	0	0	0	0	0	0	$u\bar{u}$	0	0	u	0	0	0	$\bar{u}$	1	0	0
$Z_{3,u}$	0	1	0	0	$\bar{u}$	0	0	0	u	$u\bar{u}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$Z_{4,u}$	0	0	0	0	0	0	u	0	0	0	0	$u\bar{u}$	0	0	0	0	0	0	$\bar{u}$	0	0	0	0	1
$Z_{5,u}$	$u\bar{u}$	0	0	0	0	u	0	0	0	0	0	0	1	0	0	0	0	$\bar{u}$	0	0	0	0	0	0
$Z_{6,u}$	0	$u\bar{u}$	0	0	u	0	0	0	$\bar{u}$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$Z_{7,u}$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	$\bar{u}$	0	0	0	u	$u\bar{u}$	0	0
$Z_{8,u}$	0	0	0	0	0	0	$\bar{u}$	0	0	0	0	1	0	0	0	0	0	0	u	0	0	0	0	u
$Z_{9,u}$	0	0	0	0	0	0	0	$\bar{u}$	0	$u\bar{u}$	0	0	0	0	0	0	0	0	0	u	0	1	0	0
$Z_{10,u}$	0	1	0	$\bar{u}$	0	0	0	0	0	0	0	0	0	$u\bar{u}$	0	u	0	0	0	0	0	0	0	0
$Z_{11,u}$	0	0	0	0	0	0	0	u	0	1	0	0	0	0	0	0	0	0	0	$\bar{u}$	0	$u\bar{u}$	0	0
$Z_{12,u}$	0	$u\bar{u}$	0	u	0	0	0	0	0	0	0	0	0	1	0	$\bar{u}$	0	0	0	0	0	0	0	0
$Z_{13,u}$	1	0	u	0	0	0	0	0	0	0	$\bar{u}$	$u\bar{u}$	0	0	0	0	0	0	0	0	0	0	0	0
$Z_{14,u}$	0	0	0	0	0	0	0	0	0	0	0	0	$u\bar{u}$	0	u	0	0	0	0	0	0	0	$\bar{u}$	1
$Z_{15,u}$	$u\bar{u}$	0	$\bar{u}$	0	0	0	0	0	0	0	u	1	0	0	0	0	0	0	0	0	0	0	0	0
$Z_{16,u}$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	$\bar{u}$	0	0	0	0	0	0	0	u	u

§ 10. *The Sixteen Inequalities (37) Form a Fundamental Set of Bilinear Inequalities When $J'=\breve{J}'$ and $J''=\breve{J}''$.*

PROOF. In this field, $J'=\breve{J}'$ and $J''=\breve{J}''$, equations (27) are valid, and they indicate the new coefficients,

$$\left.\begin{aligned} w_1&=z_{1234}+z_{1243}+z_{2134}+z_{2143}, & w_2&=z_{1324}+z_{1342}+z_{3124}+z_{3142}, \\ w_3&=z_{1423}+z_{1432}+z_{4123}+z_{4132}, & w_4&=z_{2341}+z_{3241}+z_{2314}+z_{3214}, \\ w_5&=z_{2431}+z_{4231}+z_{2413}+z_{4213}, & w_6&=z_{3421}+z_{4321}+z_{3412}+z_{4312}, \end{aligned}\right\} \tag{38}$$

Accordingly the general inequality which we desire to build is

$$(w_1J'_{12}J''_{34}+w_2J'_{13}J''_{24}+w_3J'_{14}J''_{23}+w_4J'_{23}J''_{41}+w_5J'_{24}J''_{31}+w_6J'_{34}J''_{21})\xi\bar{\xi}\eta\bar{\eta}\geqq 0$$
$$(\xi;\eta;J'=\breve{J}';J''=\breve{J}''). \tag{39}$$

The known inequalities from which (39) is to be built in this field, $J'=\breve{J}'$ and $J''=\breve{J}''$, are the four inequalities to which (37) reduce on account of (38), and are tabulated in (40) according to their coefficients $w_1, \ldots, w_6$.

	w_1	w_2	w_3	w_4	w_5	w_6
$W_{1,u}$	1	0	$\bar{u}$	u	0	$u\bar{u}$
$W_{2,u}$	$u\bar{u}$	0	u	$\bar{u}$	0	1
$W_{3,u}$	1	u	0	0	$\bar{u}$	$u\bar{u}$
$W_{4,u}$	$u\bar{u}$	$\bar{u}$	0	0	u	1

(40)

As before, it is possible to secure the necessary conditions from the form corresponding to binary operators. Use of

$$J'=\begin{pmatrix} j'_{11} & 0 \\ 0 & j'_{22} \end{pmatrix}, \quad J''=\begin{pmatrix} j''_{11} & 0 \\ 0 & j''_{22} \end{pmatrix},$$

gives as an instance of (39):

$$\left.\begin{aligned} &w_1(j'_{11}x_1\bar{x}_1+j'_{22}x_2\bar{x}_2)(j''_{11}y_1\bar{y}_1+j''_{22}y_2\bar{y}_2)\\ &+w_2(j'_{11}x_1y_1+j'_{22}x_2y_2)(j''_{11}\bar{x}_1\bar{y}_1+j''_{22}\bar{x}_2\bar{y}_2)\\ &+w_3(j'_{11}x_1\bar{y}_1+j'_{22}x_2\bar{y}_2)(j''_{11}\bar{x}_1y_1+j''_{22}\bar{x}_2y_2)\\ &+w_4(j'_{11}\bar{x}_1y_1+j'_{22}\bar{x}_2y_2)(j''_{11}x_1\bar{y}_1+j''_{22}x_2\bar{y}_2)\\ &+w_5(j'_{11}\bar{x}_1\bar{y}_1+j'_{22}\bar{x}_2\bar{y}_2)(j''_{11}x_1y_1+j''_{22}x_2y_2)\\ &+w_6(j'_{11}y_1\bar{y}_1+j'_{22}y_2\bar{y}_2)(j''_{11}x_1\bar{x}_1+j''_{22}x_2\bar{x}_2) \end{aligned}\right\}\geqq 0 \quad \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ j'_{11}\geqq 0 & j'_{22}\geqq 0 \\ j''_{11}\geqq 0 & j''_{22}\geqq 0 \end{pmatrix}. \qquad (41)$$

The cases $(x_1, x_2;\ y_1, y_2;\ j'_{11}, j'_{22};\ j''_{11}, j''_{22})$
$= (0, 1;\ 1, 0;\ 0, 1;\ 1, 0),\ (1, 0;\ 0, 1;\ 0, 1;\ 1, 0)$

give as necessary conditions,

$$w_1 \geqq 0, \quad w_6 \geqq 0. \qquad (42)$$

The case $(x_1, x_2;\ y_1, y_2;\ j'_{11}, j'_{22};\ j''_{11}, j''_{22}) = (1, k_1+ik_2;\ 1, k_1+ik_2;\ 0, 1;\ 1, 0)$, where k_1 and k_2 are real, gives as a special case of (41),

$$(w_1+w_3+w_4+w_6)(k_1^2+k_2^2)+(w_2+w_5)(k_1^2-k_2^2)$$
$$+2i(w_2-w_5)k_1k_2 \geqq 0 \quad (k_1^R, k_2^R). \qquad (43)$$

In (43) the values $(k_1, k_2) = (1, 0),\ (0, 1)$ give the necessary conditions

$$w_1+w_2+w_3+w_4+w_5+w_6 \geqq 0, \quad w_1-w_2+w_3+w_4-w_5+w_6 \geqq 0. \qquad (44)$$

Hence, from (42) and (44),

$$(w_2+w_5)^R, \quad (w_3+w_4)^R,$$

while from (43) it is now seen that w_2-w_5 is pure imaginary, and, therefore,

$$w_2 = \bar{w}_5. \qquad (45)$$

Similarly, the case

$$(x_1, x_2;\ y_1, y_2;\ j'_{11}, j'_{22};\ j''_{11}, j''_{22}) = (1, k_1-ik_2;\ 1, k_1+ik_2;\ 0, 1;\ 1, 0)$$

yields the conditions

$$w_1+w_2-w_3-w_4+w_5+w_6 \geqq 0, \quad (w_3-w_4)^I,$$

whence

$$w_3 = \bar{w}_4. \qquad (46)$$

The case $(x_1, x_2;\ y_1, y_2;\ j'_{11}, j'_{22};\ j''_{11}, j''_{22}) = (1, k_1+ik_2;\ 1, l_1+il_2;\ 0, 1;\ 1, 0)$, where k_1, k_2, l_1, l_2 are real, gives as an instance of (41),

$$\left.\begin{aligned}&w_6l_1^2+l_1\{(w_2+w_3+w_4+w_5)k_1+i(w_2+w_3-w_4-w_5)k_2\}\\&+w_6l_2^2+l_2\{i(w_2-w_3+w_4-w_5)k_1-(w_2-w_3-w_4+w_5)k_2\}\\&+w_1(k_1^2+k_2^2)\end{aligned}\right\}\geq 0\quad\begin{pmatrix}k_1^R & k_2^R\\ l_1^R & l_2^R\end{pmatrix}. \tag{47}$$

Inequality (47) shows that $w_6=0$ demands:

$$\left.\begin{aligned}w_2+w_3+w_4+w_5=0,\quad w_2+w_3-w_4-w_5=0,\\ w_2-w_3+w_4-w_5=0,\quad w_2-w_3-w_4+w_5=0.\end{aligned}\right\} \tag{48}$$

For if not, suppose any one of (48) is not zero, say

$$w_2+w_3+w_4+w_5\neq 0.$$

Then by taking $(k_1, k_2, l_2)=(1, 0, 0)$ we have,

$$l_1(w_2+w_3+w_4+w_5)+w_1\geq 0,\quad (l_1^R),$$

which obviously is not true. Similarly, the other equations of (48) are proved when $w_6=0$. Also, equations (48) follow when $w_1=0$. Hence,

$$w_1=0 \text{ or } w_6=0 \text{ demands } w_2=w_3=w_4=w_5=0. \tag{49}$$

Thus for the remaining cases we may suppose $w_1\neq 0$ and $w_6\neq 0$ and omit w_1 or w_6 as a factor in any expression which is positive or zero, each of whose terms contains as factor one or more of the coefficients $w_2, \ldots, w_5$, or w_1w_6.

When $l_1=0$, (47) becomes a quadratic in l_2, whose discriminant is a quadratic in k_1 and k_2, viz.:

$$\left.\begin{aligned}\{4w_1w_6+(w_2-w_3+w_4-w_5)^2\}k_1^2+\{4w_1w_6-(w_2-w_3-w_4+w_5)^2\}k_2^2\\ +2i(w_2-w_3+w_4-w_5)(w_2-w_3-w_4+w_5)k_1k_2\end{aligned}\right\}\geq 0\quad (k_1^Rk_2^R).$$

The discriminant of this quadratic in k_1 and k_2 gives the condition

$$16w_1w_6\{w_1w_6+(w_2-w_3)(w_4-w_5)\}\geq 0,$$

which by (49) gives as a condition always holding

$$w_1w_6+(w_2-w_3)(w_4-w_5)\geq 0. \tag{50}$$

Similarly, $l_2=0$ reduces (47) to a quadratic in l_1 whose discriminant is a quadratic in k_1 and k_2, which gives the necessary condition

$$w_1w_6-(w_2+w_3)(w_4+w_5)\geq 0. \tag{51}$$

Conditions (50) and (51) combine to give

$$\left.\begin{aligned}&& w_1w_6-w_3w_4-w_2w_5\geq 0,\\ &\text{whence}\qquad& w_1w_6+w_3w_4-w_2w_5\geq 0,\\ &\text{and}\qquad& w_1w_6-w_3w_4+w_2w_5\geq 0.\end{aligned}\right\} \tag{52}$$

The discriminant for (47) as a quadratic in l_1 is a quadratic in l_2, viz.:

$$\left.\begin{aligned} 4w_6^2l_2^2+4w_6l_2\{i(w_2-w_3+w_4-w_5)k_1-(w_2-w_3-w_4+w_5)k_2\} \\ +\{4w_1w_6-(w_2+w_3+w_4+w_5)^2\}k_1^2 \\ +\{4w_1w_6+(w_2+w_3-w_4-w_5)^2\}k_2^2 \\ -2i(w_2+w_3+w_4+w_5)(w_2+w_3-w_4-w_5)k_1k_2 \end{aligned}\right\}\geqq 0 \quad \begin{pmatrix} k_1^R & k_2^R \\ & l_2^R \end{pmatrix},$$

whose discriminant, aside from a factor $16w_6^2$, reduces to

$$\left.\begin{aligned} \{w_1w_6-(w_2+w_4)(w_3+w_5)\}k_1^2+\{w_1w_6+(w_2-w_4)(w_3-w_5)\}k_2^2 \\ -2i(w_2w_3-w_4w_5)k_1k_2 \end{aligned}\right\}\geqq 0 \quad (k_1^R,\ k_2^R).$$

Hence,

$$\{w_1w_6-(w_2+w_4)(w_3+w_5)\}\{w_1w_6+(w_2-w_4)(w_3-w_5)\}+(w_2w_3-w_4w_5)^2\geqq 0,$$

or

$$(w_1w_6-w_3w_4-w_2w_5)^2-4w_2w_3w_4w_5\geqq 0. \tag{53}$$

Apart from exceptional cases, we have the desired general inequality (39) in the form

$$\frac{d_1}{2w_6}W_{1,\,2w_6w_4/d_1}+\frac{d_2}{2w_1}W_{4,\,2w_1w_5/d_2},$$

where

$$d_1=w_1w_6-w_2w_5+w_3w_4+\{(w_1w_6-w_2w_5-w_3w_4)^2-4w_2w_3w_4w_5\}^{\frac{1}{2}},$$
$$d_2=w_1w_6+w_2w_5-w_3w_4+\{(w_1w_6-w_2w_5-w_3w_4)^2-4w_2w_3w_4w_5\}^{\frac{1}{2}}.$$

The exceptional cases in which the above method will not be permissible are: (a) When $w_1w_6+w_2w_5-w_3w_4=0$; (b) when $w_1w_6-w_2w_5+w_3w_4=0$; (c) when $w_1w_6-w_2w_5-w_3w_4=0$; (d) when $w_1=0$; (e) when $w_6=0$.

In case $w_1w_6+w_2w_5-w_3w_4=0$ it follows that

$$w_2=w_5=0, \quad w_1w_6-w_3w_4=0,$$

and the desired inequality is

$$w_1W_{1,\,w_4/w_1}.$$

Similarly, $w_1w_6-w_2w_5+w_3w_4=0$ demands

$$w_3=w_4=0, \quad w_1w_6-w_2w_5=0;$$

and the desired inequality is

$$w_6W_{4,\,w_5/w_6}.$$

In case $w_1w_6-w_2w_5-w_3w_4=0$, it is seen by (53) that

$$w_2=w_5=0 \quad \text{or} \quad w_3=w_4=0.$$

In case $w_2=w_5=0$, the desired inequality is

$$w_1W_{1,\,w_4/w_1}.$$

In case $w_3=w_4=0$, the desired inequality is

$$w_6W_{4,\,w_5/w_6}.$$

When $w_1=0$ or $w_6=0$, we have $w_2=w_3=w_4=w_5=0$, by (49), hence the desired inequality is

$$w_1 W_{1,0}+w_6 W_{4,0}.$$

§11. *The Sixteen Bilinear Inequalities* (37) *Form a Fundamental Set When ξ and η Are Real.*

PROOF. Equations (31) which are valid in this case $(\xi^R;\ \eta^R)$ indicate the coefficients,

$$\left.\begin{aligned}
v_1&=z_{1234}+z_{1243}+z_{2134}+z_{2143}, & v_2&=z_{1324}+z_{1423}+z_{2413}+z_{2314},\\
v_3&=z_{1342}+z_{1432}+z_{2341}+z_{2431}, & v_4&=z_{4213}+z_{3214}+z_{4123}+z_{3124},\\
v_5&=z_{3241}+z_{4231}+z_{4132}+z_{3142}, & v_6&=z_{3421}+z_{4321}+z_{3412}+z_{4312},
\end{aligned}\right\} \tag{54}$$

Further, let

$$v=v_1+v_2+v_3+v_4+v_5+v_6.$$

The general inequality desired is

$$(v_1J'_{12}J''_{34}+v_2J'_{13}J''_{24}+v_3J'_{13}J''_{42}+v_4J'_{31}J''_{24}+v_5J'_{31}J''_{42}+v_6J'_{34}J''_{12})\,\xi\bar{\xi}\eta\bar{\eta}\geqq 0$$
$$(\xi^R;\ \eta^R;\ J';\ J''). \tag{55}$$

When ξ and η are real, the given inequalities (37) reduce, in accordance with (54), to four inequalities (56).

(56)

	v_1	v_2	v_3	v_4	v_5	v_6
$V_{1,u}$	1	0	$\bar u$	u	0	$u\bar u$
$V_{2,u}$	1	u	0	0	$\bar u$	$u\bar u$
$V_{3,u}$	$u\bar u$	0	$\bar u$	u	0	1
$V_{4,u}$	$u\bar u$	u	0	0	$\bar u$	1

Binary operators give necessary and sufficient conditions on $v_1, \ldots, v_6$.

$$J'=\begin{pmatrix} j'_{11} & j'_{12}\\ j'_{21} & j'_{22}\end{pmatrix};\quad J''=\begin{pmatrix} j''_{11} & j''_{12}\\ j''_{21} & j''_{22}\end{pmatrix}$$

give, as an instance of (55),

$$\left.\begin{aligned}
&v_1\{j'_{11}x_1^2+(j'_{12}+j'_{21})x_1x_2+j'_{22}x_2^2\}\\
&\qquad\qquad\{j''_{11}y_1^2+(j''_{12}+j''_{21})y_1y_2+j''_{22}y_2^2\}\\
&+v_6\{j'_{11}y_1^2+(j'_{12}+j'_{21})y_1y_2+j'_{22}y_2^2\}\\
&\qquad\qquad\{j''_{11}x_1^2+(j''_{12}+j''_{21})x_1x_2+j''_{22}x_2^2\}\\
&+v_2\{j'_{11}x_1y_1+j'_{12}x_1y_2+j'_{21}x_2y_1+j'_{22}x_2y_2\}\\
&\qquad\qquad\{j''_{11}x_1y_1+j''_{12}x_1y_2+j''_{21}x_2y_1+j''_{22}x_2y_2\}\\
&+v_3\{j'_{11}x_1y_1+j'_{12}x_1y_2+j'_{21}x_2y_1+j'_{22}x_2y_2\}\\
&\qquad\qquad\{j''_{11}x_1y_1+j''_{12}x_2y_1+j''_{21}x_1y_2+j''_{22}x_2y_2\}\\
&+v_4\{j'_{11}x_1y_1+j'_{12}x_2y_1+j'_{21}x_1y_2+j'_{22}x_2y_2\}\\
&\qquad\qquad\{j''_{11}x_1y_1+j''_{12}x_1y_2+j''_{21}x_2y_1+j''_{22}x_2y_2\}\\
&+v_5\{j'_{11}x_1y_1+j'_{12}x_2y_1+j'_{21}x_1y_2+j'_{22}x_2y_2\}\\
&\qquad\qquad\{j''_{11}x_1y_1+j''_{12}x_2y_1+j''_{21}x_1y_2+j''_{22}x_2y_2\}
\end{aligned}\right\}\geqq 0\quad
\begin{pmatrix}
x_1^R & x_2^R\\
y_1^R & y_2^R\\
j'_{11}\geqq 0 & j'_{22}\geqq 0\\
j''_{11}\geqq 0 & j''_{22}\geqq 0\\
j'_{21}=\bar{j}'_{12} & \\
j''_{21}=\bar{j}''_{12} & \\
j'_{11}j'_{22}-j'_{12}j'_{21}\geqq 0 & \\
j''_{11}j''_{22}-j''_{12}j''_{21}\geqq 0 &
\end{pmatrix}. \tag{57}$$

The cases $(x_1, x_2;\ y_1, y_2;\ j'_{11}, j'_{12}, j'_{22};\ j''_{11}, j''_{12}, j''_{22}) = (1, 0;\ 0, 1;\ 1, 0, 0;\ 0, 0, 1)$, $(0, 1; 1, 0; 1, 0, 0; 0, 0, 1)$, $(1, 0; 1, 0; 1, 0, 0; 1, 0, 0)$, $(1, 0; 0, 1; 1, i, 1; 1, i, 1)$, $(1, 0;\ 0, 1;\ 1, i, 1;\ 1, -i, 1)$, $(1, 1; 1, -1; 0, 0, 1; 1, 0, 0)$, give as necessary conditions

$$\left.\begin{array}{ll} v_1 \geqq 0, \quad v_6 \geqq 0, & \\ v_1+v_2+v_3+v_4+v_5+v_6 \geqq 0, & v_1-v_2+v_3+v_4-v_5+v_6 \geqq 0, \\ v_1+v_2-v_3-v_4+v_5+v_6 \geqq 0, & v_1-v_2-v_3-v_4-v_5+v_6 \geqq 0, \end{array}\right\} \tag{58}$$

whence

$$(v_2+v_5)^R, \quad (v_3+v_4)^R.$$

The cases $(0, 1; 1, 0; 1, e', e'^2; 1, e'', e''^2)$, $(1, 0; 0, 1; 1, e', e'^2; 1, -e''i, e''^2)$, $(0, 1; 1, 0; 1, -e'i, e'^2; 1, e'', e''^2)$, $(1, 0; 0, 1; 1, -e'i, e'^2; 1, -e''i, e''^2)$ give the four quadratic forms

$$\left.\begin{array}{l} v_1 e'^2 + (v_2+v_3+v_4+v_5) e'e'' + v_6 e''^2 \geqq 0, \\ v_1 e''^2 + (v_3+v_5-v_2-v_4) e'e''i + v_6 e'^2 \geqq 0, \\ v_1 e'^2 + (v_2+v_3-v_4-v_5) e'e''i + v_6 e''^2 \geqq 0, \\ v_1 e''^2 + (v_3+v_4-v_2-v_5) e'e'' + v_6 e'^2 \geqq 0, \end{array}\right\} \tag{59}$$

whose discriminants give the necessary conditions,

$$\left.\begin{array}{ll} 4v_1v_6 - (v_2+v_3+v_4+v_5)^2 \geqq 0, & 4v_1v_6 + (v_3+v_5-v_2-v_4)^2 \geqq 0, \\ 4v_1v_6 + (v_2+v_3-v_4-v_5)^2 \geqq 0, & 4v_1v_6 - (v_3+v_4-v_2-v_5)^2 \geqq 0. \end{array}\right\} \tag{60}$$

From (59) it is seen also that v_2-v_5 and v_3-v_4 are pure imaginary, which with (58) gives the conditions,

$$v_2 = \overline{v}_5, \quad v_3 = \overline{v}_4 \tag{61}$$

Using the values

$$(x_1, x_2; y_1, y_2; j'_{11}, j'_{12}, j'_{22}; j''_{11}, j''_{12}, j''_{22}) = (1, 1; -1, 1; 0, 0, 1; 1, -e''i, e''^2),$$

the inequality (57) becomes

$$ve''^2 - 2e''i(v_2-v_3+v_4-v_5) + (v_1+v_6-v_2-v_3-v_4-v_5) \geqq 0 \quad (e''^R),$$

whence, since the discriminant of this quadratic in e'' must be positive or zero,

$$(v_1+v_6)^2 - 4(v_2+v_4)(v_3+v_5) \geqq 0. \tag{62}$$

The values $(1, 1; -1, 1; 1, -e'i, e'^2; 1, -e''i, e''^2)$ give:

$$\left.\begin{array}{l} e''^2 \{ ve'^2 - 2(v_2+v_3-v_4-v_5)e'i + (v_1+v_6-v_2-v_3-v_4-v_5) \} \\ \quad + 2e'' \{ -2(v_2-v_3-v_4+v_5)e' + (v_2-v_3+v_4-v_5)i(1-e'^2) \} \\ \quad + \{ (v_1+v_6-v_2-v_3-v_4-v_5)e'^2 + 2(v_2+v_3-v_4-v_5)e'i + v \} \end{array}\right\} \geqq 0 \quad (e'^R, e''^R),$$

whose discriminant may be written as a homogeneous quadratic expression in $(1-e'^2, 2e')$. Since for e' real, even between the values -1, and $+1$,

$2e'/(1-e'^2)$ takes every real value, the coefficient of $4e'^2$ and the discriminant of this homogeneous quadratic expression must be positive or zero. Hence,

$$\left.\begin{aligned} (v_1+v_6)^2-4(v_2-v_4)(v_5-v_3) &\geqq 0, \\ \{(v_1+v_6)^2-4(v_2v_5+v_3v_4)\}^2-64v_2v_3v_4v_5 &\geqq 0. \end{aligned}\right\} \tag{63}$$

From (62) and (63) it follows that

$$(v_1+v_6)^2-4(v_2v_5+v_3v_4)\geqq 0,$$

whence

$$(v_1+v_6)^2-4(v_2v_5-v_3v_4)\geqq 0,$$

and

$$(v_1+v_6)^2\geqq 4v_3v_4,$$

and by (58)

$$v_1+v_6\geqq \pm 2(v_3v_4)^{\frac{1}{2}}. \tag{64}$$

The second condition of (63) can be written

$$\{(v_1+v_6)^2-4(v_2v_5-v_3v_4)\}^2-16v_3v_4(v_1+v_6)^2\geqq 0,$$

and hence by (64),

$$(v_1+v_6)^2-4(v_2v_5-v_3v_4)\geqq \pm 4(v_1+v_6)(v_3v_4)^{\frac{1}{2}},$$

and

$$\{v_1+v_6\pm 2(v_3v_4)^{\frac{1}{2}}\}^2\geqq 4v_2v_5,$$

giving, by (64),

$$v_1+v_6\pm 2(v_3v_4)^{\frac{1}{2}}\geqq \pm 2(v_2v_5)^{\frac{1}{2}}. \tag{65}$$

In building the general inequality (55) as the sum of positive or zero multiples of the fundamental inequalities (56), it is desirable first to build those for which $v_1=v_6=\frac{1}{2}(v_1+v_6)\neq 0$. For such, aside from exceptional cases, the general inequality has the form

$$\tfrac{1}{2}(v_3v_4)^{\frac{1}{2}}(V_{1,(v_4/v_3)^{\frac{1}{2}}}+V_{3,(v_4/v_3)^{\frac{1}{2}}})+\tfrac{1}{4}d(V_{2,2v_2/d}+V_{4,2v_2/d}),$$

where

$$d=v_1+v_6-2(v_3v_4)^{\frac{1}{2}}+[\{v_1+v_6-2(v_3v_4)^{\frac{1}{2}}\}^2-4v_2v_5]^{\frac{1}{2}}.$$

When $v_1+v_6-2(v_3v_4)^{\frac{1}{2}}=0$, it follows from (65) that $v_2=v_5=0$, hence the desired inequality may be expressed as

$$\tfrac{1}{4}(v_1+v_6)(V_{1,2v_4/(v_1+v_6)}+V_{3,2v_4/(v_1+v_6)}).$$

In case $v_1\neq v_6$, the desired inequality is secured by adding to that already built

$$(v_1-v_6)V_{1,0} \quad \text{or} \quad (v_6-v_1)V_{3,0},$$

according as v_1 is greater than or less than v_6.

If $v_1=0$, or $v_6=0$, then, by (60), $v_2=v_3=v_4=v_5=0$, hence the desired inequality has the form

$$v_1V_{1,0}+v_6V_{3,0}.$$

All portions of this paper which refer to ξ^R, η^R; namely, § 4, § 8, and § 11, are valid for ξ and η pure imaginary, as an inequality for ξ and η pure imaginary reduces at once to the same inequality for ξ and η real.

On the three corresponding larger problems, retaining general operator and general operand, the writer has made considerable progress. In each case the nature of the coefficients a_{ijkl} and z_{ijkl} has been determined, and many necessary conditions on the coefficients have been secured, by use of binary, ternary, quaternary and quinary operators. As to the nature of the coefficients it has been found that in the problems corresponding to those of Part I and Part II,

$$\begin{array}{llll} a_{1234} \geqq 0, & a_{1243} \geqq 0, & a_{3421} \geqq 0, & a_{4321} \geqq 0, \\ a_{1342}{}^R, & a_{1432}{}^R, & a_{2341}{}^R, & a_{2431}{}^R, \\ & a_{1324} = \overline{a_{4231}}, & a_{1423} = \overline{a_{3241}}, & \end{array}$$

while in the problem on bilinear forms,

$$\begin{array}{llll} z_{1234} \geqq 0, & z_{1243} \geqq 0, & z_{3421} \geqq 0, & z_{4321} \geqq 0, \\ z_{3412} \geqq 0, & z_{4312} \geqq 0, & z_{2134} \geqq 0, & z_{2143} \geqq 0, \\ z_{1324} = \overline{z_{4231}}, & z_{1342} = \overline{z_{4213}}, & z_{1423} = \overline{z_{3241}}, & z_{1432} = \overline{z_{3214}}, \\ z_{2341} = \overline{z_{4123}}, & z_{2431} = \overline{z_{3124}}, & z_{2413} = \overline{z_{3142}}, & z_{2314} = \overline{z_{4132}}. \end{array}$$

A Trigonometrical Sum and the Gibbs' Phenomenon in Fourier's Series.[*]

BY H. S. CARSLAW.

§ 1. If $f(x)$ is a function which can be expanded in a Fourier's Series in the interval $0 \leqq x \leqq 2\pi$, with a discontinuity at $x=a$ such that $f(a-0)$ and $f(a+0)$ exist, the sum of the Fourier's Series for $x=a$ is $\frac{1}{2}\{f(a+0)+f(a-0)\}$.

Denoting by $S_n(x)$ the sum of the terms up to and including those in $\sin nx$ and $\cos nx$, the curves $y=S_n(x)$ are usually spoken of as the approximation curves for the series. Up till 1899 it was believed that each approximation curve, for large values of n, passes at a steep gradient from a point near $(a, f(a-0))$ to a point near $(a, \frac{1}{2}(f(a+0)+f(a-0)))$, and then on to a point near $(a, f(a+0))$, afterwards oscillating about the curve $y=f(x)$ till another discontinuity of $f(x)$ is met.

In 1899 J. Willard Gibbs pointed out[†] that the approximation curves for the sine series

$$2(\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \ldots.)$$

do not behave in this way.

In the interval $0 \leqq x \leqq 2\pi$, the sum of this series is $f(x)$, where $f(x)$ is defined as follows:

$$\left.\begin{array}{l} f(0)=f(2\pi)=0, \\ f(x)=x, \ 0<x<\pi, \\ f(x)=x-2\pi, \ \pi<x<2\pi. \end{array}\right\}$$

Gibbs stated in effect that the curve $y=S_n(x)$ for this series, for large values of n, rises just before $x=\pi$ to a point very nearly at a height $2\int_0^\pi \frac{\sin x}{x}\,dx$ above the axis of x, passes from that point through $(\pi, 0)$ at a steep gradient to a point very nearly at the same distance $2\int_0^\pi \frac{\sin x}{x}\,dx$ below the axis, while in the rest of the interval $0 \leqq x \leqq 2\pi$ it oscillates above and below the curve $y=f(x)$.

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[*] Communicated to the American Mathematical Society, October, 1915.

[†] *Nature*, Vol. LIX (1899), p. 606.

His statement was not accompanied by any proof. Though the remainder of the correspondence, of which his letter formed a part, attracted no little attention, this remarkable observation remained practically unnoticed for several years. In 1906 Bôcher returned to the subject in a memoir on Fourier's Series,* and extended Gibbs' results in two directions.

First of all he discussed the series

$$\sin x+\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x+\ldots,$$

which, in the interval $0\leqq x\leqq 2\pi$, represents the function $f(x)$ defined as follows:

$$\left.\begin{aligned} &f(0)=f(2\pi)=0,\\ &f(x)=\tfrac{1}{2}(\pi-x)\ldots 0<x<2\pi.\end{aligned}\right\}$$

He showed that the approximation curve

$$y=\sin x+\frac{1}{2}\sin 2x+\ldots+\frac{1}{n}\sin nx=S_n(x),$$

in the interval $0\leqq x\leqq 2\pi$, and for large values of n, takes the form of a wavy curve which keeps crossing and recrossing the line $y=\frac{1}{2}(\pi-x)$ and reaches its greatest distance from that line (distances being measured parallel to the axis of y) at the points $x_1, x_2, \ldots, x_{2n}$; where $x_r=\dfrac{2r\pi}{2n+1}$.

Further, these greatest distances in the r-th wave are, for large values of n, approximately equal to their limiting values

$$\tfrac{1}{2}\pi-\int_0^{r\pi}\frac{\sin x}{x}\,dx.$$

In the second place he showed that the phenomenon in question also appears in the Fourier's Series for a large class of functions. How large this class is will appear in the concluding section of this paper.

In Bôcher's own words:† If $S_n(x)$ denotes the sum of the Fourier's expansion of $f(x)$, the curve $y=S_n(x)$ will, for large values of n, pass in almost a vertical direction through a point whose abscissa is a and whose ordinate is almost $\frac{1}{2}\{f(a+0)+f(a-0)\}$. The curve then rises and falls abruptly on the two sides of this point to the neighborhood of the curve

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* *Annals of Mathematics* (2), Vol. VII (1906). See also a recent paper in *Crelle's Journal*, Bd. 144 (1914), entitled "On Gibbs' Phenomenon."

Reference should also be made to Runge's "Theorie und Praxis der Reihen" (1904), pp. 170–180. A certain series is there discussed, and the nature of the jump in the approximation curve described. But no reference is made to Gibbs, and the example seems to have been regarded as quite an isolated one.

† *Loc. cit.*, *Annals of Mathematics*, p. 131. Both Theorems I and II, p. 131, should be noted.

$y=f(x)$, and oscillates about this curve, lying alternately above and below it. The highest (or lowest) point of the k-th wave to the right and left of a will, for large values of n, be approximately at the points

$$a \pm \frac{2k\pi}{2n+1},$$

and the height of these waves will be approximately

$$\frac{f(a+0)-f(a-0)}{\pi} P_k,$$

$\left(\text{where } P_k = \tfrac{1}{2}\pi - \int_0^{k\pi} \frac{\sin x}{x}\,dx\right)$.

In the above sentence, a is a value of x at which $f(x)$ has an ordinary discontinuity, and Bôcher adds a footnote to the effect that "it must be borne in mind that we measure the height of a wave from the curve $y=f(x)$ in a direction parallel to the axis of y."

§ 2. In the function studied by Bôcher, $\frac{1}{2}(\pi-x)$,

$$\begin{aligned} S_n(x) &= \sin x + \frac{1}{2}\sin 2x + \dots + \frac{1}{n}\sin nx \\ &= \int_0^x (\cos\alpha + \cos 2\alpha + \dots + \cos n\alpha)\,d\alpha \\ &= \tfrac{1}{2}\int_0^x \frac{\sin(n+\frac{1}{2})\alpha}{\sin\frac{1}{2}\alpha}\,d\alpha - \tfrac{1}{2}x. \end{aligned}$$

The properties of the maxima and minima of $S_n(x)$ are not so easy to obtain, nor are they so useful in this case as those of

$$\begin{aligned} R_n(x) &= \tfrac{1}{2}(\pi-x) - S_n(x) \\ &= \tfrac{1}{2}\pi - \tfrac{1}{2}\int_0^x \frac{\sin(n+\frac{1}{2})\alpha}{\sin\frac{1}{2}\alpha}\,d\alpha. \end{aligned}$$

It is the maxima and minima of $R_n(x)$ with which Bôcher deals in his memoir.

Gronwall has discussed[*] the somewhat complicated properties of the maxima and minima of $S_n(x)$ for this series, and deduced Gibbs' Phenomenon for the first wave, and the general case of Fourier's Series.

In this paper I use the series

$$2(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \dots),$$

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[*] *Mathematische Annalen*, Bd. LXXII (1912). Some of Gronwall's results had been published a few months earlier by Dunham Jackson in a paper in the *Rendiconti di Circolo Matematico di Palermo*, T. XXXII (1911).

which, in the interval $-\pi \leqq x \leqq \pi$,* represents the function $f(x)$ defined by the equations

$$\left.\begin{array}{l} f(-\pi)=f(0)=f(\pi)=0, \\ f(x)=-\tfrac{1}{2}\pi \ldots . -\pi<x<0, \\ f(x)=\tfrac{1}{2}\pi \ldots . 0<x<\pi. \end{array}\right\}$$

I take $S_n(x)$ for the sum of the first n terms of this series, so that

$$S_n(x)=2\left(\sin x+\tfrac{1}{3}\sin 3x+\ldots .+\frac{1}{2n-1}\sin(2n-1)x\right).$$

A number of interesting properties of the turning points of $y=S_n(x)$ are obtained by quite simple methods; and all the features of Gibbs' Phenomenon for this series follow immediately from these properties.

In the concluding article the extension to the general case of Fourier's Series is given; but the method does not differ materially from that of Bôcher.

It is, of course, not an unusual thing that the curves $y=S_n(x)$ for a series, whose terms are continuous functions, differ considerably, even for large values of n, from the curve $y=\underset{n\to\infty}{\text{Lt.}}(S_n(x))=S(x)$.

They certainly do so in the neighborhood of a point where $S(x)$, the sum of the series, is discontinuous. And they also may do so in the neighborhood of a point where $S(x)$ is continuous.†

Ex. (i). If

$$S_n(x)=\frac{nx}{1+n^2x^2}, \quad x \geqq 0,$$

we have

$$S(x)=0, \qquad x \geqq 0.$$

In this case $S_n(x)$ has a maximum value $\frac{1}{2}$ at $x=\frac{1}{n}$. As n gets larger and larger, this summit is pushed towards $x=0$ but its height remains equal to $\frac{1}{2}$.

Ex. (ii). If

$$S_n(x)=\frac{n^2x}{1+n^3x^2}, \quad x \geqq 0,$$

we have

$$S(x)=0, \qquad x \geqq 0.$$

In this case $S_n(x)$ has a maximum value $\frac{1}{2}n^{\frac{1}{2}}$ at $x=\frac{1}{n^{\frac{3}{2}}}$. As n gets larger and larger, this summit is pushed towards $x=0$ but its height increases without limit.

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* It is more convenient for my purpose to take the interval $-\pi \leqq x \leqq \pi$ than $0 \leqq x \leqq 2\pi$. This does not affect the argument to any extent.

† Cf. Osgood, *Bulletin of the American Math. Soc.* (2), Vol. III (1897), p. 63; AMERICAN JOURNAL OF MATHEMATICS, Vol. XIX (1897), p. 155; Hobson, "Theory of Functions of a Real Variable," p. 480; Bromwich, "Theory of Infinite Series," p. 110; Carslaw, "Fourier's Series and Integrals," Ch. III.

The existence of maxima (or minima) of $S_n(x)$, the abscissæ of which tend towards a (a being a value of x for which the sum of the series is discontinuous), while their ordinates remain at a finite distance from $S(a+0)$ and $S(a-0)$, as n increases, is the chief feature of the Gibbs' Phenomenon in Fourier's Series. And it is most remarkable that its occurrence in Fourier's Series remained undiscovered till so recent a date.*

THE TRIGONOMETRICAL SUM

$$S_n(x)=2\left(\sin x+\tfrac{1}{3}\sin 3x+\ldots\ldots+\frac{1}{2n-1}\sin(2n-1)x\right).$$

§ 3. If we define $f(x)$ by the equations

$$\left.\begin{aligned} f(-\pi)&=f(0)=f(\pi),\\ f(x)&=\tfrac{1}{2}\pi\ldots\ldots 0<x<\pi,\\ f(x)&=-\tfrac{1}{2}\pi\ldots\ldots -\pi<x<0,\end{aligned}\right\}$$

the Fourier's Series for $f(x)$ in the interval $-\pi\leqq x\leqq\pi$ is

$$2(\sin x+\tfrac{1}{3}\sin 3x+\tfrac{1}{5}\sin 5x+\ldots\ldots).$$

We denote by $S_n(x)$ this sum up to and including the term in $\sin(2n-1)x$. Then

$$S_n(x)=2\int_0^x(\cos x+\cos 3x+\ldots\ldots+\cos(2n-1)x)\,dx=\int_0^x\frac{\sin 2n\alpha}{\sin\alpha}\,d\alpha.$$

Since $S_n(x)$ is an *odd* function, we need only consider the interval $0\leqq x\leqq\pi$.

We proceed to obtain the properties of the maxima and minima of this function $S_n(x)$.

I. Since, for any integer m, $\sin(2m-1)(\tfrac{1}{2}\pi+x')=\sin(2m-1)(\tfrac{1}{2}\pi-x')$, it follows from the series that *$S_n(x)$ is symmetrical about $x=\tfrac{1}{2}\pi$, and when $x=0$ and $x=\pi$ it is zero.*

II. *When $0<x<\pi$, $S_n(x)$ is positive.*

From (I) we need only consider $0<x\leqq\tfrac{1}{2}\pi$. We have

$$S_n(x)=\int_0^x\frac{\sin 2n\alpha}{\sin\alpha}\,d\alpha=\frac{1}{2n}\int_0^{2nx}\frac{\sin\alpha}{\sin\dfrac{\alpha}{2n}}\,d\alpha,\quad 0<x\leqq\tfrac{1}{2}\pi.$$

The denominator in the integrand is positive and continually increases in the interval of integration. By considering the successive waves in the graph of $\sin\alpha \operatorname{cosec}\dfrac{\alpha}{2n}$, the last of which may or may not be completed, it is clear that the integral is positive.

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* Cf. also Weyl: (i) "Die Gibbs'sche Erscheinung in der Theorie der Kugel-Funktionen," *Rend. Circ. Mat. d. Palermo,* T. XXIX (1910); (ii) "Über die Gibbs'sche Erscheinung und verwandte Konvergenzphänomene," *ibid.,* T. XXX (1910).

III. *The turning points of* $y=S_n(x)$ *are given by*

$$\begin{cases} x_1=\dfrac{\pi}{2n},\ x_3=\dfrac{3\pi}{2n},\ \ldots,\ x_{2n-1}=\dfrac{2n-1}{2n}\pi \ (\textit{maxima}), \\ x_2=\dfrac{\pi}{n},\ x_4=\dfrac{2\pi}{n},\ \ldots,\ x_{2(n-1)}=\dfrac{n-1}{n}\pi \ (\textit{minima}). \end{cases}$$

We have

$$y=\int_0^x \frac{\sin 2n\alpha}{\sin\alpha}\,d\alpha, \text{ and } \frac{dy}{dx}=\frac{\sin 2nx}{\sin x}.$$

The result follows at once.

IV. *As we proceed from* $x=0$ *to* $x=\frac{1}{2}\pi$, *the heights of the maxima continually diminish, and the heights of the minima continually increase, n being kept fixed.*

Consider two consecutive maxima in the interval $0<x\leqq\frac{1}{2}\pi$, namely, $S_n\left(\dfrac{2m-1}{2n}\pi\right)$ and $S_n\left(\dfrac{2m+1}{2n}\pi\right)$, m being a positive integer less than or equal to $\frac{1}{2}(n-1)$. We have

$$S_n\left(\frac{2m-1}{2n}\pi\right)-S_n\left(\frac{2m+1}{2n}\pi\right)=\frac{1}{2n}\int_{(2m+1)\pi}^{(2m-1)\pi}\frac{\sin\alpha}{\sin\dfrac{\alpha}{2n}}\,d\alpha$$

$$=-\frac{1}{2n}\left\{\int_{(2m-1)\pi}^{2m\pi}\frac{\sin\alpha}{\sin\dfrac{\alpha}{2n}}\,d\alpha+\int_{2m\pi}^{(2m+1)\pi}\frac{\sin\alpha}{\sin\dfrac{\alpha}{2n}}\,d\alpha\right\}.$$

The denominator in both integrands is positive and it continually increases in the interval $(2m-1)\pi\leqq\alpha\leqq(2m+1)\pi$; also the numerator in the first is continually negative and in the second continually positive; the absolute values for elements at equal distances from $(2m-1)\pi$ and $2m\pi$ being the same.

Thus the result follows. Similarly for the minima, we have to examine the sign of

$$S_n\left(\frac{m-1}{n}\pi\right)-S_n\left(\frac{m}{n}\pi\right),$$

where m is a positive integer less than or equal to $\frac{1}{2}n$.

V. *The first maximum to the right of* $x=0$ *is at* $x=\dfrac{\pi}{2n}$ *and its height continually diminishes as n increases. When n tends to infinity, its limit is*

$$\int_0^\pi \frac{\sin x}{x}\,dx.$$

We have

$$S_n\left(\frac{\pi}{2n}\right)=\int_0^{\frac{\pi}{2n}}\frac{\sin 2n\alpha}{\sin\alpha}\,d\alpha=\frac{1}{2n}\int_0^\pi \sin\alpha\operatorname{cosec}\frac{\alpha}{2n}\,d\alpha,$$

and

$$S_n\left(\frac{\pi}{2n}\right)-S_{n+1}\left(\frac{\pi}{2n+2}\right)=\int_0^\pi \sin\alpha\left(\frac{1}{2n}\operatorname{cosec}\frac{\alpha}{2n}-\frac{1}{2n+2}\operatorname{cosec}\frac{\alpha}{2n+2}\right)d\alpha.$$

Since $\alpha/\sin\alpha$ continually increases from 1 to ∞, as α passes from 0 to π, it is clear that in the interval with which we have to deal

$$\frac{1}{2n}\operatorname{cosec}\frac{\alpha}{2n}-\frac{1}{2n+2}\operatorname{cosec}\frac{\alpha}{2n+2}>0.$$

Thus
$$S_n\left(\frac{\pi}{2n}\right)-S_{n+1}\left(\frac{\pi}{2(n+1)}\right)>0.$$

But, from (I), $S_n(x)$ is positive when $0<x<\pi$.

It follows that $S_n\left(\dfrac{\pi}{2n}\right)$ tends to a limit as n tends to infinity.

The value of this limit can be obtained by the method used by Bôcher for the integral $\displaystyle\int_0^\pi \frac{\sin(n+\frac{1}{2})\alpha}{\sin\frac{1}{2}\alpha}\,d\alpha$.* But it is readily obtained from the definition of the Definite Integral as the limit of a sum.

For we have

$$S_n\left(\frac{\pi}{2n}\right)=2\left(\frac{\pi}{2n}\right)\left\{\frac{2n}{\pi}\sin\frac{\pi}{2n}+\frac{2n}{3\pi}\sin\frac{3\pi}{2n}+\dots+\frac{2n}{(2n-1)\pi}\sin\frac{2n-1}{2n}\pi\right\}$$
$$=2\sum_{\substack{m=1\\2nh=\pi}}^{2n}\left(\frac{\sin mh}{mh}h\right)-\sum_{\substack{m=1\\nh=\pi}}^{n}\left(\frac{\sin mh}{mh}h\right).$$

Therefore

$$\operatorname*{Lt.}_{n\to\infty} S_n\left(\frac{\pi}{2n}\right)=2\int_0^\pi\frac{\sin x}{x}\,dx-\int_0^\pi\frac{\sin x}{x}\,dx=\int_0^\pi\frac{\sin x}{x}\,dx.$$

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* *Loc. cit., Annals of Mathematics,* p. 124. Also Hobson, *loc. cit.,* p. 649.

VI. The result obtained in (V) for the first wave is a special case of the following:

The r-th maximum to the right of $x=0$ *is at* $x_{2r-1}=\frac{2r-1}{2n}\pi$, *and its height continually diminishes as n increases, r being kept constant. When n tends to infinity, its limit is* $\int_0^{(2r-1)\pi}\frac{\sin x}{x}\,dx$, *which is greater than* $\frac{1}{2}\pi$.

The r-th minimum to the right of $x=0$ *is at* $x_{2r}=\frac{r}{n}\pi$, *and its height continually increases as n increases, r being kept constant. When n tends to infinity, its limit is* $\int_0^{2r\pi}\frac{\sin x}{x}\,dx$, *which is less than* $\frac{1}{2}\pi$.

To prove these theorems we consider first the integral

$$\int_0^{m\pi}\sin\alpha\left(\frac{1}{2n}\operatorname{cosec}\frac{\alpha}{2n}-\frac{1}{2n+2}\operatorname{cosec}\frac{\alpha}{2n+2}\right)d\alpha,$$

m being a positive integer less than or equal to $2n-1$, so that $0<\frac{\alpha}{2n}<\pi$ in the interval of integration.

Then

$$F(\alpha)=\frac{1}{2n}\operatorname{cosec}\frac{\alpha}{2n}-\frac{1}{2n+2}\operatorname{cosec}\frac{\alpha}{2n+2}>0$$

in this interval. (Cf. (V)).

Further,

$$\begin{aligned}F'(\alpha)&=\frac{1}{(2n+2)^2}\cos\frac{\alpha}{2n+2}\operatorname{cosec}^2\frac{\alpha}{2n+2}-\frac{1}{(2n)^2}\cos\frac{\alpha}{2n}\operatorname{cosec}^2\frac{\alpha}{2n}\\&=\alpha^{-2}\{\phi^2\cos\phi\operatorname{cosec}^2\phi-\psi^2\cos\psi\operatorname{cosec}^2\psi\},\end{aligned}$$

where $\phi=\alpha/(2n+2)$ and $\psi=\alpha/2n$.

But

$$\frac{d}{d\phi}(\phi^2\cos\phi\operatorname{cosec}^2\phi)=-\phi\operatorname{cosec}^3\phi\,[\phi(1\mp\cos\phi)^2\pm2\cos\phi(\phi\mp\sin\phi).$$

And the right-hand side of the equation will be seen to be negative, choosing the upper signs for $0<\phi<\frac{1}{2}\pi$ and the lower for $\frac{1}{2}\pi<\phi<\pi$.

Therefore $\phi^2\cos\phi\operatorname{cosec}^2\phi$ diminishes as ϕ increases from 0 to π.

It follows, from the expression for $F'(\alpha)$, that $F'(\alpha)>0$, and $F(\alpha)$ increases with α in the interval of integration.

The curve

$$y = \sin x \left(\frac{1}{2n} \operatorname{cosec} \frac{x}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{x}{2n+2}\right), \ldots, 0 < x < m\pi,$$

thus consists of a succession of waves of length π, alternately above and below the axis, and the absolute values of the ordinates at points at the same distance from the beginning of each wave continually increase.

It follows that, when m is equal to $2, 4, \ldots, 2(n-1)$, the integral

$$\int_0^{m\pi} \sin \alpha \left(\frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2}\right) d\alpha$$

is negative; and, when m is equal to $1, 3, \ldots, 2n-1$, this integral is positive.

Returning to the maxima and minima, we have, for the r-th maximum to the right of $x=0$,

$$\begin{aligned} S_n(x_{2r-1}) - S_{n+1}(x_{2r-1}) &= \int_0^{\frac{2r-1}{2n}\pi} \frac{\sin 2n\alpha}{\sin \alpha} d\alpha - \int_0^{\frac{2r-1}{2(n+1)}\pi} \frac{\sin 2(n+1)\alpha}{\sin \alpha} d\alpha \\ &= \int_0^{(2r-1)\pi} \sin \alpha \left(\frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2}\right) d\alpha. \end{aligned}$$

Therefore from the above argument, $S_n(x_{2r-1}) > S_{n+1}(x_{2r-1})$.

Also for the r-th minimum to the right of $x=0$, we have

$$S_n(x_{2r}) - S_{n+1}(x_{2r}) = \int_0^{2r\pi} \sin \alpha \left(\frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2}\right) d\alpha,$$

and $S_n(x_{2r}) < S_{n+1}(x_{2r})$.

By an argument similar to that at the close of (V) we have

$$\underset{n \to \infty}{\text{Lt.}}\ S_n(x_r) = \int_0^{r\pi} \frac{\sin x}{x} dx.^*$$

It is clear that these limiting values are all greater than $\frac{1}{2}\pi$ for the maxima, and positive and less than $\frac{1}{2}\pi$ for the minima.

THE GIBBS' PHENOMENON FOR THE SERIES

$2(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots)$.

§ 4. From the Theorems (I)–(VI) of § 3 all the features of the Gibbs' Phenomenon for the series

$$2(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \ldots) \ldots -\pi \leqq x \leqq \pi$$

follow immediately.

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* For the values of $\int_0^{r\pi} \frac{\sin x}{x} dx$, see Bôcher, *loc. cit.*, *Annals of Mathematics*, p. 129.

It is obvious that we need only examine the interval $0 \leqq x \leqq \pi$, and that the discontinuity occurs at $x=0$.

For large values of n, the curve

$$y=S_n(x),$$

where $S_n(x)=2\left(\sin x+\frac{1}{3}\sin 3x+\ldots+\frac{1}{2n-1}\sin(2n-1)x\right)$, rises at a steep gradient from the origin to its first maximum, which is very near, but above, the point $\left(0, \int_0^{\pi} \frac{\sin x}{x}dx\right)$ {§ 3, V}. The curve, then, falls at a steep gradient, without reaching the axis of x {§ 3, II}, to its first minimum, which is very near, but below, the point $\left(0, \int_0^{2\pi} \frac{\sin x}{x}dx\right)$ {§ 3, VI}. It then oscillates above and below the line $y=\frac{1}{2}\pi$, the heights (and depths) of the waves continually diminishing as we proceed from $x=0$ to $x=\frac{1}{2}\pi$ {§ 3, IV}; and from $x=\frac{1}{2}\pi$ to $x=\pi$, the procedure is reversed, the curve in the interval $0 \leqq x \leqq \pi$ being symmetrical about $x=\frac{1}{2}\pi$ {§ 3, I}.

The highest (or lowest) point of the r-th wave to the right of $x=0$ will, for large values of n, be at a point whose abscissa is $\frac{r\pi}{2n}$ {§ 3, III} and whose ordinate is very nearly $\int_0^{r\pi} \frac{\sin x}{x}dx$ {§ 3, VI}.

By increasing n the curve for $0 \leqq x \leqq \pi$ can be brought as close as we please to the lines

$$\left.\begin{array}{ll} x=0, & 0<y<\int_0^{\pi} \frac{\sin x}{x}dx, \\ 0<x<\pi, & y=\frac{\pi}{2}, \\ x=\pi, & 0<y<\int_0^{\pi} \frac{\sin x}{x}dx. \end{array}\right\}$$

We may state these results more definitely as follows:

(i) If ϵ is any positive number, as small as we please, there is a positive integer ν' such that

$$\left|\tfrac{1}{2}\pi-S_n(x)\right|<\epsilon \text{ for } n \geqq \nu',\ \epsilon \leqq x \leqq \tfrac{1}{2}\pi.$$

This follows from the uniform convergence of the Fourier's Series for $f(x)$ in an interval which does not include a discontinuity of $f(x)$.

(ii) Since the height of the first maximum to the right of $x=0$ tends from above to $\int_0^\pi \frac{\sin x}{x}\,dx$ as n tends to infinity, there is a positive integer ν'' such that

$$0<S_n\left(\frac{\pi}{2n}\right)-\int_0^\pi \frac{\sin x}{x}\,dx<\varepsilon \text{ for } n\geqq \nu''.$$

(iii) Let ν''' be the integer next greater than $\frac{\pi}{2\varepsilon}$. Then the abscissa of the first maximum to the right of $x=0$, for $n\geqq\nu'''$, is less than ε.

It follows from (i), (ii) and (iii) that, if ν is the greatest of the positive integers ν', ν'' and ν''', the curve $y=S_n(x)$, *for* $n\geqq\nu$, behaves as follows:

It rises at a steep gradient from the origin to its first maximum, which is above $\int^\pi \frac{\sin x}{x}\,dx$ and within the rectangle

$$0<x<\varepsilon,\ 0<y<\int_0^\pi \frac{\sin x}{x}\,dx+\varepsilon.$$

After leaving this rectangle, in which there may be many oscillations about $y=\frac{1}{2}\pi$, it remains within the rectangle $\varepsilon<x<\pi-\varepsilon$, $\frac{1}{2}\pi-\varepsilon<y<\frac{1}{2}\pi+\varepsilon$.

Finally, it enters the rectangle

$$\pi-\varepsilon<x<\pi,\ 0<y<\int_0^\pi \frac{\sin x}{x}\,dx+\varepsilon,$$

and the procedure in the first region is repeated.[*]

The Gibbs' Phenomenon for the Fourier's Series in General.

§ 5. Let $f(x)$ be a function with an ordinary discontinuity when $x=a$, which can be expanded in a Fourier's Series in the interval $-\pi\leqq x\leqq\pi$.

Denote as usual by $f(a+0)$ and $f(a-0)$, the values towards which $f(x)$ tends as x approaches a from above or below. It will be convenient to consider $f(a+0)$ as greater than $f(a-0)$ in the description of the curve, but this restriction is in no way necessary.

Let $\phi(x-a)=2\sum_1^\infty \frac{1}{(2r-1)}\sin(2r-1)(x-a)$. Then

$$\left.\begin{aligned}\phi(x-a)&=\tfrac{1}{2}\pi, \text{ when } a<x<\pi+a,\\ \phi(x-a)&=-\tfrac{1}{2}\pi, \text{ when } -\pi+a<x<a,\\ \phi(+0)&=\tfrac{1}{2}\pi,\ \phi(-0)=-\tfrac{1}{2}\pi,\\ \phi(0)&=0 \text{ and } \phi(x)=\phi(x+2\pi).\end{aligned}\right\}$$

---

[*] The cosine series

$$\frac{\pi}{4}-\left[\cos x-\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x+\dots\right]$$

which represents 0 in the interval $0\leqq x<\frac{\pi}{2}$ and $\frac{\pi}{2}$ in the interval $\frac{\pi}{2}<x\leqq\pi$ can be treated in the same way as the series discussed in this article.

Now put

$$\psi(x)=f(x)-\tfrac{1}{2}\{f(a+0)+f(a-0)\}-\frac{1}{\pi}\{f(a+0)-f(a-0)\}\phi(x-a),$$

and let $f(x)$, for $x=a$, be defined as $\tfrac{1}{2}\{f(a+0)+f(a-0)\}$.

Then $\psi(a+0)=\psi(a-0)=\psi(a)=0$, and $\psi(x)$ is continuous at $x=a$.

The following distinct steps in the argument are numbered for the sake of clearness:

(i) Since $\psi(x)$ is continuous at $x=a$ and $\psi(a)=0$, if ε is a positive number, as small as we please, a number δ exists such that

$$|\psi(x)|<\frac{\varepsilon}{4} \text{ for } |x-a|<\delta.$$

If δ is not originally less than ε, we can choose this part of δ for our interval.

(ii) $\psi(x)$ can be expanded in a Fourier's Series, this series being uniformly convergent in an interval $\alpha\leqq x\leqq\beta$ which includes no other discontinuity of $f(x)$ and $\phi(x-a)$ than $x=a$.

Let $s_n(x)$, $\phi_n(x-a)$ and $\sigma_n(x)$ be the sums of the terms up to and including those in $\sin nx$ and $\cos nx$ in the Fourier's Series for $f(x)$, $\phi(x-a)$ and $\psi(x)$. Then ε being the positive number of (i), as small as we please, there exists a positive integer ν' such that

$$|\sigma_n(x)-\psi(x)|<\frac{\varepsilon}{4} \text{ for } n\geqq\nu' \text{ in } \alpha\leqq x\leqq\beta.$$

Also

$$|\sigma_n(x)|\leqq|\sigma_n(x)-\psi(x)|+|\psi(x)|<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}<\frac{\varepsilon}{2}$$

in $|x-a|<\delta$, if $\alpha<a-\delta<a<a+\delta<\beta$.

(iii) Now if n is even, the first maximum in $\phi_n(x-a)$ to the right of $x=a$ is at $a+\frac{\pi}{n}$; and if n is odd, it is at $a+\frac{\pi}{n+1}$. In either case there exists a positive integer ν'' such that the height of the first maximum lies between $\int_0^\pi \frac{\sin x}{x}dx$ and $\int_0^\pi \frac{\sin x}{x}dx+\frac{\pi\varepsilon}{2\{f(a+0)-f(a-0)\}}$ for $n\geqq\nu''$.

(iv) This first maximum will have its abscissa between a and $a+\delta$, provided that $\frac{\pi}{n}<\delta$.

Let ν''' be the first positive integer which satisfies this inequality.

(v) In the interval $a+\delta \leqq x \leqq \beta$, $s_n(x)$ converges uniformly to $f(x)$. Therefore, there exists a positive integer $\nu^{(\mathrm{iv})}$ such that for $n \geqq \nu^{(\mathrm{iv})}$

$$|f(x)-s_n(x)| < \varepsilon \text{ in this interval.}$$

Now from the equation defining $\psi(x)$ we have

$$s_n(x)=\tfrac{1}{2}(f(a+0)+f(a-0))+\frac{f(a+0)-f(a-0)}{\pi}\phi_n(x-a)+\sigma_n(x).$$

It follows from (i)–(v) that if ν is the first positive integer greater than ν', ν'', ν''' and $\nu^{(\mathrm{iv})}$, the curve $y=s_n(x)$ in the interval $a \leqq x \leqq \beta$, behaves as follows:

When $x=a$, it passes through a point whose ordinate is within $\frac{\varepsilon}{2}$ of $\frac{1}{2}(f(a+0)+f(a-0))$, and ascends at a steep gradient to its first maximum, which is at a height within ε of

$$\tfrac{1}{2}\{f(a+0)+f(a-0)\}+\frac{f(a+0)-f(a-0)}{\pi}\int_0^\pi \frac{\sin x}{x}\,dx.$$

This may be written

$$f(a+0)-\frac{f(a+0)-f(a-0)}{\pi}\int_\pi^\infty \frac{\sin x}{x}\,dx,$$

and, from Bôcher's table, referred to in § 3, we have

$$\int_\pi^\infty \frac{\sin x}{x}\,dx=-0.2811.$$

It then oscillates about $y=f(x)$ from $x=a$ to $a+\delta$, the character of the waves being determined by the function $\phi_n(x-a)$, since the term $\sigma_n(x)$ only adds a quantity less than $\frac{\varepsilon}{2}$ to the ordinate.

And on passing beyond $x=a+\delta$, the curve enters and remains within, the strip of width 2ε enclosing $y=f(x)$ from $x=a+\delta$ to $x=\beta$.

On the other side of the point a a similar set of circumstances can be established.

APPENDIX.

The following diagrams are the approximation curves

$$y = S_n(x)$$

of § 3 above, for $n = 1, 2, 3$ and 4. They illustrate the results (I–VI) of that section.

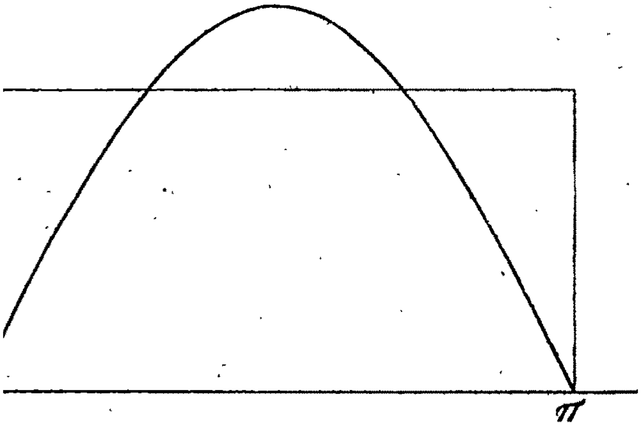

FIG. 1.

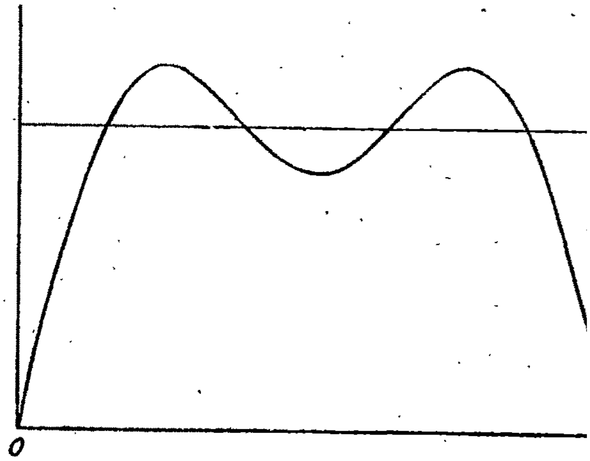

FIG. 2.

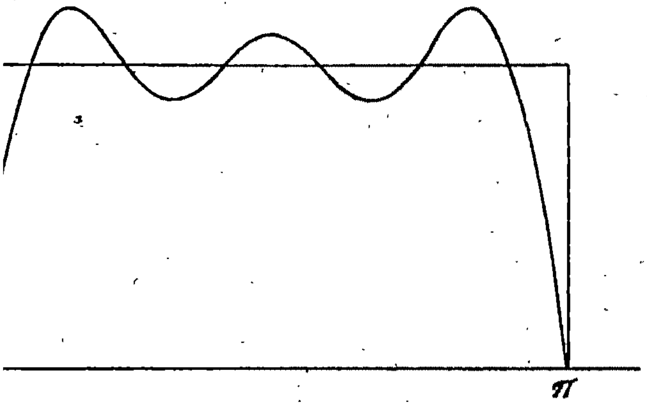

FIG. 3.

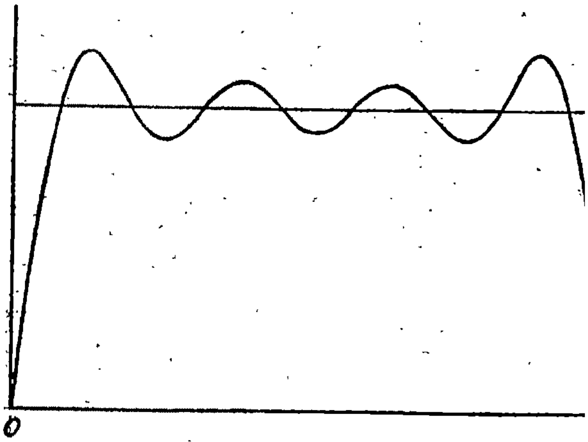

FIG. 4.

SYDNEY, AUSTRALIA, *August*, 1915.

On the Relation between Some Important Notions of Projective and Metrical Differential Geometry.[*]

BY F. M. MORRISON.

I. INTRODUCTION.

In his papers on the "Projective Differential Geometry of Curved Surfaces"[†] Professor E. J. Wilczynski has shown that the projective differential geometry of a surface may be based upon the consideration of a completely integrable system of two linear partial differential equations of the second order, which may be reduced to what he calls the intermediate form

$$y_{uu}+2ay_u+2by_v+cy=0, \quad y_{vv}+2a'y_u+2b'y_v+c'y=0. \tag{1}$$

Such a system of equations has just four linearly independent solutions, $y', y'', y^{(3)}, y^{(4)}$. If these be taken as the homogeneous coordinates of a point P_y in space, as u and v assume all of their values, the point P_y will describe a surface S, an integral surface of equations (1). This surface will be non-degenerate and non-developable, and will have the curves $u=$const. and $v=$const. as asymptotic lines. The most general integral surface of equations (1) is a projective transformation of any particular one, so that the coefficients of equations (1), in so far as they are intrinsically connected with the surface at all, will involve only its projective properties. But these coefficients also depend upon certain accidental elements of the analytic representation of the surface. To eliminate these accidental elements the notions of invariants and covariants are introduced.[‡] The projective properties of the surface are then expressed by invariant equations or system of equations. Other surfaces and other geometrical configurations which have a projective relation to it will be given by the covariants.

The four fundamental semi-covariants of system (1) are

$$y, \quad z=y_u+ay, \quad \rho=y_v+b'y, \quad \sigma=y_{uv}+b'y_u+ay_v+\tfrac{1}{2}(a_v+b'_u+2ab')y. \tag{2}$$

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[*] Presented to the American Mathematical Society, San Francisco Section, Seattle, Wash., May, 1914.

[†] *Transactions of the American Mathematical Society*, Vols. VIII, IX and X.

[‡] *Transactions*, Vol. VIII, p. 241, *et seq.*

If four linearly independent solutions $y', y'', y^{(3)}, y^{(4)}$ of (1) be put for y in the semi-covariants (2), three points P_z, P_ρ, P_σ will be obtained which are semi-covariantly connected with the point P_y.

By a properly chosen transformation of the form $y=\lambda\bar{y}$ equations (1) may be changed into the canonical form

$$y_{uu}+2by_v+fy=0, \quad y_{vv}+2a'y_u+gy=0. \tag{3}$$

The semi-covariants of (1) and (3) are connected with each other by the equations $y=\lambda\bar{y}$, $z=\lambda\bar{z}$, $\rho=\lambda\bar{\rho}$, $\sigma=\lambda\bar{\sigma}$. The semi-covariant points are the same for both (1) and (3). These points $P_y, P_z, P_\rho, P_\sigma$ form in general a non-degenerate tetrahedron which Professor Wilczynski has systematically used as tetrahedron of reference, for the purpose of studying the properties of a surface in the neighborhood of any one of its points.*

If the Cartesian coordinates of the surface are regarded as known, we may identify them with $y', y'', y^{(3)}$ and make $y^{(4)}$ equal unity. In the resulting system of partial differential equations of form (1) we shall then have $c=c'=0$, and system (1) becomes

$$y_{uu}+2ay_u+2by_v=0, \quad y_{vv}+2a'y_u+2b'y_v=0. \tag{4}$$

Let us now consider the familiar Gauss' equations of the metrical theory of surfaces, viz.:

$$\left.\begin{aligned} x_{uu}&=\begin{Bmatrix}11\\1\end{Bmatrix}x_u+\begin{Bmatrix}11\\2\end{Bmatrix}x_v+DX, \quad x_{uv}=\begin{Bmatrix}12\\1\end{Bmatrix}x_u+\begin{Bmatrix}12\\2\end{Bmatrix}x_v+D'X,\\ x_{vv}&=\begin{Bmatrix}22\\1\end{Bmatrix}x_u+\begin{Bmatrix}22\\2\end{Bmatrix}x_v+D''X. \end{aligned}\right\} \tag{5}$$

If D and D'' are equal to zero, the curves $u=$const. and $v=$const. are asymptotic lines, and we observe that the first and third of these equations are now of exactly the same form as equations (4). We can therefore express directly the coefficients of equations (4) in terms of the Christoffel symbols occurring in (5). We shall then be able to change from the homogeneous coordinate system with a semi-covariant tetrahedron of reference to any suitable Cartesian system. The coefficients of these transformations will be functions of the quantities E, F, G and D' and their derivatives with respect to u and v, quantities which are fundamental in the metrical theory of surfaces.

In this paper we shall determine these transformations and study some of the metrical properties of certain geometrical configurations associated with a point on a surface, which have been defined and investigated from a projective point of view by Professor Wilczynski, but which have not as yet been studied metrically. We can define by means of these configurations some new classes of special points on a surface and some new kinds of surfaces. We shall also apply these notions to a well-known special surface, the Minimal Surface of Enneper.

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* *Transactions*, Vol. IX, p. 79.

II. THE FUNDAMENTAL TRANSFORMATIONS OF COORDINATES.

The integrability conditions of system (4) are

$$\begin{aligned}
&a_v - b'_u = 0,\\
&a'_{uu} - 2a'a_u - 2aa'_u - (a_{vv} + 2b'a_v - 2ba'_v - 4a'b_v) = 0,\\
&b_{vv} - 2bb'_v - 2b'b_v - (b'_{uu} + 2ab'_u - 2a'b_u - 4ba'_u) = 0.
\end{aligned}$$

We shall assume that they are satisfied, so that (4) has four linearly independent solutions y', y'', $y^{(3)}$, 1, which we shall identify with the Cartesian coordinates of a point of a surface.

Comparing equations (4) and the first and third of equations (5) we have the relations

$$a = -\tfrac{1}{2}\left\{\begin{matrix}1\,1\\1\end{matrix}\right\}, \quad a' = -\tfrac{1}{2}\left\{\begin{matrix}2\,2\\1\end{matrix}\right\}, \quad b = -\tfrac{1}{2}\left\{\begin{matrix}1\,1\\2\end{matrix}\right\}, \quad b' = -\tfrac{1}{2}\left\{\begin{matrix}2\,2\\2\end{matrix}\right\}, \tag{6}$$

which express the values of the quantities a, b, a', b' in terms of the Christoffel symbols of the classical theory.

The *homogeneous* Cartesian coordinates of the points P_y, P_z, P_ρ and P_σ are obtained by substituting in order y', y'', $y^{(3)}$, $y^{(4)} = 1$ in each of the expressions (2). The *non-homogeneous* Cartesian coordinates of each of these points are the ratios of the corresponding four homogeneous coordinates concerned. The values obtained by this method for the Cartesian coordinates of the four points P_y, P_z, P_ρ, P_σ are given in the following table:

$$\begin{aligned}
&P_y\colon\ y',\ y'',\ y^{(3)}; \qquad P_z\colon\ y' + y'_u/a,\ y'' + y''_u/a,\ y^{(3)} + y^{(3)}_u/a;\\
&P_\rho\colon\ y' + y'_v/b',\ y'' + y''_v/b',\ y^{(3)} + y^{(3)}_v/b';\\
&P_\sigma\colon\ y' + (b'y'_u + ay'_v + y'_{uv})/R,\ y'' + (b'y''_u + ay''_v + y''_{uv})/R,\\
&\qquad\qquad\qquad\qquad\qquad y^{(3)} + (b'y^{(3)}_u + ay^{(3)}_v + y^{(3)}_{uv})/R,
\end{aligned}$$

where

$$R = \tfrac{1}{2}(a_v + b'_u + 2ab').$$

In speaking of y', y'', $y^{(3)}$ as the Cartesian coordinates of a point of the surface, we imply that some Cartesian system of coordinates, fixed in space, has been chosen as system of reference. Referred to this fixed Cartesian system, let the coordinates of a general point P of space be $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ and let the coordinates of the same point with reference to the semi-covariant surface tetrahedron $P_y, P_z, P_\rho, P_\sigma$ be x_1, x_2, x_3, x_4. Then the relations between the two sets of coordinates of the point P are*

$$\left.\begin{aligned}
\omega\bar{\xi} &= x_1 y' + x_2(y'_u + ay') + x_3(y'_v + b'y') + x_4(y'_{uv} + b'y'_u + ay'_v + Ry'),\\
\omega\bar{\eta} &= x_1 y'' + x_2(y''_u + ay'') + x_3(y''_v + b'y'') + x_4(y''_{uv} + b'y''_u + ay''_v + Ry''),\\
\omega\bar{\zeta} &= x_1 y^{(3)} + x_2(y^{(3)}_u + ay^{(3)}) + x_3(y^{(3)}_v + b'y^{(3)}) + x_4(y^{(3)}_{uv} + b'y^{(3)}_u + ay^{(3)}_v + Ry^{(3)}),\\
\omega &= x_1 + x_2 a + x_3 b' + x_4 R,
\end{aligned}\right\} \quad (7)$$

where we have written these relations in homogeneous form by introducing a factor of proportionality, ω.

The determinant of the right members of equations (7), which will be designated by Δ, is different from zero for any non-developable surface. As we shall see presently, it is equal to $-HD'$ (see equations (10)).

Let us write $y' = x$, $y'' = y$, $y^{(3)} = z$ in agreement with the usual notation for the Cartesian coordinates of a point. Then equations (7) become

$$\left.\begin{aligned}
\omega\bar{\xi} &= x_1 x + x_2(x_u + ax) + x_3(x_v + b'x) + x_4(x_{uv} + b'x_u + ax_v + Rx),\\
\omega\bar{\eta} &= x_1 y + x_2(y_u + ay) + x_3(y_v + b'y) + x_4(y_{uv} + b'y_u + ay_v + Ry),\\
\omega\bar{\zeta} &= x_1 z + x_2(z_u + az) + x_3(z_v + b'z) + x_4(z_{uv} + b'z_u + az_v + Rz),\\
\omega &= x_1 + x_2 a + x_3 b' + x_4 R.
\end{aligned}\right\} \quad (8)$$

If there be substituted in equations (8) the values of x_{uv}, y_{uv}, z_{uv} as given by the second Gauss equation

$$x_{uv} = \begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix} x_u + \begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix} x_v + D'X,$$

which holds also if y, z and Y, Z be substituted for x and X, respectively, these equations become

$$\left.\begin{aligned}
\omega\bar{\xi} &= x_1 x + x_2(x_u + ax) + x_3(x_v + b'x)\\
&\qquad + x_4[(b' - 2c)x_u + (a - 2d)x_v + Rx + D'X],\\
\omega\bar{\eta} &= x_1 y + x_2(y_u + ay) + x_3(y_v + b'y)\\
&\qquad + x_4[(b' - 2c)y_u + (a - 2d)y_v + Ry + D'Y],\\
\omega\bar{\zeta} &= x_1 z + x_2(z_u + az) + x_3(z_v + b'z)\\
&\qquad + x_4[(b' - 2c)z_u + (a - 2d)z_v + Rz + D'Z],\\
\omega &= x_1 + x_2 a + x_3 b' + x_4 R,
\end{aligned}\right\} \quad (9)$$

where

$$c = -\tfrac{1}{2}\begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix}, \quad d = -\tfrac{1}{2}\begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix}, \quad \Delta = -HD', \qquad (10)$$

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* *Transactions*, Vol. IX, p. 80.

and where X, Y, Z, H are the quantities usually denoted by these letters in the metrical theory of surfaces.

Solving equations (9) for x_1, x_2, x_3, x_4, we find

$$\left.\begin{aligned}
\omega x_1 = {} & \{SH^2X+D'[(aG-b'F)x_u-(aF-b'E)x_v]\}(\bar\xi-x) \\
& +\{SH^2Y+D'[(aG-b'F)y_u-(aF-b'E)y_v]\}(\bar\eta-y) \\
& +\{SH^2Z+D'[(aG-b'F)z_u-(aF-b'E)z_v]\}(\bar\zeta-z)-H^2D', \\
\omega x_2 = {} & \{(b'-2c)H^2X-D'(Gx_u-Fx_v)\}(\bar\xi-x) \\
& +\{(b'-2c)H^2Y-D'(Gy_u-Fy_v)\}(\bar\eta-y) \\
& +\{(b'-2c)H^2Z-D'(Gz_u-Fz_v)\}(\bar\zeta-z), \\
\omega x_3 = {} & \{(a-2d)H^2X+D'(Fx_u-Ex_v)\}(\bar\xi-x) \\
& +\{(a-2d)H^2Y+D'(Fy_u-Ey_v)\}(\bar\eta-y) \\
& +\{(a-2d)H^2Z+D'(Fz_u-Ez_v)\}(\bar\zeta-z), \\
\omega x_4 = {} & -H^2X(\bar\xi-x)-H^2Y(\bar\eta-y)-H^2Z(\bar\zeta-z),
\end{aligned}\right\} \quad (11)$$

where we have put

$$S=\tfrac{1}{2}(a_v+b'_u-2ab')+2(ac+bd).$$

In equations (9) and (11) we have transformations of the kind desired for the purposes of this paper, but we shall find it of advantage to make use also of the moving Cartesian system made up of the surface normal and the lines of curvature tangents of a general point on the surface. We shall speak of this system of coordinate axes as the *surface trihedral.*

The direction cosines of the tangents to the asymptotic curves $v=\text{const.}$ and $u=\text{const.}$ are

$$x_u/\sqrt{E},\ y_u/\sqrt{E},\ z_u/\sqrt{E} \quad\text{and}\quad x_v/\sqrt{G},\ y_v/\sqrt{G},\ z_v/\sqrt{G},$$

respectively. Therefore, the direction cosines of the tangents to the lines of curvature are

$$(x_v\sqrt{E}+x_u\sqrt{G})/2\beta\sqrt{EG},\quad (y_v\sqrt{E}+y_u\sqrt{G})/2\beta\sqrt{EG},\quad (z_v\sqrt{E}+z_u\sqrt{G})/2\beta\sqrt{EG}$$

and

$$(x_v\sqrt{E}-x_u\sqrt{G})/2\alpha\sqrt{EG},\quad (y_v\sqrt{E}-y_u\sqrt{G})/2\alpha\sqrt{EG},\quad (z_v\sqrt{E}-z_u\sqrt{G})/2\alpha\sqrt{EG},$$

where

$$\alpha=\sqrt{(\sqrt{EG}-F)/2\sqrt{EG}},\quad \beta=\sqrt{(\sqrt{EG}+F)/2\sqrt{EG}}. \qquad (12)$$

Let the coordinates of any point referred to the surface trihedral be ξ, η, ζ. We select the positive directions of these axes in such a way that the transformation of coordinates from the fixed Cartesian system to the surface trihedral is given by the following equations:

$$\left.\begin{aligned}\bar{\xi}&=\frac{x_v\sqrt{E}+x_u\sqrt{G}}{2\beta\sqrt{EG}}\xi+\frac{x_v\sqrt{E}-x_u\sqrt{G}}{2\alpha\sqrt{EG}}\eta+X\zeta+x,\\ \bar{\eta}&=\frac{y_v\sqrt{E}+y_u\sqrt{G}}{2\beta\sqrt{EG}}\xi+\frac{y_v\sqrt{E}-y_u\sqrt{G}}{2\alpha\sqrt{EG}}\eta+Y\zeta+y,\\ \bar{\zeta}&=\frac{z_v\sqrt{E}+z_u\sqrt{G}}{2\beta\sqrt{EG}}\xi+\frac{z_v\sqrt{E}-z_u\sqrt{G}}{2\alpha\sqrt{EG}}\eta+Z\zeta+z.\end{aligned}\right\}\tag{13}$$

Applying equations (13) to (11) we obtain

$$\left.\begin{aligned}\omega x_1&=\alpha D'(a\sqrt{G}+b'\sqrt{E})\xi-\beta D'(a\sqrt{G}-b'\sqrt{E})\eta+HS\zeta-HD',\\ \omega x_2&=-\alpha D'\sqrt{G}\xi+\beta D'\sqrt{G}\eta+H(b'-2c)\zeta,\\ \omega x_3&=-\alpha D'\sqrt{E}\xi-\beta D'\sqrt{E}\eta+H(a-2d)\zeta,\\ \omega x_4&=-H\zeta.\end{aligned}\right\}\tag{14}$$

These equations express the relations between the homogeneous coordinates x_1, x_2, x_3, x_4 *of a point, referred to the surface tetrahedron* $P_y, P_z, P_\rho, P_\sigma$ *and the Cartesian coordinates* ξ, η, ζ *of the same point, referred to the surface trihedral composed of the surface normal and the lines of curvature tangents.*

III. THE OSCULATING LINEAR COMPLEXES OF THE ASYMPTOTIC CURVES.

The osculating linear complexes of the asymptotic curves were first defined and discussed in detail from the standpoint of projective geometry by Professor Wilczynski.* If we denote by C' and C'' the osculating linear complexes of the asymptotic curves $v=\text{const.}$ and $u=\text{const.}$, respectively, he found the equations of these complexes to be

$$-b_v\,\bar{\omega}_{34}-b\,\bar{\omega}_{14}+b\,\bar{\omega}_{23}=0\tag{15}$$

and

$$-a'_u\,\bar{\omega}_{42}+a'\,\bar{\omega}_{14}+a'\,\bar{\omega}_{23}=0,\tag{16}$$

respectively, where the quantities $\bar{\omega}_{ik}$, the Plückerian homogeneous line coordinates, are connected with the homogeneous point coordinates of the system $P_y\,P_z\,P_\rho\,P_\sigma$ in the usual way. We wish to investigate some of the metrical properties of these complexes and shall use the special surface coordinate system which has been defined.

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* *Transactions*, Vol. IX, p. 90, *et seq.*

From equations (14) the transformations for the line coordinates are found to be

$$\left.\begin{aligned}
\overline{\omega}_{34}&=-\alpha HD'\sqrt{E}\omega_{31}+\beta HD'\sqrt{E}\omega_{23},\\
\overline{\omega}_{42}&=\alpha HD'\sqrt{G}\omega_{31}+\beta HD'\sqrt{G}\omega_{23},\\
\overline{\omega}_{14}&=\alpha HD'(a\sqrt{G}+b\sqrt{E})\omega_{31}+\beta HD'(a\sqrt{G}-b'\sqrt{E})\omega_{23}-H^2D'\omega_{34},\\
\overline{\omega}_{23}&=HD'^2\omega_{12}-\alpha HD'[(b'-2c)\sqrt{E}-(a-2d)\sqrt{G}]\omega_{31}\\
&\qquad\qquad+\beta HD'[(b'-2c)\sqrt{E}+(a-2d)\sqrt{G}]\omega_{23},
\end{aligned}\right\}\tag{17}$$

where the coordinates ω_{ik} are defined by the relations

$$\left.\begin{aligned}
&\omega_{14}=\xi_1-\xi_2, && \omega_{24}=\eta_1-\eta_2, && \omega_{34}=\zeta_1-\zeta_2,\\
&\omega_{23}=\eta_1\zeta_2-\eta_2\zeta_1, && \omega_{31}=\xi_2\zeta_1-\xi_1\zeta_2, && \omega_{12}=\xi_1\eta_2-\xi_2\eta_1;
\end{aligned}\right\}\tag{18}$$

ξ_1, η_1, ζ_1 and ξ_2, η_2, ζ_2 being two points of the line.

By transformations (17) the equation of complex C', equation (15), becomes

$$bD'\omega_{12}-\alpha M_1\omega_{31}+\beta M_2\omega_{23}+bH\omega_{34}=0,\tag{19}$$

where

$$M_1=M'\sqrt{E}+2bd\sqrt{G},\quad M_2=M'\sqrt{E}-2bd\sqrt{G},\quad M'=2bb'-2bc-b_v.\tag{20}$$

If we use the quantities ω_{ik} as given in (18), the equation of any linear complex may be written in the form

$$a_{12}\omega_{12}+a_{31}\omega_{31}+a_{14}\omega_{14}+a_{23}\omega_{23}+a_{34}\omega_{34}+a_{24}\omega_{24}=0,$$

where we have used the homogeneous notation for the coefficients also. The equations of the axis of the complex will then be*

$$\frac{\xi-\dfrac{a_{24}a_{12}-a_{34}a_{31}}{a_{23}^2+a_{31}^2+a_{12}^2}}{a_{23}}=\frac{\eta-\dfrac{a_{34}a_{23}-a_{14}a_{12}}{a_{23}^2+a_{31}^2+a_{12}^2}}{a_{31}}=\frac{\zeta-\dfrac{a_{14}a_{31}-a_{24}a_{23}}{a_{23}^2+a_{31}^2+a_{12}^2}}{a_{12}}.\tag{21}$$

Therefore, from (21) the equations of the axis of the complex C' are

$$\frac{\xi-(\alpha bHM_1/V'^2)}{\beta M_2}=\frac{\eta-(\beta bHM_2/V'^2)}{-\alpha M_1}=\frac{\zeta}{bD'},\tag{22}$$

where

$$V'^2=\alpha^2M_1^2+\beta^2M_2^2+b^2D'^2=M'^2E-4bdFM'+4b^2d^2G+b^2D'^2.\tag{23}$$

We shall call the plane through the surface point perpendicular to the axis of the complex the *principal plane*, and its intersection with the tangent plane the *principal line*. The equation of the principal plane with respect to the complex C' is found to be

$$\beta M_2\xi-\alpha M_1\eta+bD'\zeta=0,\tag{24}$$

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* Plücker, "Neue Geometrie des Raumes," p. 32.

and, therefore, the equations of the principal line are

$$\beta M_2\xi - \alpha M_1\eta = 0, \quad \zeta = 0. \tag{25}$$

The point on the principal line where the axis of the complex intersects the tangent plane is at a distance δ' from the surface point, where

$$\delta' = bHU'/V'^2 \text{ and } U'^2 = \alpha^2 M_1^2 + \beta^2 M_2^2. \tag{26}$$

In order to find the *parameter* of the complex we change from the surface trihedral to another rectangular system, ξ', η', ζ', whose origin is the point of intersection of the axis of the complex with the tangent plane, the ξ' axis coïnciding with the principal line and the ζ' axis with the axis of the complex.

From equations (25) and (22) the direction cosines of the ξ' and ζ' axes are

$$\alpha M_1/U', \ \beta M_2/U', \ 0, \ \text{and} \ \beta M_2/V', \ -\alpha M_1/V', \ bD'/V',$$

respectively. Consequently, the direction cosines of the η' axis are

$$-\beta bD'M_2/U'V', \ \alpha bD'M_1/U'V', \ U'/V',$$

the matter of the positive direction of the lines being left arbitrary. Therefore, the equations of transformation from the surface trihedral to the special coordinate system just defined are

$$\begin{aligned}
\xi &= (\alpha M_1/U')\xi' - (\beta bD'M_2/U'V')\eta' + (\beta M_2/V')\zeta' + (\alpha bHM_1/V'^2),\\
\eta &= (\beta M_2/U')\xi' + (\alpha bD'M_1/U'V')\eta' - (\alpha M_1/V')\zeta' + (\beta bHM_2/V'^2),\\
\zeta &= (U'/V')\eta' + (bD'/V')\zeta'.
\end{aligned}$$

The corresponding transformations for the line coordinates are found to be

$$\left.\begin{aligned}
\omega_{12} &= (bD'/V')\omega'_{12} + (U'/V')\omega'_{31} - (b^2HD'U'/V'^3)\omega'_{24} + (bHU'^2/V'^3)\omega'_{34},\\
\omega_{23} &= (\beta M_2/V')\omega'_{12} - (\beta bD'M_2/U'V')\omega'_{31} + (\alpha M_1/U')\omega'_{23}\\
&\qquad - (\beta bHM_2U'/V'^3)\omega'_{24} - (\beta b^2HD'M_2/V'^3)\omega'_{34},\\
\omega_{34} &= (U'/V')\omega'_{24} + (bD'/V')\omega'_{34},\\
\omega_{31} &= -(\alpha M_1/V')\omega'_{12} + (\alpha bD'M_1/U'V')\omega_{31} + (\beta M_2/U')\omega'_{23}\\
&\qquad + (\alpha bHM_1U'/V'^3)\omega'_{24} + (\alpha b^2HD'M_1/V'^3)\omega'_{34}.
\end{aligned}\right\} \tag{27}$$

Substituting the values (27) in equation (19) we have as the transformed equation of the complex the equation

$$\omega'_{12} + (b^2HD'/V'^2)\omega'_{34} = 0.$$

Therefore, the parameter of complex C' is

$$P' = b^2HD'/V'^2. \tag{28}$$

We now consider the corresponding properties of the complex C''. The equation of this complex referred to the surface trihedral is

$$a' D' \omega_{12} + \alpha N_1 \omega_{31} + \beta N_2 \omega_{23} - a' H \omega_{34} = 0, \tag{29}$$

where

$$N_1 = N' \sqrt{G} + 2a'c\sqrt{E}; \quad N_2 = N' \sqrt{G} - 2a'c\sqrt{E}, \quad N' = 2aa' - 2a'd - a'_u, \tag{30}$$

and the equations of its axis are

$$\frac{\xi - (\alpha a' H N_1 / V''^2)}{\beta N_2} = \frac{\eta + (\beta a' H N_2 / V''^2)}{\alpha N_1} = \frac{\zeta}{a'D'}, \tag{31}$$

where

$$V''^2 = \alpha^2 N_1^2 + \beta^2 N_2^2 + a'^2 D'^2 = N'^2 G - 4a'cFN' + 4a'^2 c^2 E + a'^2 D'^2. \tag{32}$$

The principal plane with respect to the complex C'' is given by

$$\beta N_2 \xi + \alpha N_1 \eta + a' D' \zeta = 0,$$

and, therefore, the principal line with respect to C'' has the equations

$$\beta N_2 \xi + \alpha N_1 \eta = 0, \quad \zeta = 0.$$

The distance from the surface point to the point in which the axis of this complex intersects the tangent plane is given by

$$\delta'' = a' H U'' / V''^2, \text{ where } U''^2 = \alpha^2 N_1^2 + \beta^2 N_2^2. \tag{33}$$

We make a similar change of coordinate system in order to find the parameter of complex C'' as in the case of complex C'. The transformed equation of the complex is

$$\omega''_{12} - (a'^2 H D' / V''^2) \omega''_{34} = 0,$$

so that the parameter has the value

$$P'' = -a'^2 H D' / V''^2. \tag{34}$$

Comparing the parameters P' and P'' of the two complexes we see that their values are opposite in sign, if the surface and the two asymptotic curves concerned are real. Therefore, *the linear complexes which osculate two real asymptotic curves of a real surface at their point of intersection are oppositely twisted.*

If ϕ is the angle between the axes of the complexes C' and C'' and if K denotes the perpendicular distance between them, we find from equations (22) and (31) that these quantities are given by

$$\cos \phi = Q^2 / V' V'', \tag{35}$$

$$K = HD' (a'^2 V'^2 - b^2 V''^2) Q^2 / V'^2 V''^2 \sqrt{V'^2 V''^2 - Q^4}, \tag{36}$$

where

$$Q^2 = \beta^2 M_2 N_2 - \alpha^2 M_1 N_1 + b a' D'^2. \tag{37}$$

IV. THE PENCIL OF LINEAR COMPLEXES DETERMINED BY THE COMPLEXES C' AND C''.

The equation of any complex C_λ of the pencil determined by C' and C'' is a linear combination of equations (19) and (29) and is, therefore, of the form

$$D'(b+\lambda a')\omega_{12}-\alpha(M_1-\lambda N_1)\omega_{31}+\beta(M_2+\lambda N_2)\omega_{23}+H(b-\lambda a')\omega_{34}=0. \quad (38)$$

The equation of the principal plane with respect to any complex C_λ of the pencil is

$$\beta(M_2+\lambda N_2)\xi-\alpha(M_1-\lambda N_1)\eta+D'(b+\lambda a')\zeta=0,$$

and, therefore, the equations of the principal line with respect to C_λ are

$$\beta(M_2+\lambda N_2)\xi-\alpha(M_1-\lambda N_1)\eta=0, \ \zeta=0. \quad (39)$$

The line conjugate to the surface normal with respect to C_λ is given by the equations

$$\alpha(M_1-\lambda N_1)\xi+\beta(M_2+\lambda N_2)\eta=0, \ \zeta=0. \quad (40)$$

All of these lines form a pencil in the tangent plane whose vertex is the point

$$\xi=\beta P_2/A, \ \eta=-\alpha Q_2/A, \ \zeta=0,$$

where we have put

$$A=M'N'-4ba'cd, \quad (41)$$

and

$$\left.\begin{aligned} P_2&=a'\sqrt{E}(M'-2bc)+b\sqrt{G}(N'-2a'd),\\ Q_2&=a'\sqrt{E}(M'-2bc)-b\sqrt{G}(N'-2a'd). \end{aligned}\right\} \quad (42)$$

Corresponding lines of the pencils given by equations (39) and (40) are perpendicular to each other. Therefore, the locus of their points of intersection is a circle. Its equations are

$$\xi^2+\eta^2-(\beta P_2/A)\xi+(\alpha Q_2/A)\eta=0, \ \zeta=0.$$

We may state the theorem:

The line which corresponds to the surface normal in any one of the complexes C_λ of the pencil is in the tangent plane of the corresponding surface point and perpendicular to the principal line with respect to that complex.

The special complexes of the pencil (38) are given by the values of λ which satisfy the equation of the form

$$a_{12}a_{34}+a_{31}a_{42}+a_{14}a_{23}=0.$$

These values are $-b/a'$ and b/a'. We shall call the corresponding special complexes the special complexes of the first and second kind, respectively.

If we put $\lambda = -b/a'$ in equation (38) we obtain the equation of the special complex of the first kind. It is

$$\alpha P_1 \omega_{31} - \beta Q_1 \omega_{23} - 2ba'H\omega_{34} = 0, \tag{43}$$

where

$$\left.\begin{aligned} P_1 &= a'\sqrt{E}(M'+2bc) + b\sqrt{G}(N'+2a'd), \\ Q_1 &= a'\sqrt{E}(M'+2bc) - b\sqrt{G}(N'+2a'd). \end{aligned}\right\} \tag{44}$$

The equations of the axis of this complex, or of the directrix of the first kind, are found to be

$$\frac{\xi - (2\alpha b a' H P_1/V_1^2)}{\beta Q_1} = \frac{\eta - (2\beta b a' H Q_1/V_1^2)}{-\alpha P_1} = \frac{\zeta}{0}, \tag{45}$$

where

$$V_1^2 = \alpha^2 P_1^2 + \beta^2 Q_1^2 = a'^2 V'^2 + b^2 V''^2 - 2ba'Q^2. \tag{46}$$

The equation of the special complex of the second kind is

$$2ba'D'\omega_{12} - \alpha Q_2 \omega_{31} + \beta P_2 \omega_{23} = 0, \tag{47}$$

and the equations of its axis, or of the directrix of the second kind, are

$$\xi/\beta P_2 = \eta/-\alpha Q_2 = \zeta/2ba'D', \tag{48}$$

in connection with which equations we introduce the quantity V_2 defined by

$$V_2^2 = \alpha^2 Q_2^2 + \beta^2 P_2^2 + 4b^2 a'^2 D'^2 = a'^2 V'^2 + b^2 V''^2 + 2ba'Q^2. \tag{49}$$

Equations (45) and (48) enable us to compute a number of important quantities. Let ψ be the angle between the two directrices, I the perpendicular distance between them, l, m, n the direction cosines of their common perpendicular, $\underline{\xi}, \underline{\eta}, \underline{\zeta}$ the coordinates of the middle point of the congruence. Then we shall have

$$\left.\begin{gathered} \cos\psi = (a'^2 V'^2 - b^2 V''^2)/V_1 V_2, \quad I = 2ba'HD'/V_4, \\ l = \alpha D' P_1/V_4, \quad m = \beta D' Q_1/V_4, \quad n = -HA/V_4, \\ \underline{\xi} = H(\alpha b a' D'^2 P_1 + \beta H A P_2)/V_4^2, \quad \underline{\eta} = H(\beta b a' D'^2 Q_1 - \alpha H A Q_2)/V_4^2, \\ \underline{\zeta} = ba'H^2 D' A/V_4^2, \end{gathered}\right\} \tag{50}$$

where

$$V_4^2 = V_1^2 D'^2 + H^2 A^2.$$

V. THE CYLINDROID FORMED BY THE AXES OF THE LINEAR COMPLEXES OF THE PENCIL DETERMINED BY THE COMPLEXES C' AND C''.

The ruled surface formed by the axes of a pencil of linear complexes is a special ruled surface of the third degree, usually known as a cylindroid. In order to obtain a simple form for the equation of the cylindroid, which belongs to the pencil of linear complexes C_λ, we introduce a new rectangular system of coordinates, whose origin is the middle point of the congruence, whose ζ''' axis is the common perpendicular of the two directrices, and whose ξ''' and η''' axes bisect the angles made by the lines parallel to the directrices through the origin.

From equations (45) and (48) we find the direction cosines of the ξ''' and η''' axes to be

$\beta S_1/W_1,\ -\alpha R_1/W_1,\ 2ba'D'V_1/W_1$ and $\beta S_2/W_2,\ -\alpha R_2/W_2,\ -2ba'D'V_1/W_2$,

respectively, where

$$\begin{aligned}
&V_3^2 = \alpha^2 P_1 Q_2 + \beta^2 P_2 Q_1,\\
&S_1 = Q_1 V_2 + P_2 V_1, \quad S_2 = Q_1 V_2 - P_2 V_1,\\
&R_1 = P_1 V_2 + Q_2 V_1, \quad R_2 = P_1 V_2 - Q_2 V_1,\\
&W_1^2 = 2 V_1 V_2 (V_1 V_2 + V_3^2), \quad W_2^2 = 2 V_1 V_2 (V_1 V_2 - V_3^2).
\end{aligned}$$

The direction cosines of the ζ''' axis were given among the relations (50). The positive directions in regard to these new coordinate axes are left arbitrary. Then the necessary transformations of the point coordinates from the surface trihedral to this new system of coordinates are given by

$$\begin{aligned}
\xi &= (\beta S_1/W_1)\xi''' + (\beta S_2/W_2)\eta''' + (\alpha D'P_1/V_4)\zeta''' + H(\alpha ba'D'^2P_1 + \beta H\Delta P_2)/V_4^2,\\
\eta &= (-\alpha R_1/W_1)\xi''' - (\alpha R_2/W_2)\eta''' + (\beta D'Q_1/V_4)\zeta''' + H(\alpha ba'D'^2Q_1 - \beta H\Delta Q_2)/V_4^2,\\
\zeta &= (2ba'D'V_1/W_1)\xi''' - (2ba'D'V_1/W_2)\eta''' - (H\Delta/V_4)\zeta''' + ba'H^2D'\Delta/V_4^2.
\end{aligned}$$

We shall not write down the corresponding transformations of the line coordinates.

We find the equations of the complexes C' and C'', referred to this new system of coordinates, to be

$$\begin{aligned}
&[ba'D'H(V_2+V_1)/W_1]\omega_{14}''' + [ba'D'H(V_2-V_1)/W_2]\omega_{24}'''\\
&\qquad + [(V_2+V_1)V_4/W_2]\omega_{23}''' - [(V_2-V_1)V_4/W_1]\omega_{31}''' = 0, \qquad (51)
\end{aligned}$$

and

$$\begin{aligned}
&[ba'D'H(V_2-V_1)/W_1]\omega_{14}''' + [ba'D'H(V_2+V_1)/W_2]\omega_{24}'''\\
&\qquad + [(V_2-V_1)V_4/W_2]\omega_{23}''' - [(V_2+V_1)V_4/W_1]\omega_{31}''' = 0, \qquad (52)
\end{aligned}$$

respectively.

According to the general theory of a pencil of linear complexes, when the equations of the fundamental complexes are in the form of equations (51) and (52), the equation of the cylindroid formed by the axes of the complexes of the pencil is of the form*

$$(\xi'''^2 + \eta'''^2)\zeta''' = (a_{14}/a_{23} - a_{24}/a_{31})\xi'''\eta''',$$

a_{14}, a_{23}, a_{24}, a_{31} being taken from the equation of either of the fundamental complexes. We find from either equation (51) or (52) that

$$\frac{a_{14}}{a_{23}} = ba'D'HW_2/V_4W_1 \text{, and } \frac{a_{24}}{a_{31}} = -ba'D'HW_1/V_4W_2.$$

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* Plücker, "Neue Geometrie des Raumes," p. 97.

Therefore, the equation of the cylindroid in this case is

$$(\xi'''^2+\eta'''^2)\zeta''' = (4ba'D'HV_1^2V_2^2/W_1W_2V_4)\xi'''\eta'''. \tag{53}$$

If we let

$$T = 2ba'D'HV_1^2V_2^2/W_1W_2V_4,$$

then the cylindroid lies between the two planes

$$\zeta''' = T \quad \text{and} \quad \zeta''' = -T;$$

that is, these planes are the tangent planes at the pinch points,

$$\xi'''=0,\ \eta'''=0,\ \zeta'''=T; \quad \xi'''=0,\ \eta'''=0,\ \zeta'''=-T.$$

VI. THE DIRECTRIX CURVES.

Professor Wilczynski has shown* that there exist, in general, two one-parameter families of curves on the surface S such that, if the point P describes one of these curves, both of its directrices will simultaneously describe developable surfaces. He called these curves *directrix curves.* The quadratic equation which determines the tangents of the two directrix curves which pass through the point P he found to be†

$$bL\delta u^2+2M\delta u\delta v-a'N\delta v^2=0,$$

where

$$\left.\begin{aligned} L &= -2a'(2ba'f+2abb_v+ba'_{uu})+ba'^2_u, \\ M &= 2ba'(a'b_{uv}-ba'_{uv})+2b^2a'_ua'_v-2a'^2b_ub_v, \\ N &= -2b(2ba'g+2ba'a'_u+a'b_{vv})+a'b_v^2, \end{aligned}\right\} \tag{54}$$

the surface S having been defined by equations (3). We wish to find the corresponding equation when the surface is defined by equations (1).

Equations (1) may be changed into equations (3) by a transformation of the form $y=\lambda\bar{y}$. Then the quantities f and g in equations (3) have the values‡

$$f=c-a_u-a^2-2bb', \quad g=c'-b'_v-b'^2-2aa'.$$

If in these expressions we make $c=c'=0$ and substitute the resulting values of f and g in the quantities (54), the corresponding equation of the directrix curves can be put in the form

$$\begin{aligned} b^2(C^2+2a'C_u+4a'^2B)\delta u^2-2[a'^2(bB_u-Bb_u)-b^2(a'C_v-Ca'_v)]\delta u\delta v \\ -a'^2(B^2+2bB_v+4b^2C)\delta v^2=0, \end{aligned} \tag{55}$$

in which the quantities B and C have the values

$$B=M'+2bc, \quad C=N'+2a'd.$$

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* *Transactions*, Vol. IX, p. 114, *et seq.*

† *Transactions*, Vol. IX, p. 116.

‡ *Transactions*, Vol. VIII, p. 246.

VII. SPECIAL POINTS.

Since the pencil of linear complexes formed from the osculating linear complexes of the asymptotic lines plays an important part in the study made in this paper, and since these complexes are indeterminate if a' or b is equal to zero, it is clear that we have tacitly assumed that neither a' nor b shall vanish. Therefore, if the surface contains straight lines, all points upon such straight lines are to be excluded and, of course, all ruled surfaces are also excluded from consideration.

We wish to discuss the values of K, $\cos\phi$ and $\cos\psi$ in (36), (35) and (50), respectively. These quantities are seen to be indeterminate in value when either a' or b is zero. If there are other values of the parameters different from those that make either a' or b zero, for which these quantities become indeterminate, the corresponding points must also be excluded. Thus, the developments of this chapter apply only to such points for which the quantity Q^2 or $a'^2V'^2-b^2V''^2$ is equal to zero while neither V', nor V'' nor $V'^2V''^2-Q^4$ is equal to zero.

a) *Points for Which the Axes of the Complexes C' and C'' Intersect.*

The axes of these complexes intersect when K is zero. This quantity is zero under the two separate conditions,

$$Q^2=0 \tag{56}$$

and

$$a'^2V'^2-b^2V''^2=0. \tag{57}$$

We observe that equation (56) makes the quantity $\cos\phi$ also zero and that $\cos\psi$ is zero under the condition (57). Therefore, we can state the following theorems:

If the axes of the osculating linear complexes of the asymptotic lines are determinate and intersect, either these axes are perpendicular to each other or the directrices of the first and second kind are perpendicular to each other, or both conditions may be fulfilled at the same time.

If the axes of the osculating linear complexes of the asymptotic lines are determinate and perpendicular to each other, they intersect.

If the directrices of the first and second kind are perpendicular to each other, the axes of the osculating linear complexes of the asymptotic lines intersect if they are determinate.

b) *Points for Which the Directrices of the First and Second Kind Are Perpendicular to Each Other.*

Such points were defined by equation (57). They are of particular interest on account of the further specialization which exists in the geometrical configuration to which the directrices belong and whose metrical properties we have developed.

If we consider the condition that the parameters P' and P'' of the complexes C' and C'' be numerically equal, we find from (28) and (34) that it is exactly equation (57). We have, therefore, the theorem:

If the asymptotic tangents of a point of a surface are real and if its directrices are perpendicular to each other, the parameters of the osculating linear complexes of the corresponding asymptotic lines are numerically equal and opposite in sign, and conversely.

The equation of the cylindroid reduces to

$$(\xi'''^2+\eta'''^2)\zeta''' = (2ba'D'/V_4)\xi'''\eta'''.$$

We see that the directrices are in this case the singular tangents to the cylindroid. Therefore, *when the directrices of the first and second kind are perpendicular to each other, they are the singular tangents to the cylindroid.*

Since the directrix of the first kind is in the tangent plane and in this case the two directrices are perpendicular to each other, the plane of the directrix of the second kind and the axis of the cylindroid must be perpendicular to the tangent plane. It is found to bisect the angle the axes of the complexes C' and C'' make with each other. Consequently, these axes must meet the tangent plane in points which are at the same distance from the surface point; that is, δ' and δ'' in (26) and (33), respectively, must be equal, which fact may easily be verified. We may state the theorem:

When the axes of the complexes C' and C'' intersect each other and the directrices of the first and second kind are perpendicular to each other, the axes make the same angle with the tangent plane and meet it in points equidistant from the surface point.

Let us write equation (57) in the form

$$(a'^2M'^2E-b^2N'^2G)-4ba'F(a'dM'-bcN')+4a'^2b^2(d^2G-c^2E)=0.$$

Evidently, it may be satisfied by means of the three separate equations

$$\left.\begin{array}{c}(a'M'\sqrt{E}-bN'\sqrt{G})(a'M'\sqrt{E}+bN'\sqrt{G})=0, \quad a'dM'-bcN'=0,\\ (d\sqrt{G}-c\sqrt{E})(d\sqrt{G}+c\sqrt{E})=0.\end{array}\right\} \qquad (58)$$

These may be divided into two sets, each of three separate equations, any two of which imply the third. They are

$$a'M'\sqrt{E}-bN'\sqrt{G}=0,\quad d\sqrt{G}-c\sqrt{E}=0,\quad a'dM'-bcN'=0, \tag{59}$$

and

$$a'M'\sqrt{E}+bN'\sqrt{G}=0,\quad d\sqrt{G}+c\sqrt{E}=0,\quad a'dM'-bcN'=0. \tag{60}$$

We observe that in the case given by equations (59) the directrix of the first kind is perpendicular to one of the principal normal planes, that consequently the directrix of the second kind and the axis of the cylindroid are in this plane, and that in the case of equations (60) a similar position exists with reference to the other principal normal plane.

c) *Points for Which the Directrix of the Second Kind Coincides with the Normal to the Surface.*

According to equation (48) the directrix of the second kind coincides with the normal to the surface if and only if P_2 and Q_2 separately are equal to zero; that is, if and only if

$$M'=2bc,\quad N'=2a'd. \tag{61}$$

Since for these points the directrices must be perpendicular to each other, these points must be included among the special points which were defined by equation (57). The fact that equation (57) is satisfied by equations (61) may easily be verified.

If in addition to equations (61) the equation

$$d\sqrt{G}-c\sqrt{E}=0$$

is satisfied, the directrix of the first kind is perpendicular to one of the principal normal planes, while if the additional equation which is satisfied is

$$d\sqrt{G}+c\sqrt{E}=0,$$

the same fact holds in regard to the other principal normal plane.

If the relations (61) be substituted in the quantities B and C which occur in equation (55), that equation of the directrix curves reduces to the form

$$E\delta u^2-G\delta v^2=0,$$

so that at these points the directrix curves coincide with the lines of curvature, as they obviously should.

By introducing in equations (61) the values of M' and N' from (20) and (30), respectively, we may write them in the form

$$b_v-2bb'+4bc=0,\quad a'_u-2aa'+4a'd=0. \tag{62}$$

If in the two general relations

$$\frac{H_u}{H}=\begin{Bmatrix}11\\1\end{Bmatrix}+\begin{Bmatrix}12\\2\end{Bmatrix},\quad \frac{H_v}{H}=\begin{Bmatrix}12\\1\end{Bmatrix}+\begin{Bmatrix}22\\2\end{Bmatrix},$$

we replace the Christoffel symbols by their values in a, b, c, d we obtain

$$c=-b'-H_v/2H,\quad d=-a-H_u/2H. \tag{63}$$

By means of these relations then, equations (62) become

$$Hb_v-2bH_v-6Hbb'=0,\quad Ha'_u-2a'H_u-6Haa'=0.$$

The conditions developed in this chapter characterize certain special kinds of special points on a surface if they are merely satisfied by special values of the parameters. They characterize special classes of surfaces if they can be satisfied identically. The existence, in general, of real points or surfaces of these several kinds will not be discussed in this paper. In the next chapter we shall show however by means of a particular surface that points of the several different kinds which have been defined actually exist.

VIII. The Minimal Surface of Enneper.

In order that the significance of the preceding developments may appear more clearly we proceed to apply them to a well-known surface, the Minimal Surface of Enneper. We shall obtain, in this way, some new properties of this surface.

The Minimal Surface of Enneper is represented by the equations

$$x=\tfrac{1}{4}(u+v)(u^2-4uv+v^2+6),\quad y=\tfrac{1}{4}(u-v)(u^2+4uv+v^2+6),\quad z=3uv, \tag{64}$$

where the curves $u=$const. and $v=$const. are the asymptotic lines, the lines of curvature being given by the equations $u+v=$const. and $u-v=$const.

The fundamental quantities E, F, G, H, D' and the quantities α, β, defined in (12), are found to have the values,

$$E=\frac{9}{8}W^2,\quad F=0,\quad G=\frac{9}{8}W^2,\quad H=\frac{9}{8}W^2,\quad D'=-3,\quad \alpha=\frac{1}{\sqrt{2}},\quad \beta=\frac{1}{\sqrt{2}},$$

where

$$W=u^2+v^2+2.$$

The quantities a, a', b, b' c, d which occur in (6) and (10) are,

$$a=-u/W,\quad a'=u/W,\quad b=v/W,\quad b'=-v/W,\quad c=-v/W,\quad d=-u/W,$$

We see from the values just given that the condition that neither a' nor b shall be zero implies for this surface that neither u nor v shall be equal to zero. The zero values of the parameters give the intersection of the surface with the XY-plane, which is made up of the two straight lines, $x+y=0$, $x-y=0$.

Therefore, every point upon these two straight lines is excluded from our considerations.

By easy substitutions we obtain the following equations and results in connection with the complexes C' and C'', the surface coordinate system being used:

Equation of complex C',

$$8v\omega_{12}-2(W-2v^2+2uv)\omega_{31}+2(W-2v^2-2uv)\omega_{23}-3vW^2\omega_{34}=0;$$

Equation of complex C'',

$$8u\omega_{12}+2(W-2u^2+2uv)\omega_{31}+2(W-2u^2-2uv)\omega_{23}+3uW^2\omega_{34}=0;$$

Equations of axis of complex C',

$$\frac{\xi+\frac{3}{4}v[W+2v(u-v)]}{-[W-2v(u+v)]}=\frac{\eta+\frac{3}{4}v[W-2v(u+v)]}{[W+2v(u-v)]}=\frac{\zeta}{4v}; \tag{65}$$

Equations of axis of complex C'',

$$\frac{\xi+\frac{3}{4}u[W-2u(u-v)]}{[W-2u(u+v)]}=\frac{\eta-\frac{3}{4}u[W-2u(u+v)]}{[W-2u(u-v)]}=\frac{\zeta}{4u}; \tag{66}$$

The parameters P' and P'' are given by

$$P'=-3v^2, \quad P''=3u^2.$$

We find that the expression Q^2 in (56) is identically zero except when the parameters have the value zero and that each of the two quantities V' and V'' has the value 9/8. No points then are excluded by the limitation that was placed upon the quantities V', V'' and Q^2. Therefore, every general point on the surface is of the type defined by equation (56), that is, for every such point the axes of the osculating linear complexes of the asymptotic lines intersect and are perpendicular to each other.

The equations of the special complexes of the first and second kind and of the two directrices are as follows:

Equation of special complex of first kind,

$$(u+v)\omega_{31}-(u-v)\omega_{23}+3uvW\omega_{34}=0;$$

Equation of special complex of second kind,

$$8uv\omega_{12}-(u-v)(W+4uv)\omega_{31}+(u+v)(W-4uv)\omega_{23}=0;$$

Equations of directrix of first kind,

$$\frac{\xi+(3uv(u+v)W/2(u^2+v^2))}{u-v}=\frac{\eta+(3uv(u-v)W/2(u^2+v^2))}{-(u+v)}=\frac{\zeta}{0}; \tag{67}$$

Equations of directrix of second kind,

$$\frac{\xi}{(u+v)(W-4uv)}=\frac{\eta}{-(u-v)(W+4uv)}=\frac{\zeta}{8uv}. \tag{68}$$

The expressions for the several quantities in (50) and (54) are given by,

$$\left.\begin{aligned}&\cos\psi=(u-v)(u+v)/(u^2+v^2),\quad I=6uv,\quad l=2(u+v)/W,\\ &\quad m=2(u-v)/W,\quad \eta=(W-4)/W,\\ &\quad \underline{\xi}=3(u+v)[(u-v)^4-4(u^2v^2+1)]/4W,\\ &\quad \underline{\eta}=-3(u-v)[(u+v)^4-4(u^2v^2+1)]/4W,\\ &\quad \underline{\zeta}=3uv(W-4)/W,\quad T=-3(u^2+v^2)/2.\end{aligned}\right\} \tag{69}$$

Equation (57) here assumes the form

$$(u-v)(u+v)=0.$$

Therefore, the directrices are perpendicular to each other at the points given by the two conditions, $u=v\neq 0$ and $u=-v\neq 0$. But the condition $u=v\neq 0$ satisfies equations (59) for this surface and the condition $u=-v\neq 0$ equations (60). Therefore, the points given by these two conditions are of the more special types defined by equations (59) and (60).

The curves defined on the surface by the two conditions $u=v\neq 0$ and $u=-v\neq 0$ are found from (64) to be given by

$$x^2=\frac{z}{27}(9-z)^2,\quad y=0, \tag{70}$$

and

$$x=0,\quad y^2=\frac{-z}{27}(9+z)^2, \tag{71}$$

respectively. Every point of these two plane cubic curves, except the origin, is of the special kind described.

For this surface the two equations (61) reduce to

$$u^2-3v^2+2=0,\quad 3u^2-v^2-2=0,$$

which have the solutions $u=\pm 1$ and $v=\pm 1$. These values of the parameters give four points of this very special type, for which the directrix of the second kind coincides with the normal to the surface. The points are

$$\begin{array}{llll} x=2, & x=-2, & x=0, & x=0,\\ y=0, & y=0, & y=2, & y=-2,\\ z=3, & z=3, & z=-3, & z=-3.\end{array}$$

Of course these points are on the curves given in (70) and (71) and are found to be the maximum and minimum points on the ovals of those curves with reference to their axes of symmetry.

Equation (55), the differential equation of the directrix curves, becomes $\delta u^2/u^2 - \delta v^2/v^2 = 0$, so that these curves are given by $uv = c_1$ and $u/v = c_2$. They may be represented parametrically by

$$\begin{aligned} x &= (u^2+c_1)\,[u^4+(6-4c_1)\,u^2+c_1^2]/4u^3, \\ y &= (u^2-c_1)\,[u^4+(6+4c_1)\,u^2+c_1^2]/4u^3, \\ z &= 3c_1, \end{aligned}$$

and

$$\begin{aligned} x &= u\,(1+c_2)\,[u^2(1-4c_2+c_2^2)+6c_2^2]/4c_2^3, \\ y &= u\,(1-c_2)\,[u^2(1+4c_2+c_2^2)+6c_2^2]/4c_2^3, \\ z &= 3u^2/c_2, \end{aligned}$$

respectively.

We observe that the directrix curve directions are conjugate, that along a curve given by $uv = c_1$, a plane curve, the two directrices are a constant distance apart and that for the points on the two curves (70) and (71) the directrix curve directions are the same as those of the lines of curvature.

It has been found that in general the axes of the two complexes C' and C'', whose equations were given in (72) and (73) intersect. Their point of intersection proves to be the middle point of the congruence, the point ξ, η, ζ given in (69), which we will call the point C.

The two directrices whose equations were given in (67) and (68) intersect the axis of the cylindroid, the line which goes through the point ξ, η, ζ and has the direction cosines l, m, n, as given in (69), in the points C_1 and C_2 which are as follows:

Point C_1,

$$\xi = \tfrac{3}{4}(u+v)\,[W-4\,(uv+1)], \quad \eta = -\tfrac{3}{4}(u-v)\,[W+4\,(uv-1)], \quad \zeta = 0,$$

Point C_2,

$$\begin{aligned} \xi &= 3\,(u+v)\,(W-4)\,(W-4uv)/4W, \\ \eta &= -3\,(u-v)\,(W-4)\,(W+4uv)/4W, \\ \zeta &= 6uv\,(W-4)/W. \end{aligned}$$

As the point P traces the surface each of the three points C, C_1, C_2 traces a surface which is covariant with the original surface.

From equations (13) the equations of transformation from the surface trihedral to the fixed trihedral, with reference to which the equations of the surface are given, are found to be

$$\begin{aligned} \bar{\xi} &= \frac{-2\,(uv-1)}{W}\,\xi - \frac{(u^2-v^2)}{W}\,\eta + \frac{2\,(u+v)}{W}\,\zeta + x, \\ \bar{\eta} &= \frac{(u^2-v^2)}{W}\,\xi - \frac{2\,(uv+1)}{W}\,\eta - \frac{2\,(u-v)}{W}\,\zeta + y, \\ \bar{\zeta} &= \frac{2\,(u+v)}{W}\,\xi + \frac{2\,(u-v)}{W}\,\eta + \frac{(W-4)}{W}\,\zeta + z. \end{aligned}$$

By means of these equations we find that the coordinates of the points C, C_1, C_2, referred to the fixed coordinate system, have the following form:

$$\begin{array}{llllll} C. & x=u^3+v^3, & C_1. & x=u^3+v^3, & C_2. & x=u^3+v^3, \\ & y=u^3-v^3, & & y=u^3-v^3, & & y=u^3-v^3, \\ & z=0, & & z=-3uv, & & z=3uv. \end{array}$$

These expressions for the coordinates of these points show several important relations: That the axis of the cylindroid is always perpendicular to the XY-plane, and consequently the axes of the complexes and the directrices are always parallel to this plane; that the middle point of the congruence is in the XY-plane and, therefore, the axes of the complexes are in this plane; that the covariant surfaces of the points C_1 and C_2 are exactly like in character, being reciprocally symmetrical surfaces with references to the XY-plane.

The covariant surfaces of the points C_1 and C_2 are the simple cubic surfaces

$$x^2-y^2=-\frac{4}{27}z^3, \text{ and } x^2-y^2=\frac{4}{27}z^3,$$

respectively. These surfaces have in common the lines $x+y=0$ and $x-y=0$ in the XY-plane, lines which are upon the Surface of Enneper itself and the points of which were excluded from our considerations.

It is easy to show that if the point P moves along a line of curvature, an asymptotic line or a directrix curve of the Enneper Surface, the three points C, C_1, C_2, which correspond to P in every case trace a plane curve.

We have studied in this paper the metrical properties of certain geometrical configurations which have been built up in connection with a point on a surface. We have made a study of these configurations for all the points of the Surface of Enneper to which the developments apply and have found the following properties of this surface:

a) *The axis of the cylindroid is always perpendicular to the XY-plane, with the middle point of the congruence always in this plane. Consequently, the axes of the osculating linear complexes of the asymptotic lines and the two directrices are always parallel to the XY-plane.*

b) *The axes of these complexes intersect and are perpendicular to each other. Their point of intersection is the middle point of the congruence and consequently they are in the XY-plane.*

c) *The surface contains two plane curves, the intersections of the surface by the XZ- and YZ-planes, at all of whose points, excepting the origin, the* *directrices of the first and second kind are perpendicular to each other, the axis of the cylindroid in each case lying in the plane of the curve. For all these points the parameters of the complexes are numerically equal and opposite in sign.*

d) *There are four points for which the directrix of the second kind coincides with the normal to the surface. Two of these points are upon each of the curves in the XZ- and YZ-planes and are the maximum and minimum points on the oval of these curves with reference to the axes of symmetry.*

By means of the consideration of this special Surface of Enneper we have verified the results of the paper and have established the existence of real points of all of the special types that have been defined; points for which the axes of the osculating linear complexes of the asymptotic lines intersect and are perpendicular to each other, other points for which these axes intersect while the directrices of the two kinds are perpendicular to each other, and points for which the directrix of the second kind coincides with the normal to the surface. In the absence of any general existence theorems the fact that such points are shown to exist has an important bearing.

THE UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON, *August*, 9, 1915.

A Comitant Curve of the Plane Quartic.[*]

By Teresa Cohen.

I. The Conic on a Line and the Four Tangents at its Intersections with a Plane Quartic.

If a quartic curve be cut by a line ξ and the tangents to the quartic at the intersections be drawn, the four tangents together with ξ itself determine a conic. This conic is to be expressed in terms of known comitants of the quartic.

For this it is necessary to know the degree of the conic both in ξ and in the coefficients of the quartic. While not known directly, it may be obtained indirectly by finding the contact of ξ with the conic. A line conic may be considered as the locus of lines cutting four fixed lines in a given double ratio. Now suppose two lines ξ and ξ' to cut the quartic in A, B, C, D and A', B', C', D', respectively, so that

$$(AB/CD) = (A'B'/C'D'). \tag{1}$$

Then the lines AA', BB', CC', DD', ξ, ξ' touch a conic. Now allow ξ' to approach ξ, $(A'B'/C'D')$ being kept constantly equal to (AB/CD). AA', BB', CC', DD' approach the tangents to the quartic at A, B, C, D, respectively, the conic approaches the required conic, and the intersection of ξ and ξ' approaches the contact of ξ with the latter. But ξ and ξ', cutting the quartic in a constant double ratio, are lines of a known locus. Therefore, the contact of ξ with the required conic is the same as its contact with a known curve.

A quartic $(ax)^4$ has two contravariants, $(s\xi)^4$ and $(t\xi)^6$, the loci of lines cut by the quartic in a self-apolar set and in harmonic pairs, respectively.[†] The locus of lines cutting the quartic in a given double ratio is

$$(s\xi)^4(s'\xi)^4(s''\xi)^4 - \lambda(t\xi)^6(t'\xi)^6 = 0,$$

or, more briefly,

$$s^3 - \lambda t^2 = 0,$$

---

[*] Presented to the American Mathematical Society, December 28, 1916.

[†] These are understood to be taken as given in Salmon, "Higher Plane Curves," so that in symbolic notation $(s\xi)^4 = \frac{1}{2}|\alpha\beta\xi|^4$ and $(t\xi)^6 = \frac{1}{6}|\beta\gamma\xi|^2|\gamma\alpha\xi|^2|\alpha\beta\xi|^2$.

where λ is a constant depending on the given double ratio. The polar point of ξ with respect to this curve is

$$(s\xi)^3(s\eta)(s'\xi)^4(s''\xi)^4-\lambda(t\xi)^5(t\eta)(t'\xi)^6=0,$$

or

$$s^2s_1-\lambda tt_1=0.$$

If ξ is on the curve, we may eliminate λ between these two equations. Throwing out the factor s^2t, since ξ is in general on neither s nor t, we obtain for the polar point, or point of contact, of ξ

$$(s\xi)^3(s\eta)(t\xi)^6-(s\xi)^4(t\xi)^5(t\eta)=0, \text{ or } s_1t-st_1=0.$$

This is of degree 5 in the coefficients of the quartic and 9 in ξ; therefore, the conic itself should be of degree 5 in the coefficients of the quartic and 8 in ξ, which may be stated symbolically by saying that the conic is an $A^5\xi^8\eta^2$.

Under certain circumstances the degree in ξ will be lowered. The fact that the polar of ξ is s_1t-st_1 indicates that if $(s\xi)^4$ or $(t\xi)^6$ has a repeated factor, this factor appears in the $A^5\xi^8\eta^2$. Also suppose the curve to have a double point and ξ to be a line through it. Then ξ intersects the quartic in points whose double ratio is 0, 1, ∞ according to the order in which they are taken. But since any line through the double point can be regarded as an improper sort of tangent, the question arises, What lines should be taken as the two tangents here? Since with this choice of ξ in equation (1) $A=B$, then $A'=B'$. Then AA' is the same line as BB' and since the tangents at A and B are the limiting positions of these as ξ' approaches ξ, then the two tangents at the double point are to be taken as coincident. But then any line η cuts the four tangents where ξ meets the quartic in the same double ratio as ξ does and the conic becomes arbitrary. Therefore, the double points factor out of the $A^5\xi^8\eta^2$ at least once. The degree in ξ is not high enough for them to occur more than once.

There are only five independent forms of proper order to make up the $A^5\xi^8\eta^2$. Therefore,

$$\begin{aligned}A^5\xi^8\eta^2=\lambda(s\xi)^2(s\eta)^2(t\xi)^6+\mu(s\xi)^3(s\eta)(t\xi)^5(t\eta)+\nu(s\xi)^4(t\xi)^4(t\eta)^2\\ +\rho(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)\,|\alpha\xi\eta|^2+\sigma(s\xi)^4(s'\xi)^2(s'\alpha)^2\,|\alpha\xi\eta|^2,\end{aligned}\tag{2}$$

where λ, μ, ν, ρ, σ are constant coefficients, the value of which is to be determined. Since the conic is on ξ,

$$\begin{gathered}\lambda(s\xi)^4(t\xi)^6+\mu(s\xi)^4(t\xi)^6+\nu(s\xi)^4(t\xi)^6=0,\\ \lambda+\mu+\nu=0.\end{gathered}\tag{I}$$

This is one relation on the coefficients. For further relations special quartics for which the conic is known must be used.

Special Cases.

A. The simplest quartic for which the conic is known is $4x_0(x_1^3+x_2^3)=0$, representing four lines, three of which pass through the point $(1, 0, 0)$. The conic for any line ξ is made up of this point and the point where ξ cuts the fourth line, $x_0=0$. Its equation is

$$\eta_0(\eta_1\xi_2-\eta_2\xi_1)=0.$$

The double points of the quartic and the repeated factors of $(s\xi)^4$ and $(t\xi)^6$ account for an outstanding factor of degree 7 in ξ. Therefore,

$$A^5\xi^8\eta^2=\xi_0^4(\xi_1^3-\xi_2^3)(\eta_0\eta_1\xi_2-\eta_0\eta_2\xi_1).$$

By substitution in equation (2),

$$\begin{aligned}A^5\xi^8\eta^2=\eta_0^2(\xi_0^3\xi_1^4\xi_2+\xi_0^3\xi_1\xi_2^4)(8\lambda+12\mu+\tfrac{48}{5}\nu+36\rho+48\sigma)\\ +\sum^{2}\eta_0\eta_1[\xi_0^4\xi_1^3\xi_2(16\lambda+18\mu+\tfrac{144}{5}\nu-54\rho-96\sigma)+\xi_0^4\xi_2^4(16\lambda+6\mu-18\rho)]\\ +\sum^{2}\eta_1^2\cdot\xi_0^5\xi_1^2\xi_2(6\mu+\tfrac{48}{5}\nu+18\rho+48\sigma)+\eta_1\eta_2(\xi_0^5\xi_1^3+\xi_0^5\xi_2^3)(8\lambda+6\mu+18\rho).\end{aligned}$$

From this come the relations

$$4\lambda+3\mu+9\rho=0, \tag{II}$$

and

$$\mu+\frac{8}{5}\nu+3\rho+8\sigma=0. \tag{III}$$

B. The same relations would have been obtained by use of the four general lines

$$3[x_0^4+x_1^4+x_2^4-2x_1^2x_2^2-2x_2^2x_0^2-2x_0^2x_1^2]=0,$$

for which the conic is evidently

$$\begin{vmatrix}\eta_0^2 & \eta_1^2 & \eta_2^2\\ \xi_0^2 & \xi_1^2 & \xi_2^2\\ 1 & 1 & 1\end{vmatrix}=0.$$

By including the six double points

$$\begin{aligned}A^5\xi^8\eta^2&=(\xi_1^2-\xi_2^2)(\xi_2^2-\xi_0^2)(\xi_0^2-\xi_1^2)\sum^{3}\eta_0^2(\xi_1^2-\xi_2^2)\\ &=\sum^{3}\eta_0^2(-\xi_0^4\xi_1^4+2\xi_0^4\xi_1^2\xi_2^2-\xi_0^4\xi_2^4+\xi_0^2\xi_1^6-\xi_0^2\xi_1^4\xi_2^2-\xi_0^2\xi_1^2\xi_2^4+\xi_0^2\xi_2^6-\xi_1^6\xi_2^2+2\xi_1^4\xi_2^4-\xi_1^2\xi_2^6).\end{aligned}$$

C. The lemniscate

$$6(x_1^2x_2^2+x_2^2x_0^2+x_0^2x_1^2)=0$$

is another case for which the conic can be found. Since it has three double points, $\xi_0\xi_1\xi_2$, and $(s\xi)^4=3(\xi^2)^2$, where $(\xi^2)=\xi_0^2+\xi_1^2+\xi_2^2$, the $A^5\xi^8\eta^2$ has the factor $\xi_0\xi_1\xi_2(\xi^2)$ and the conic reduces to the third degree in ξ.

That the conic can be found in this case is due to the property of the lemniscate that a line satisfying the equation $(\xi^2)=0$ cuts the lemniscate so that the four tangents at the intersections meet in the point $(\eta/\xi)=0$, or $(\eta_0\xi_1\xi_2)=0$. The conic, then, breaks up into two points, of which this is one; the other, found from s_1t-st_1, is $(\xi\eta)=0$. Therefore, for such lines as obey the relation $(\xi^2)=0$ the conic is

$$(\xi\eta)(\eta_0\xi_1\xi_2)=0.$$

But the conic in general can differ from this only by terms containing (ξ^2), and must be of the form

$$(\xi\eta)(\eta_0\xi_1\xi_2)+\varkappa(\xi^2)(\eta_1\eta_2\xi_0)=0.$$

Since it must be on ξ, $\varkappa=-1$. Therefore, the conic is

$$(\xi\eta)(\eta_0\xi_1\xi_2)-(\xi^2)(\eta_1\eta_2\xi_0)=0, \text{ or } \xi_0\xi_1\xi_2(\eta^2)-(\eta_1\eta_2\xi_0^3)=0.$$

It can furthermore be shown by direct proof that this conic touches the four tangents at the intersections of ξ with the quartic. Let y be one of these points. Then

$$(\xi y)=0, \quad \left(\frac{1}{y^2}\right)=0.$$

The tangent at y is $(x/y^3)=0$; that is, its coordinates are $1/y_0^3$, $1/y_1^3$, $1/y_2^3$. Substituting these values in the equation of the conic and remembering that if $a+b+c=0$, then $a^3+b^3+c^3=3abc$, we find

$$\xi_0\xi_1\xi_2\left(\frac{1}{y^6}\right)-\left(\frac{\xi_0^3}{y_1^3y_2^3}\right)=\frac{3\xi_0\xi_1\xi_2}{y_0^2y_1^2y_2^2}-\frac{(\xi^3y^3)}{y_0^3y_1^3y_2^3}=\frac{3\xi_0\xi_1\xi_2y_0y_1y_2-3\xi_0\xi_1\xi_2y_0y_1y_2}{y_0^3y_1^3y_2^3}=0.$$

Therefore, the tangent to the quartic at y is a line of the conic.

To obtain the $A^5\xi^8\eta^2$ the conic must be multiplied by the outstanding factor $(\xi^2)\xi_0\xi_1\xi_2$.

$$A^5\xi^8\eta^2=\overset{3}{\Sigma}\eta_0(\xi_0^4\xi_1^2\xi_2^2+\xi_0^2\xi_1^4\xi_2^2+\xi_0^2\xi_1^2\xi_2^4)+\overset{3}{\Sigma}\eta_1\eta_2\xi_1\xi_2(-\xi_0^6-\xi_0^4\xi_1^2-\xi_0^4\xi_2^2).$$

The relations on λ, μ, ν, ρ, σ, given by this special qnartic, however, are all more simply found from the following case.

D. The type of the conic for the quartic $x_0^4+x_1^4+x_2^4=0$ can be obtained from consideration of

$$s_1t-st_1=\xi_0\xi_1\xi_2[(\xi^3\eta)\xi_0\xi_1\xi_2-\tfrac{1}{3}(\xi^4)(\eta_0\xi_1\xi_2)].$$

To give such a polar as to ξ (the repeated factor $\xi_0\xi_1\xi_2$ of $(t\xi)^6$ being dropped) the conic must be of the form

$$l(\xi^2\eta^2)\xi_0\xi_1\xi_2+m(\xi^3\eta)(\eta_0\xi_1\xi_2)+n(\xi^4)(\eta_1\eta_2\xi_0),$$

where l, m, n are to be determined. Since the conic is on ξ,

$$l+3m+3n=0.$$

For further relations on l, m, n take a special line, say through the reference point $\xi_0=0$. The line is then of the form $\varkappa x_1+x_2=0$, and it meets the curve in four points given by

$$x_0^4+(1+\varkappa^4)x_1^4=0.$$

Let α be any one of the fourth roots of $-(1+\varkappa)^4$. Then the four points are

$$(\alpha, 1, -\varkappa),\ (-\alpha, 1, -\varkappa),\ (i\alpha, 1, -\varkappa),\ (-i\alpha, 1, -\varkappa);$$

the four tangents to the curve at these points are

$$\alpha^3x_0+x_1-\varkappa^3x_2=0,\ -\alpha^3x_0+x_1-\varkappa^3x_2=0,\ -i\alpha^3x_0+x_1-\varkappa^3x^4=0,\ i\alpha^3x_0+x_1-\varkappa^3x_2=0.$$

These are all satisfied by the point $(0, \varkappa^3, 1)$; for the line $\varkappa x_1+x_2=0$, $s_1t-st_1=\eta_0$. Then for the line $(0, \varkappa, 1)$

$$m(\varkappa^3\eta_1+\eta_2)\cdot\varkappa\eta_0+n(\varkappa^4+1)(\eta_0\eta_1+\varkappa\eta_0\eta_2)\equiv\eta_0(\varkappa^3\eta_1+\eta_2).$$
$$\therefore\ n=0,\ m=-1,\ l=3.$$

The conic is

$$3(\xi^2\eta^2)\xi_0\xi_1\xi_2-(\xi^3\eta)(\eta_0\xi_1\xi_2)=0.$$

By a method similar to that for the lemniscate it can be shown satisfied by $(xy^3)=0$, the tangent at y, when $(y^4)=0$, $(\xi y)=0$. To obtain the $A^6\xi^8\eta^2$ multiply by $\xi_0\xi_1\xi_2$, the repeated factor of $(t\xi)^6$.

$$A^6\xi^8\eta^2=\sum^{3}\eta_0^2\cdot 2\xi_0^4\xi_1^2\xi_2^2+\sum^{3}\eta_1\eta_2\xi_1\xi_2(-\xi_0^2\xi_1^4-\xi_0^2\xi_2^4).$$

Substituting in (2),

$$A^6\xi^8\eta^2=\sum^{3}\eta_0^2[\xi_0^4\xi_1^2\xi_2^2(\lambda+\tfrac{1}{3}\mu+\tfrac{1}{15}\nu+2\sigma)+(\xi_1^6\xi_2^2+\xi_1^2\xi_2^6)(\tfrac{1}{15}\nu+\rho+2\sigma)]$$
$$+\sum^{3}\eta_1\eta_2\xi_1\xi_2[(\xi_0^2\xi_1^4+\xi_0^2\xi_2^4)(\tfrac{1}{3}\mu+\tfrac{4}{15}\nu-2\sigma)+\xi_0^6(\tfrac{4}{15}\nu-2\rho-2\sigma)].$$

Then $$\tfrac{1}{15}\nu+\rho+2\sigma=0. \qquad \text{(IV)}$$

This equation together with (I), (II) and (III) is sufficient to determine λ, μ, ν, ρ, σ (or rather their ratios).

$$\lambda=0,\ \mu=15,\ \nu=-15,\ \rho=-5,\ \sigma=3.$$

E. The formula

$$A^6\xi^8\eta^2=15(s\xi)^3(s\eta)(t\xi)^5(t\eta)-15(s\xi)^4(t\xi)^4(t\eta)^2-5(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)|\alpha\xi\eta|^2$$
$$+3(s\xi)^4(s'\xi)^2(s'\alpha)^2|\alpha\xi\eta|^2$$

may be tested on the general quartic

$$ax_0^4+4a_1x_0^3x_1+4a_2x_0^3x_2+6hx_0^2x_1^2+12lx_0^2x_1x_2+6gx_0^2x_2^2+4b_0x_0x_1^3+12mx_0x_1^2x_2$$
$$+12nx_0x_1x_2^2+4c_0x_0x_2^3+bx_1^4+4b_2x_1^3x_2+6fx_1^2x_2^2+4c_1x_1x_2^3+cx_2^4=0 \qquad (3)$$

for a special reference triangle and line ξ. Let the line ξ be $x_0=0$ and let two of the tangents at its intersections be $x_1=0$ and $x_2=0$. This requires that

$$b=c=b_0=c_0=0.$$

Then beside the two reference points $(0,1,0)$ and $(0,0,1)$ the line $x_0=0$ cuts the quartic in two points given by

$$4b_2x_1^2+6fx_1x_2+4c_1x_2^2=0. \tag{4}$$

Let these two points be $(0,A,1)$ and $(0,B,1)$. Then the tangent at $(0,A,1)$ is

$$x_0(3mA^2+3nA)+x_1(3b_2A^2+3fA+c_1)+x_2(b_2A^3+3fA^2+3c_1A)=0,$$

or, making use of the fact that $(0,A,1)$ must satisfy (4),

$$x_0(3mA+3n)+x_1(2b_2A+\tfrac{3}{2}f)+x_2(\tfrac{3}{2}fA+2c_1)=0. \tag{5}$$

Similarly, the tangent at $(0,B,1)$ is

$$x_0(3mB+3n)+x_1(2b_2B+\tfrac{3}{2}f)+x_2(\tfrac{3}{2}fB+2c_1)=0. \tag{6}$$

The conic, being on the reference lines, must be of the form

$$\alpha_0\eta_1\eta_2+\alpha_1\eta_2\eta_0+\alpha_2\eta_0\eta_1=0.$$

Requiring it to be satisfied by the lines (5) and (6) gives two homogeneous equations in α_0, α_1, α_2, which can be solved for them in terms of A and B. But A and B enter symmetrically, so that they can be gotten rid of by use of symmetric functions of the roots of (4). The conic is obtained in the form

$$\eta_0\eta_1(-4b_2c_1n+3c_1fm)+\eta_0\eta_2(-4b_2c_1m+3b_2fn)+\eta_1\eta_2(6b_2n^2+6c_1m^2-9fmn)=0,$$

when the factor $-16b_2c_1+9f^2$ has been dropped.

By use of equation (2)*

$$\begin{aligned} A^5\xi^3\eta^2=\eta_0^2[&b_2^2c_1^2f(-8\mu-8\nu\equiv0)\\ &+b_2c_1f^3(10\mu+10\nu\equiv0)\\ &+f^5(-3\mu-3\nu\equiv0)]\\ +\sum^{2}2\eta_0\eta_1[&b_2^2c_1^2n(\tfrac{4}{3}\mu+\tfrac{8}{3}\nu\equiv-20)\\ &+b_2c_1^2fm(5\mu+4\nu\equiv15)\\ &+b_2c_1f^2n(-6\mu-6\nu\equiv0)\\ &+c_1f^3m(-3\mu-3\nu\equiv0)\\ &+f^4n(3\mu+3\nu\equiv0)] \end{aligned}$$

---

* For the coefficients of s and t see the note at the end of the article.

$$
\begin{aligned}
&+\sum^{2}\eta_1^2[b_2^2c_1^2g(-\tfrac{8}{15}\nu+16\rho+24\sigma\equiv 0)\\
&\qquad+b_2c_1^3h(\tfrac{24}{15}\nu+8\sigma\equiv 0)\\
&\qquad+b_2c_1^2fl(-\tfrac{24}{15}\iota-8\sigma\equiv 0)\\
&\qquad+b_2c_1^2mn(-2\mu-\tfrac{48}{15}\nu-6\rho-16\sigma\equiv 0)\\
&\qquad+b_2c_1f^2g(\tfrac{18}{15}\nu-24\rho-34\sigma\equiv 0)\\
&\qquad+c_1^2f^2h(-\tfrac{18}{15}\nu-6\sigma\equiv 0)\\
&\qquad+b_2c_1fn^2(2\mu+\tfrac{48}{15}\nu+6\rho+16\sigma\equiv 0)\\
&\qquad+c_1^2fm^2(-3\mu-9\rho\equiv 0)\\
&\qquad+c_1f^3l(\tfrac{18}{15}\nu+6\sigma\equiv 0)\\
&\qquad+c_1f^2mn(6\mu+\tfrac{36}{15}\nu+18\rho+12\sigma\equiv 0)\\
&\qquad+f^4g(-\tfrac{9}{15}\nu+9\rho+12\sigma\equiv 0)\\
&\qquad+f^3n^2(-3\mu-\tfrac{36}{15}\nu-9\rho-12\sigma\equiv 0)]\\
&+2\eta_1\eta_2[b_2^2c_1^2l(-\tfrac{16}{15}\nu-16\rho-32\sigma\equiv 0)\\
&\qquad\quad+(b_2^2c_1fg+b_2c_1^2fh)(-\tfrac{12}{15}\nu-4\sigma\equiv 0)\\
&\qquad\quad+(b_2^2c_1n^2+b_2c_1^2m^2)(-\mu-\tfrac{24}{15}\nu+15\rho+32\sigma\equiv 30)\\
&\qquad\quad+b_2c_1f^2l(\tfrac{36}{15}\nu+24\rho+52\sigma\equiv 0)\\
&\qquad\quad+b_2c_1fmn(-\mu+\tfrac{12}{15}\nu-30\rho-56\sigma=-45)\\
&\qquad\quad+(b_2f^3g+c_1f^3h)(\tfrac{9}{15}\nu+3\sigma\equiv 0)\\
&\qquad\quad+(b_2f^2n^2+c_1f^2m^2)(3\mu+\tfrac{18}{15}\nu-9\rho-24\sigma\equiv 0)\\
&\qquad\quad+f^4l(-\tfrac{18}{15}\nu-9\rho-21\sigma\equiv 0)\\
&\qquad\quad+f^3mn(-3\mu-\tfrac{9}{15}\nu+18\rho+42\sigma\equiv 0)]\\
=5b_2c_1[&\sum^{2}2\eta_0\eta_1(-4b_2c_1n+3c_1fm)+2\eta_1\eta_2(6b_2n^2+6c_1m^2-9fmn)].
\end{aligned}
$$

Therefore, *the conic on a line ξ and the tangents at its four intersections with a quartic is given by*

$$
15(s\xi)^3(s\eta)(t\xi)^5(t\eta)-15(s\xi)^4(t\xi)^4(t\eta)^2-5(s\xi)^3(s\alpha)(s'\xi)^6(s'\alpha)|\alpha\xi\eta|^2
$$
$$
+3(s\xi)^4(s'\xi)^2(s'\alpha)^2|\alpha\xi\eta|^2.
$$

II. THE LOCUS OF LINES MEETING THE QUARTIC SO THAT THREE OF THE TANGENTS AT THE INTERSECTIONS ARE ON A POINT.

For certain lines ξ the conic breaks up. These are the lines such that three of the tangents at the intersections with the quartic are on a point. The discriminant of the $A^5\xi^8\eta^2$, an $A^{15}\xi^{24}$, is therefore the locus of such lines. The line equation of the quartic does not enter into this $A^{15}\xi^{24}$, as may be seen from case E above, where $16\,b_2c_1-9\,f^2$, the condition that $x_0=0$ be tangent to the quartic, is not a factor in the discriminant of the conic.

The expression obtained for the $A^5\xi^8\eta^2$ is of the form

$$\begin{aligned}&(a_1x_0^2+b_1x_1^2+c_1x_2^2+2f_1x_1x_2+2g_1x_2x_0+2h_1x_0x_1)\\ &+(a_2x_0^2+b_2x_1^2+c_2x_2^2+2f_2x_1x_2+2g_2x_2x_0+2h_2x_0x_1)\\ &+(a_3x_0^2+b_3x_1^2+c_3x_2^2+2f_3x_1x_2+2g_3x_2x_0+2h_3x_0x_1)\\ &+(a_4x_0^2+b_4x_1^2+c_4x_2^2+2f_4x_1x_2+2g_4x_2x_0+2h_4x_0x_1).\end{aligned}$$

Therefore, if

$$\Delta_{iii}=\begin{vmatrix}a_i\,h_i\,g_i\\ h_i\,b_i\,f_i\\ g_i\,f_i\,c_i\end{vmatrix},\quad \Delta_{iij}=\sum^{3}\begin{vmatrix}a_i\,h_i\,g_j\\ h_i\,b_i\,f_j\\ g_i\,f_i\,c_j\end{vmatrix},\quad \Delta_{ijk}=\sum^{6}\begin{vmatrix}a_i\,h_j\,g_k\\ h_i\,b_j\,f_k\\ g_i\,f_j\,c_k\end{vmatrix},$$

then

$$A^{15}\xi^{24}=\begin{vmatrix}a_1+a_2+a_3+a_4 & h_1+h_2+h_3+h_4 & g_1+g_2+g_3+g_4\\ h_1+h_2+h_3+h_4 & b_1+b_2+b_3+b_4 & f_1+f_2+f_3+f_4\\ g_1+g_2+g_3+g_4 & f_1+f_2+f_3+f_4 & c_1+c_2+c_3+c_4\end{vmatrix}$$
$$=\sum^{4}\Delta_{111}+\sum^{12}\Delta_{112}+\sum^{4}\Delta_{123},$$

where

$$\Delta_{111}=0,\quad \Delta_{333}=0,\quad \Delta_{444}=0,$$
$$\Delta_{222}=-\tfrac{15^3}{6}(s\xi)^4(s'\xi)^4(s''\xi)^4(t\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2,$$
$$\begin{aligned}\Delta_{112}=-\tfrac{15^3}{24}\{&(s\xi)^4(s'\xi)^4(s''\xi)^4(t\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2\\ &-3(s\xi)^4(s'\xi)^3(s''\xi)^3(t\xi)^6(t'\xi)^4(t''\xi)^4|s't't''|\,|s''t't''|\},\end{aligned}$$
$$\Delta_{122}=\tfrac{15^3}{6}(s\xi)^4(s'\xi)^4(s''\xi)^4(t\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2,$$
$$\begin{aligned}\Delta_{113}=\tfrac{15^2.5}{4}(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)\{&(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^6(t'\xi)^6\\ &-2(s''\xi)^4(s'''\xi)^3(s'''\alpha)(t\xi)^6(t'\xi)^5(t'\alpha)\\ &+(s''\xi)^4(s'''\xi)^4(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},\end{aligned}$$
$$\Delta_{133}=\tfrac{15.5^2}{2}(s\xi)^4(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha')(s^{\text{iv}}\xi)^3(s^{\text{iv}}\alpha')|\alpha\alpha'\xi|^2(t\xi)^6,$$
$$\begin{aligned}\Delta_{114}=-\tfrac{15^2.3}{4}(s\xi)^4(s'\xi)^2(s'\alpha)^2\{&(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^6(t'\xi)^6\\ &-2(s''\xi)^4(s'''\xi)^3(s'''\alpha)(t\xi)^6(t'\xi)^5(t'\alpha)\\ &+(s''\xi)^4(s'''\xi)^4(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},\end{aligned}$$
$$\Delta_{144}=\tfrac{15.3^2}{2}(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\xi)^2(s'''\alpha)^2(s^{\text{iv}}\xi)^2(s^{\text{iv}}\alpha')^2|\alpha\alpha'\xi|^2(t\xi)^6,$$
$$\begin{aligned}\Delta_{223}=-15^2.5\,&(s\xi)^4(s'\xi)^4(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)\\ &\{(t\xi)^6(t'\xi)^4(t'\alpha)^2-(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},\end{aligned}$$
$$\Delta_{233}=-\tfrac{15.5^2}{2}(s\xi)^4(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha')(s^{\text{iv}}\xi)^3(s^{\text{iv}}\alpha')|\alpha\alpha'\xi|^2(t\xi)^6,$$
$$\begin{aligned}\Delta_{224}=15^2.3\,&(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\xi)^2(s'''\alpha)^2\\ &\{(t\xi)^6(t'\xi)^4(t'\alpha)^2-(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},\end{aligned}$$
$$\Delta_{244}=-\tfrac{15.3^2}{2}(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\xi)^2(s'''\alpha)^2(s^{\text{iv}}\xi)^2(s^{\text{iv}}\alpha')^2|\alpha\alpha'\xi|^2(t\xi)^6,$$
$$\Delta_{334}=0,\quad \Delta_{344}=0,$$
$$\begin{aligned}\Delta_{123}=15^2.5\,&(s\xi)^4(s'\xi)^4(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)\\ &\{(t\xi)^6(t'\xi)^4(t'\alpha)^2-(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},\end{aligned}$$
$$\begin{aligned}\Delta_{124}=-15^2.3\,&(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\xi)^2(s'''\alpha)^2\\ &\{(t\xi)^6(t'\xi)^4(t'\alpha)^2-(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},\end{aligned}$$

$$\Delta_{134} = -15^2(s\xi)^4(s'\xi)^4(s''\xi)^3(s''\alpha)\,(s'''\xi)^3(s'''\alpha)\,(s^{\text{iv}}\xi)^2(s^{\text{iv}}\alpha)^2|\alpha\alpha'\xi|^2(t\xi)^6,$$
$$\Delta_{234} = 15^2(s\xi)^4(s'\xi)^4(s''\xi)^3(s''\alpha)\,(s'''\xi)^3(s'''\alpha)\,(s^{\text{iv}}\xi)^2(s^{\text{iv}}\alpha)^2|\alpha\alpha'\xi|^2(t\xi)^6.$$

Then for convenience dropping the factor $-\frac{15^2}{8}$,

$$\begin{aligned} A^{15}\xi^{24} = {} & 5\,(s\xi)^4(s'\xi)^4(s''\xi)^4(t\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2 \\ & -15\,(s\xi)^4(s'\xi)^3(s''\xi)^3(t\xi)^6(t'\xi)^4(t''\xi)^4|s't't''|\,|s''t't''| \\ & -2\{5\,(s\xi)^3(s\alpha)\,(s'\xi)^3(s'\alpha)-3\,(s\xi)^4(s'\xi)^2(s'\alpha)^2\} \\ & \qquad \{(s''\xi)^3(s''\alpha)\,(s'''\xi)^3(s'''\alpha)\,(t\xi)^6(t'\xi)^6 \\ & \qquad\quad -2\,(s''\xi)^4(s'''\xi)^3(s'''\alpha)\,(t\xi)^6(t'\xi)^5(t'\alpha) \\ & \qquad\quad +(s''\xi)^4(s'''\xi)^4(t\xi)^5(t\alpha)\,(t'\xi)^5(t'\alpha)\}. \end{aligned}$$

Special Cases.

The correctness of this formula may be tested on the special cases given before for which the conic is known.

A. For the quartic $4x_0(x_1^3+x_2^3)=0$

$$A^{15}\xi^{24} \equiv 0.$$

B. For the four lines $3[x_0^4+x_1^4+x_2^4-2x_1^2x_2^2-2x_2^2x_0^2-2x_0^2x_1^2]=0$

$$A^{15}\xi^{24} = (\xi_1^2-\xi_2^2)^4(\xi_2^2-\xi_0^2)^4(\xi_0^2-\xi_1^2)^4.$$

C. For the lemniscate $6[x_1^2x_2^2+x_2^2x_0^2+x_0^2x_1^2]=0$

$$A^{15}\xi^{24} = 180\cdot 162^2(\xi^2)^4\xi_0^4\xi_1^4\xi_2^4[(\xi^2)^2-3(\xi_1^2\xi_2^2)].$$

D. For the quartic $x_0^4+x_1^4+x_2^4=0$

$$A^{15}\xi^{24} = \tfrac{10}{3}\,\xi_0^4\xi_1^4\xi_2^4[(\xi^4)(\xi_1^4\xi_2^4)-9\xi_0^4\xi_1^4\xi_2^4].$$

In each case the result obtained by the formula agrees with that obtained directly from the $A^5\xi^8\eta^2$.

It should be noted that the double points and repeated factors of s and t, which occur *once* in the $A^5\xi^8\eta^2$ and therefore would be expected to enter the $A^{15}\xi^{24}$ *three* times, actually occur *four* times.

Singular Lines of the Locus.

Of special interest in connection with the $A^{15}\xi^{24}$ are the lines ξ such that

$$(s\xi)^3(s\eta)\,(t\xi)^6-(s\xi)^4(t\xi)^5(t\eta)\equiv 0;$$

that is, the contact of ξ with the $A^5\xi^8\eta^2$ is indeterminate. This occurs 1) if s or t has a repeated factor; 2) for the common lines of s and t, the stationary lines of the quartic; 3) for the lines having the same polar point as to s and t.

1) For the general quartic neither s nor t has repeated factors. Therefore, this case need not be discussed here.

2) It is evident that the common lines of s and t are double lines of the

$A^{15}\xi^{24}$, for every term of the latter contains either s^2, st, or t^2. Furthermore, the polar conic when $s=t=0$ becomes

$$\begin{aligned}&(s\xi)^3(s\eta)(s'\xi)^3(s'\eta)\cdot-\tfrac{20}{3}(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\\&+(s\xi)^3(s\eta)(t\xi)^5(t\eta)\,[20(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t'\xi)^5(t'\alpha)\\&\qquad\qquad-15(s'\xi)^3(s''\xi)^3(t'\xi)^4(t''\xi)^4|s't't''|\,|s''t't''|\,]\\&+(t\xi)^5(t\eta)(t'\xi)^5(t'\eta)\cdot-15(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha).\end{aligned}$$

But $(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)=0$ is the condition that the point $(s\xi)^3(s\eta)$ lie on the polar conic as to the quartic of $(t\xi)^5(t\eta)$, a condition known to be fulfilled when ξ is a stationary line of the quartic, since $(t\xi)^5(t\eta)$ is the flex itself. Therefore, the coefficient of $(s\xi)^3(s\eta)(s'\xi)^3(s'\eta)$ vanishes, showing that $(t\xi)^5(t\eta)$ is one of the points of contact of ξ with the $A^{15}\xi^{24}$. Therefore ξ is not only a double line of both the $A^{15}\xi^{24}$ and the quartic $(ax)^4$, but also has one point of contact the same in each case, so that the stationary lines account for $24(2\cdot2+1)=120$ of the common lines of the two curves.

3) For a line ξ to have the same polar point as to s and t requires that

$$\frac{(s\xi)^3s_0}{(t\xi)^5t_0}=\frac{(s\xi)^3s_1}{(t\xi)^5t_1}=\frac{(s\xi)^3s_2}{(t\xi)^5t_2}.$$

These equations have $25+9+15=49$ solutions. Among these must be included the twenty-eight double lines of the quartic, for both $s_1t-st_1\equiv0$ and $s^2s_1-27tt_1\equiv0$ hold if $s^3-27t^2=0$, which is true for the double lines. The remaining twenty-one lines are those which meet the quartic so that all four tangents at the intersections are on a point; they correspond to the twenty-one double points of the Steinerian, which are points whose polar cubic is a conic and a line. For all of these lines the $A^5\xi^8\eta^2$ is degenerate, so that they are at least lines of the $A^{15}\xi^{24}$. But presumably they are more than this, and indeed the twenty-one lines are at least triple lines, for they cause the $A^5\xi^8\eta^2$ to vanish identically. If these lines are double lines, their polar points as to the $A^{15}\xi^{24}$ should vanish. But the polar point of ξ when $s_1t-st_1\equiv0$ becomes

$$\begin{aligned}-\tfrac{5}{12}(s'\xi)^4(s''\xi)^4[(t\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2(s\xi)^3(s\eta)\\-(s\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2(t\xi)^3(t\eta)],\end{aligned}$$

and it is not evident that this vanishes. Furthermore, though it can be shown that $(t'\xi)^4(t''\xi)^4|t't''x|^2$ is a numerical multiple of

$$\begin{aligned}&9(s\xi)^2(s\alpha)^2(\alpha x)^2(t\xi)^6-\tfrac{15}{4}(s\xi)^4(t\xi)^4(t\alpha)^2(\alpha x)^2\\&\quad+10(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)|\alpha\alpha'\xi|^2(\alpha'x)^2-\tfrac{39}{8}(s\xi)^4(s'\xi)^2(s'\alpha)^2|\alpha\alpha'\xi|^2(\alpha x)^2\\&+(x\xi)[-18(s\xi)^2(s\alpha)^2(t\xi)^5(t\alpha)(\alpha x)+\tfrac{75}{7}(s\xi)^4(t\xi)^3(t\alpha)^3(\alpha x)]\\&+(x\xi)^2[\tfrac{17}{6}(s\alpha)^4(t\xi)^6-\tfrac{21}{2}(s\xi)^2(s\alpha)^2(t\xi)^4(t\alpha)^2+\tfrac{50}{7}(s\xi)^3(s\alpha)(t\xi)^3(t\alpha)^3\\&\qquad\qquad-\tfrac{145}{56}(s\xi)^4(t\xi)^2(t\alpha)^4],\end{aligned}$$

the substitution of this value does not give an expression that vanishes on its face.

Let us first consider the double lines for the general quartic (3) with a convenient reference triangle. Let $x_0=0$ be a double line with contacts $(0,1,0)$ and $(0,0,1)$, and let $(1,0,0)$ be the polar point of x_0 as to both s and t.* This reference scheme requires that in the equation of the quartic

$$b=c=b_2=c_1=m=n=0.$$

Then (giving only the highest powers of ξ_0)

$$\begin{aligned}A^5\xi^8\eta^2=\eta_0^2&[270b_0c_0f^3\xi_0^6\xi_1\xi_2+\ldots.]+\eta_1^2[-90b_0c_0^2f^2\xi_0^7\xi_1+\ldots.]\\&+\eta_2^2[-90b_0^2c_0f^2\xi_0^7\xi_2+\ldots.]\\&+2\eta_1\eta_2[-45b_0c_0f^3\xi_0^8+135c_0f^3h\xi_0^7\xi_1+135b_0f^3g\xi_0^7\xi_2+\ldots.]\\&+2\eta_0\eta_2[-45b_0c_0f^3\xi_0^7\xi_1+\ldots.]+2\eta_0\eta_1[-45b_0c_0f^3\xi_0^7\xi_2+\ldots.],\end{aligned}$$

so that

$$A^{15}\xi^{24}=-90b_0^8c_0^3f^0\xi_0^{22}\xi_1\xi_2+\text{ lower powers in }\xi_0.$$

Therefore, the double lines of the quartic are also double lines of the $A^{15}\xi^{24}$ with the same contacts. They then account for $28(2\cdot2+2)=168$ common lines of the two curves, and since it has already been shown that the stationary lines account for 120, all of the $24\times12=288$ common lines of the $A^{15}\xi^{24}$ and the quartic (in lines an $A^6\xi^{12}$) are accounted for.

For the twenty-one lines giving four tangents meeting in a point (which point is the common polar point of the line as to s and t) choose a reference scheme where $x_0=0$ is one of these lines with two of its intersections as $(0,1,0)$ and $(0,0,1)$ and let the tangents at these two points be $x_2=0$ and $x_1=0$, respectively, so that $(1,0,0)$ is the meeting point of the four tangents. Then

$$b=c=b_0=c_0=m=n=0.$$

$$\begin{aligned}A^5\xi^8\eta^2=\eta_0^2&[\text{terms beginning with }\xi_0^6\xi_1^2]\\&+\eta_1^2[\xi_0^6\xi_1\xi_2(120b_2^2c_1^2g-360b_2c_1^2gh-180b_2c_1fgl+270c_1f^2gh)\\&\qquad\qquad+\xi_0^6\xi_2^2(240b_2^2c_1gl-90b_2^2fg^2-90b_2c_1fgh)+\ldots.]\\&+\eta_2^2[\xi_0^6\xi_1^2(240b_2c_1^2hl-90b_2c_1fgh-90c_1^2fh^2)\\&\qquad+\xi_0^6\xi_1\xi_2(-360b_2^2c_1gh+120b_2c_1^2h^2-180b_2c_1fhl+270b_2f^2gh)+\ldots.]\\&2\eta_1\eta_2[\xi_0^6\xi_1^2(30b_2^2c_1g^2-120b_2c_1^2gh+90c_1^3h^2+120b_2c_1^2l^2-90b_2c_1fgl\\&\qquad\qquad-180c_1^2fhl+135c_1f^2gh)\\&\qquad+\xi_0^6\xi_1\xi_2(-120b_2^2c_1gl-120b_2c_1^2hl-90b_2^2fg^2+540b_2c_1fgh\\&\qquad\qquad-90c_1^2fh^2-180b_2c_1fl^2+270b_2f^2gl+27c_1f^2hl\\&\qquad\qquad-261f^3gh-168f^3l^2)\\&\qquad+\xi_0^6\xi_2^2(90b_2^3g^2-120b_2^2c_1gh+30b_2c_1^2h^2+120b_2^2c_1l^2-180b_2^2fgl\\&\qquad\qquad-90b_2c_1fhl+135b_2f^2gh)+\ldots.]\\&+2\eta_0\eta_2[\xi_0^7\xi_1(40b_2^2c_1^2l-15b_2^2c_1fg-15b_2c_1^2fh)\\&\qquad\qquad+\xi_0^7\xi_2(-60b_2^3c_1g+20b_2^2c_1^2h-30b_2^2c_1fl+45b_2^2f^2g)+\ldots.]\\&+2\eta_0\eta_1[\xi_0^7\xi_1(20b_2^2c_1^2g-60b_2c_1^3h-30b_2c_1^2fl+45c_1^2f^2h)\\&\qquad\qquad+\xi_0^7\xi_2(40b_2^2c_1^2l-15b_2^2c_1fg-15b_2c_1^2fh)+\ldots.].\end{aligned}$$

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* It is evident that this polar point can not lie on the double line itself, for then the latter would have to be a line of both s and t, all of whose common lines are accounted for by the stationary lines of the quartic.

From this it is evident that ξ_0^{20} is the highest power of ξ_0 that can enter into the $A^{15}\xi^{24}$. Therefore, the twenty-one lines are quadruple lines of the locus. There is no reason for supposing that the $A^{15}\xi^{24}$ has any more singular lines.

NOTE.

$$(s\xi)^4 = A\xi_0^4+B\xi_1^4+C\xi_2^4+4A_1\xi_0^3\xi_1+4A_2\xi_0^3\xi_2+4B_0\xi_1^3\xi_0+4B_2\xi_1^3\xi_2+4C_0\xi_2^3\xi_0+4C_1\xi_2^3\xi_1$$
$$+6F\xi_1^2\xi_2^2+6G\xi_2^2\xi_0^2+6H\xi_0^2\xi_1^2+12L\xi_0^2\xi_1\xi_2+12M\xi_0\xi_1^2\xi_2+12N\xi_0\xi_1\xi_2^2,$$

where

$$A = bc-4b_2c_1+3f^2,$$
$$A_1 = -cb_0+b_2c_0+3c_1m-3fn, \quad A_2 = -bc_0+b_0c_1+3b_2n-3fm,$$
$$F = af-2a_1n-2a_2m+gh+2l^2,$$
$$L = b_0c_0-b_2g-c_1h+2fl-mn.$$

$$(t\xi)^6 = \overset{3}{\Sigma}A_{600}\xi_0^6+6\overset{6}{\Sigma}A_{510}\xi_0^5\xi_1+15\overset{6}{\Sigma}A_{420}\xi_0^4\xi_1^2+30\overset{3}{\Sigma}A_{411}\xi_0^4\xi_1\xi_2+20\overset{3}{\Sigma}A_{330}\xi_0^3\xi_1^3$$
$$+60\overset{6}{\Sigma}A_{321}\xi_0^3\xi_1^2\xi_2+90A_{222}\xi_0^2\xi_1^2\xi_2^2,$$

where

$$A_{600} = bcf-bc_1^2-cb_2^2+2b_2c_1f-f^3,$$
$$A_{510} = \tfrac{1}{3}[-bcn+bc_0c_1-2cb_0f+3cb_2m+2b_0c_1^2-b_2c_0f-2b_2c_1n-3c_1fm+3f^2n],$$
$$A_{420} = \tfrac{1}{15}[bcg-bc_0^2+8cb_0n-6cb_2l+6cfh-9cm^2-8b_0c_0c_1+2b_2c_1g-6c_1^2h$$
$$+4b_2c_0n+6c_2fm+6c_1fl+12c_1mn-3f^2g-12fn^2],$$
$$A_{411} = \tfrac{1}{15}[2bcl-3bc_1g-3cb_2h+bc_0n+cb_0m+9b_0c_0f-10b_0c_1n-10b_2c_0m$$
$$+4b_2c_1l+3b_2fg+3c_1fh+6b_2n^2+6c_1m^2-6f^2l-3fmn],$$
$$A_{330} = \tfrac{1}{10}[ca_2b_2-2ca_1f-2cb_0g-6chn+9clm+2a_1c_1^2+2b_0c_0^2-a_2c_1f-b_2c_0g$$
$$+6c_0c_1h-3c_0fl-3c_1gm-6c_0mn-6c_1ln+6fgn+4n^3],$$
$$A_{321} = \tfrac{1}{30}[-bca_2+3ca_1b_2+bc_0g-5cb_0l+3chm-2a_2b_2c_1-3a_1c_1f+11b_0c_1g$$
$$+8b_2c_0l-6b_0c_0n+3a_2f^2-15c_0fh+12c_1hn-9b_2gn+12c_0m^2-18c_1lm$$
$$-3fgm+15fln-6mn^2],$$
$$A_{222} = \tfrac{1}{90}[abc+2ab_2c_1+2ba_2c_0+2ca_1b_0-3af^2-3bg^2-3ch^2-20a_1b_2c_0$$
$$-20a_2b_0c_1+24a_1c_1m+24a_2b_2m+24b_0c_0l-6a_1fn-6a_2fm-6b_0gn$$
$$-6b_2gl-6c_0hm-6c_1hl+48fgh-30fl^2-30gm^2-30hn^2+48lmn].$$

All the other coefficients can be found by symmetry from those given here.

JOHNS HOPKINS UNIVERSITY, *October*, 1916.

On Two Related Transformations of Space Curves.[*]

By William Caspar Graustein

The Transformation C.

1. Two curves, corresponding point for point so that the tangents in corresponding points are parallel, are said to be related by a transformation of Combescure.[†] The general curve in 1—1 correspondence with a given space curve, $x=x(s)$,[‡] may be given by $y=x+f\alpha+\phi\beta+\psi\gamma$, where f, ϕ, ψ are functions of s. The conditions that the tangents in corresponding points be parallel are

$$\frac{d\phi}{ds}+\frac{f}{R}+\frac{\psi}{T}=0, \quad \frac{d\psi}{ds}-\frac{\phi}{T}=0.$$

If there is set $\psi=\dfrac{dC}{ds_\gamma}$, then $\phi=-\dfrac{d^2C}{ds_\gamma^2}$, $f=\dfrac{d}{ds_\alpha}\left(C+\dfrac{d^2C}{ds_\gamma^2}\right)$, and

$$y=x+\delta, \text{ where } \delta=\frac{d}{ds_\alpha}\left(C+\frac{d^2C}{ds_\gamma^2}\right)\alpha-\frac{d^2C}{ds_\gamma^2}\beta-\frac{dC}{ds_\gamma}\gamma. \tag{1}$$

This y-curve is the general curve in Combescurian correspondence with the x-curve. Since it is determined, when the function C (except for an additive constant) is given, and conversely, one may speak of the correspondence, or transformation, C. If $C=R$, the y-curve is the second polar curve of the x-curve (the first polar curve of a curve is the edge of regression of the polar developable, the second is the polar curve of the first, etc.). Now

$$\frac{dy}{ds}=\left(1+\frac{P(C)}{R}\right)\alpha, \text{ where } P(C)=\frac{d^2}{ds_\alpha^2}\left(C+\frac{d^2C}{ds_\gamma^2}\right)+\frac{d^2C}{ds_\gamma^2}\,.$$

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[*] Read before the American Mathematical Society, December 28, 1915.

[†] Aoust discusses this transformation in his "Analyse infinitesimale des courbes dans l'espace,' 1876, Ch. XVI. It comes by its present title, the writer believes, from its similarity with the Combescurian transformation of triply orthogonal systems of surfaces. Salkowski gives a geometrical treatmen of it in the *Mathematische Annalen*, Vol. LXVI (1909), and Sannia gives an analytical one, by means o the moving trihedral, in the *Rendiconti del Circolo Matematico di Palermo*, Vol. XX (1905), pp. 83-92.

[‡] In this paper x_1, x_2, x_3 are the coordinates of a point x; $x_1=x_1(s)$, $x_2=x_2(s)$, $x_3=x_3(s)$ is th parametric representation, in terms of the arc s, of the curve $x=x(s)$; α_1, α_2, α_3, β_1, β_2, β_3, γ_1, γ_2, γ_3 are the direction cosines of the directions α, β, γ of the tangent, principal normal and binormal; R an T are the radii of curvature and torsion; s_α and s_γ are the arcs of the tangent and binormal indicatrices Plane curves are excluded and the work is for the real domain.

Thus $\alpha_y = \pm\alpha$, and hence $\beta_y = \pm\beta$; if the choices $\alpha_y = \alpha$, $\beta_y = \beta$ are made, it follows that $\gamma_y = \gamma$. Further

$$\frac{ds_y}{ds} = \frac{R_y}{R} = \frac{T_y}{T} = 1 + \frac{P(C)}{R}. \tag{2}$$

Thus the rectifying lines in corresponding points are parallel also.

The surface, $z = x + r\delta$, generated by the line xy, is a developable. Its edge of regression, if its exists, is given by $\eta = x - R/P(C)\,\delta$.

2. The x- and y-curves are congruent[*] only if $P(C) = 0$. Now

$$P(C) = \left(\frac{ds_\gamma}{ds_\alpha}\right)^2 \Gamma\left(\frac{dC}{ds_\gamma}\right), \tag{3}$$

where $\Gamma(\gamma) = 0$ is the result of elimination of α and β from the Frenet formulae. The general solution of $\Gamma(\gamma) = 0$ is then $-r(c\,|\,\gamma)$,[†] where $(c\,|\,c) = 1$ and $r > 0$; hence the general solution of $P(C) = 0$ is

$$C = -r\int (c\,|\,\gamma)\,ds_\gamma, \quad \text{or} \quad \frac{dC}{ds_\gamma} = -r(c\,|\,\gamma). \tag{4}$$

Substitution of this value of C in (1) yields $y = x + rc$.

Thus the general transformation C, carrying a given curve into a congruent curve, is defined by (4) and is equivalent to a translation in the direction c through the distance r. On the other hand, if the transformation C is given, (4) becomes the defining relation for the curves congruent to their transforms under it. A parametric representation of these curves, as the curves defined by (4) and in Combescurian correspondence with an arbitrarily given curve, the ξ-curve, may be found by a method of Salkowski.[‡] Since $\gamma = \gamma_\xi$, $ds_\gamma = (ds_\gamma)_\xi$, the first of the formulae (4) may be written

$$C = -r\int (c\,|\,\gamma_\xi)\,(ds_\gamma)_\xi. \tag{5}$$

If C involves s, R and T, (5) and the last of the equations

$$\frac{ds}{(ds)_\xi} = \frac{R}{R_\xi} = \frac{T}{T_\xi}, \tag{6}$$

may be solved for R and T as functions of s and the parameter of the ξ-curve, and thus either of the remaining equations of (6) may be solved for s as a function of the parameter of the ξ-curve. Hence

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[*] In the *Bulletin de la Société Mathématique*, Vol. VII (1878), pp. 143–154, Aoust set up a differential equation of the curves congruent to their second polar curves, which Hoppe recognized later, *Archiv der Mathematik und Physik*, Vol. LXVI (1881), pp. 386–396, as that which the direction cosines of the binormal of a space curve satisfy, and thus easily integrated. It is this idea of Hoppe that is being used here in the case of the general transformation C.

[†] N. B. $(c\,|\,\gamma) = c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3$, $(c\,|\,c) = c_1^2 + c_2^2 + c_3^2$.

[‡] *Sitzungsberichte der Berliner Mathematischen Gesellschaft*, Vol. IV (1905), pp. 64–69.

$$x=\int\frac{ds}{(ds)_\xi}\,d\xi,\ \text{ or }\ x=\int\frac{R}{R_\xi}\,d\xi,\ \text{ or }\ x=\int\frac{T}{T_\xi}\,d\xi$$

yields a parametric representation of the required curves. If, in particular, $C=R\,F\left(\frac{R}{T}\right)$, this method results in the representation

$$x=r\int\frac{\int(c\,|\,\gamma_\xi)\,(ds_\gamma)_\xi}{C_\xi}\,d\xi.$$

For $C=R$ this becomes the parametric representation of the curves congruent to their second polar curves.

The y-curve is similar to the x-curve,* if R_y and T_y are constant multiples of R and T; that is, by (2), if $P(C)=kR$, where k is a constant, not -1. By (3) this condition becomes

$$\Gamma\left(\frac{dC}{ds_\gamma}\right)=kR\left(\frac{ds_a}{ds_\gamma}\right)^2.$$

The general solution of this equation is

$$\frac{dC}{ds_\gamma}=r(c\,|\,\gamma)-k(x\,|\,\gamma). \tag{7}$$

For this value of C (1) may be written ($k\not\equiv 0$)

$$y-\frac{rc}{k}=(1+k)\left(x-\frac{rc}{k}\right).$$

Thus the general transformation C, carrying a given curve into a similar curve, is defined by (7) and is equivalent to a stretching from the point rc/k with coefficient of enlargement $1+k$. Work similar to that leading to this result yields a new representation of the general y-curve: The general transformation C, carrying a given x-curve into a y-curve such that R_y and T_y are given multiples of R and T—$R_y/R=T_y/T=1+K$, where K is a given function, $\not\equiv -1$—is defined by $dC/ds_\gamma=r(c\,|\,\gamma)-(\gamma\,|\int K\,dx)$, and the most general such y-curve is then $y=x+\int K\,dx-rc$.

The curves similar to their transforms under a given transformation C are defined by (7). The problem of setting up a parametric representation for these curves is, in general, incapable of explicit solution.†

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* Salkowski discusses this problem for a curve and its second polar curve, *Archiv der Mathematik und Physik*, Vol. XIV (1908), pp. 231–239.

† For the transformation $C=R$, it depends on the solution of three simultaneous linear differential equations of the second order.

3. If the x- and y-curves are congruent, the distance $xy=d$ is constant. The converse, however, is not true;* differentation of $d^2=(\delta|\delta)$ with respect to s yields the product of $2P(C)$ and

$$\frac{d}{ds_\alpha}\left(C+\frac{d^2C}{ds_\gamma^2}\right). \tag{8}$$

Thus the transformation $C=c_0+c\sin(s_\gamma+k)$, $c\neq 0$, applied to any given x-curve, yields a non-congruent y-curve at a constant distance. The representation of the curves, which transform into non-congruent curves at constant distances under a given transformation C, may be found by Salkowski's method. Thus the curves, which are at constant distances from their second polar curves but not congruent to them, are those whose first polar curves are of constant curvature, while they themselves are not.

Whereas constant length of xy is not a sufficient condition for the congruence of the x- and y-curves, constant direction of xy is. This may be shown by differentiating the direction cosines δ/d of xy.

The direction xy is relatively fixed with respect to the trihedral in x only if the factors of combination of α, β, γ in δ/d are constant. This is true only if the x-curve is a helix and C is such that $dC/ds_\gamma=ke^{ms_\gamma}$. If $m=1$, the two curves are congruent helices on the same cylinder.

4. The transformations C, such that the point tracing one of the curves lies always in the corresponding rectifying plane of the other, are given by $dC/ds_\gamma=a$, a constant; then (1) becomes $y=x+a(\alpha R/T-\gamma)$. Thus the two curves are parallel geodesics on the developable z; conversely, two parallel geodesics on an arbitrary developable are related by such a transformation C.

The curves whose rectifying planes contain the corresponding points of their second polar curves are those for which the distance from the center of curvature to the center of the osculating sphere is constant. The only such curves, which, further, are congruent to their second polar curves are helices for which $R^2=bs$; these helices lie on cylinders with involutes of circles as directrices.

The transformations C, such that the normal plane of one curve contains always the corresponding point of the other, are given by (8), set equal to zero. The two curves are orthogonal trajectories of the rulings of the developable z; conversely, two orthogonal trajectories of the rulings of an arbitrary developable are related by such a transformation C. The curves whose normal planes contain the corresponding points of their second polar curves are those whose first polar curves are of constant curvature.

---

* Salkowski supposes the converse true in his geometric determination of the curves congruent to their second polar curves, *Sitzungsberichte der Berliner Mathematischen Gesellschaft*, Vol. VI (1907), pp. 54–59.

The only y-curve in correspondence C with an x-curve such that y always lies in the osculating plane of x is the x-curve itself.

The Transformation A.

5. The transformation of Combescure finds an analytic parallel in the point-to-point correspondence of two curves, in which the tangents in corresponding points have as their common perpendicular direction that of the principal normal of the given curve and make with one another an angle whose cosine equals the ratio of the corresponding elements of arc—so that the element of arc of the given curve is the projection of that of the transformed curve. The y-curve is the transform of a given x-curve by this transformation, if $(\alpha_y|\beta)=0$, $(\alpha_y|\alpha)ds_y=ds$; hence it has the general form

$$y=x+\delta, \text{ where } \delta=-\frac{dA}{ds_\alpha}\alpha-\frac{d^2A}{ds_\alpha^2}\beta+\frac{d}{ds_\gamma}\left(A+\frac{d^2A}{ds_\alpha^2}\right)\gamma. \tag{9}$$

When the function A (except for an additive constant) is given, this y-curve is determined, and, conversely. Thus one may speak of the correspondence, or transformation, A.

A comparison of (9) with (1) shows that analytically the transformation A becomes the transformation C if α and γ, ds_α and ds_γ, R and T, are interchanged. This relationship between the two transformations subsists to a great extent throughout. Thus, from (9),

$$\frac{dy}{ds}=\alpha+\frac{Q(A)}{T}\gamma, \text{ where } Q(A)=\frac{d^2}{ds_\gamma^2}\left(A+\frac{d^2A}{ds_\alpha^2}\right)+\frac{d^2A}{ds_\alpha^2}\,; \tag{10}$$

$Q(A)$ becomes $P(C)$ by replacing A by C and interchanging ds_α and ds_γ.

If θ, where $\tan\theta=Q(A)/T$, is introduced, then for the y-curve

$$\left.\begin{aligned}
\alpha_y&=\cos\theta\,\alpha+\sin\theta\,\gamma,\\
\sqrt{\theta'^2+M^2}\,\beta_y&=\quad M\beta+\theta'(\cos\theta\,\gamma-\sin\theta\,\alpha),\\
\sqrt{\theta'^2+M^2}\,\gamma_y&=-\theta'\beta+M(\cos\theta\,\gamma-\sin\theta\,\alpha),\\
\frac{1}{R_y}&=\cos\theta\sqrt{\theta'^2+M^2},\\
\frac{1}{T_y}&=\cos\theta\frac{\theta'M'-M\theta''-N\theta'^2-M^2N}{\theta'^2+M^2},
\end{aligned}\right\} \tag{11}$$

where
$$M=\frac{\cos\theta}{R}+\frac{\sin\theta}{T},\quad N=\frac{\sin\theta}{R}-\frac{\cos\theta}{T},$$

and the primes denote differentiation with respect to s.

The ruled surface, $z=x+r\delta$, generated by the line xy, is a developable only when $Q(A)=0$ or $A=as_\alpha+b$. The common perpendicular direction to two neighboring rulings of the surface is parallel always to the normal plane of the x-curve. The x-curve is the line of striction on the surface only if $d^2A/ds_\alpha^2+A=\text{const}$.

6. If the x- and y-curves are to be, further, in Combescurian correspondence, it is evident from (10), that $Q(A)=0$; but then the curves are obviously congruent. Now

$$Q(A)=\left(\frac{ds_\alpha}{ds_\gamma}\right)^2\Delta\left(\frac{dA}{ds_\alpha}\right), \tag{12}$$

where $\Delta(\alpha)=0$ is the differential equation of the direction cosines of the direction α and thus has, as its general solution, $-r(a\,|\,\alpha)$, where $(a\,|\,a)=1$, $r>0$. Hence the general solution of $Q(A)=0$ is

$$\frac{dA}{ds_\alpha}=-r(a\,|\,\alpha). \tag{13}$$

Substitution of this value in (9) gives $y=x+ra$.

The general transformation A, carrying a given curve into a congruent curve, is defined by (13) and is equivalent to a translation in the direction a through the distance r. On the other hand, if the transformation A is given, (13) becomes the defining relation for the curves congruent to their transforms under it. A parametric representation of these curves may be obtained by Salkowski's method.

7. The ratio of the elements of arc of the x- and y-curves is constant, or, what is equivalent, the angle θ formed by the directions of their tangents is constant, only if $Q(A)/T\,(=\tan\theta)$ is constant. By (12) this condition becomes

$$\Delta\left(\frac{dA}{ds_\alpha}\right)=\tan\theta\, T\left(\frac{ds_\gamma}{ds_\alpha}\right)^2.$$

The general solution of this equation is

$$\frac{dA}{ds_\alpha}=r(a\,|\,\alpha)-\tan\theta\left(\textstyle\int\gamma ds\,|\,\alpha\right). \tag{14}$$

For this value of dA/ds_α, (9) reduces to $y=x+\tan\theta\int\gamma ds-ra$.

The corresponding elements of arc of the x-curve and its transform by A are in constant ratio, or the directions of their corresponding tangents form a constant angle θ, only when A is given by (14). Work similar to that leading to this result yields a new means of representing the general y-curve: The general transformation A, carrying a given x-curve into a y-curve whose element of arc is a given multiple of that of the x-curve——$ds_y = \sec\theta ds$, where θ is a given function——is defined by $dA/ds_a = r(a|\alpha) - (\int \tan\theta\,\gamma ds|\alpha)$, and the most general such y-curve is then $y = x + \int \tan\theta\,\gamma ds - ra$.

If θ is constant, the formulae (11) become

$$\alpha_y = \cos\theta\,\alpha + \sin\theta\,\gamma, \quad \beta_y = \beta, \quad \gamma_y = \cos\theta\,\gamma - \sin\theta\,\alpha,$$
$$\frac{\sec\theta}{R_y} = \frac{\cos\theta}{R} + \frac{\sin\theta}{T}, \quad \frac{\sec\theta}{T_y} = \frac{\cos\theta}{T} - \frac{\sin\theta}{R}, \quad ds_y = \sec\theta\,ds.$$

Thus, if the y-curve results from the x-curve by a transformation A, preserving the ratio of the elements of arc, then the principal normals of the two curves are parallel, and conversely; their rectifying lines are parallel and the elements of arc of their principal normal indicatrices are in constant ratio. Further, if the given curve is a curve of constant curvature or torsion, the transformed curve is a curve of Bertrand; conversely, a curve of Bertrand may be carried over by a suitable such transformation A into a curve of constant curvature or torsion. (Thus the determination of the curves of Bertrand is reduced to the determination of the curves of constant curvature or torsion.) Finally, under such a transformation A, a helix goes into a helix and a circular helix into a circular helix.

The surface z is, for constant θ, $z = x + \rho(\tan\theta \int \gamma ds - ra)$. The curve $\rho = 1$ on it is the transform of the x-curve by (14). Evidently an arbitrary curve $\rho = \text{const.}$ is also such a transform. Thus the curves $\rho = \text{const.}$ have their principal normals along a ruling of the surface parallel. Conversely, if a ruled non-developable surface contains a family of curves having parallel principal normals along each ruling, these curves are the transforms of one of their number by transformations of the type (14).

8. The transformations A, such that the point tracing the y-curve lies always in the corresponding rectifying plane of the x-curve, are given by $dA/ds_a = a$, a constant; the y-curve is then $y = x - a(\alpha - T/R\,\gamma)$. Evidently the surface z is the rectifying developable of the x-curve and thus the x-curve is a geodesic on it. Conversely, a geodesic on a developable may be carried into any second curve on the developable, whose tangent is perpendicular to the corresponding principal normal of the geodesic, by such a transformation A. The y-curve is also a geodesic on the developable only when the x-curve is a helix; the two curves are then congruent.

The transformations A, such that the osculating plane of the y-curve contains always the corresponding point of the x-curve, are given by

$$\frac{dA}{ds_\alpha} = c\cos(s_\alpha+k). \tag{15}$$

The y-curve is then $y=x+\delta$, where $\delta=-c[\cos(s_\alpha+k)\alpha-\sin(s_\alpha+k)\beta]$. The x-curve is an asymptotic line and also the line of striction on the surface $z=x+r\delta$. The curves $r=\text{const.}$ on the surface are the curves at constant distances, along the rulings, from the x-curve and each of them is a transform of it by a suitable transformation (15). Conversely, if the line of striction of a ruled surface is asymptotic, the curves at constant distances from it are its transforms by transformations A of the form (15). If these curves are to be asymptotic also, the line of striction must be a curve of the category defined by $dT/ds_\gamma=c\cos(s_\alpha+k)$.

The only y-curve in correspondence A with an x-curve so that y always lies in the normal plane of x is the x-curve itself.

9. Here again, as in the case of the transformation C, a constant distance $xy=d$ is not sufficient for the congruence of the x- and y-curves. In fact, the distance d is constant, and the x- and y-curves are still not congruent, in case of the special transformation (15).

On the other hand, and again as in the case of the transformation C, constant direction of xy is a sufficient condition for congruence of the x- and y-curves.

If the direction xy is relatively fixed with respect to the trihedral in x, the x-curve is a helix and the transformation is given by $dA/ds_\alpha=ke^{ms_\alpha}$, and conversely.

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The Space Problem of the Calculus of Variations in Terms of Angle.

By Paul R. Rider.

Introduction.

In her Göttingen thesis* Gernet treated the problem of the calculus of variations for three dimensions, considering an integral of the form

$$I=\int f\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right)dx.$$

Bliss and Mason† have developed the theory for the integral

$$I=\int f(x, y, z, x_t, y_t, z_t)\,dt,$$

where the arguments of f are functions of a parameter t.

It is the object of this paper to develop the theory for an integral in a different form from either of these, and to use the results in considering certain generalized definitions of angle and of solid angle. The first six sections treat the general theory. In § 1 the Euler equations are derived and the form of the equations of the extremals is shown. By means of the family of extremals through a given point the Jacobi necessary condition is derived in § 2. The Hilbert invariant integral and the Weierstrass E-function are taken up in § 3, and the necessary conditions of Weierstrass and Legendre are given in this section and in § 4. As the F_1-function, which first appears in § 1, is a definite expression instead of a factor of proportionality, the Legendre condition is more easily developed and is simpler than in the Weierstrass form of the problem in three-dimensional space as developed by Bliss and Mason. However, sufficient conditions are obtained for a weak extremum only—§ 5. The so-called transversality condition that must hold when one end-point is allowed to vary along a curve is given in § 6. In § 7 a transversal surface is studied and two generalized definitions of angle are made. Incidentally the differential equations of geodesics on the transversal surface are obtained. Finally, § 8 deals with a generalization of the notion of solid angle.

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* "Untersuchungen zur Variationsrechnung," Dissertation, Göttingen (1902).

† "The Properties of Curves in Space Which Minimize a Definite Integral," *Transactions of the American Mathematical Society*, Vol. IX (1908), p. 440.

§ 1. *The Euler Equations.*

Suppose that there is a curve defined by the functions

$$x=\phi(t),\quad y=\psi(t),\quad z=\chi(t). \tag{C}$$

The integral to be studied is

$$I=\int f(x, y, z, \tau, \sigma)\sqrt{x_t^2+y_t^2+z_t^2}\,dt, \tag{1}$$

where τ and σ are defined by the equations

$$\left.\begin{aligned}\frac{x_t}{\sqrt{x_t^2+y_t^2+z_t^2}}&=\cos\sigma\cos\tau,\\ \frac{y_t}{\sqrt{x_t^2+y_t^2+z_t^2}}&=\cos\sigma\sin\tau,\\ \frac{z_t}{\sqrt{x_t^2+y_t^2+z_t^2}}&=\sin\sigma.\end{aligned}\right\} \tag{2}$$

The geometric significance of τ and σ is evident from equations (2). If at any point on the curve C the positive tangent to the curve is constructed, then σ is the angle that this tangent makes with its own projection in the xy-plane, and τ is the angle that this projection makes with the x-axis.* It is assumed that the function $f(x, y, z, \tau, \sigma)$ is of class C^{iv}† with respect to each of its five arguments in a region $R(x, y, z, \tau, \sigma)$, and throughout the discussion the variables will be confined to this region. The curves considered will be represented in the parametric form (C), where ϕ, ψ, χ are functions of class C^{iv} defining values (x, y, z, τ, σ) interior to the region R for all values of t in the interval $t_0 \leqq t \leqq t_1$. Since τ and σ are to be definitely determinate for every value of t in this interval, it is assumed that x_t, y_t, z_t do not all vanish simultaneously, *i. e.*, that the curves under consideration are without singular points.

If the integral I is taken along a curve of the family

$$x=\phi(t, a),\quad y=\psi(t, a),\quad z=\chi(t, a), \tag{C_a}$$

where $\phi(t, a)$, $\psi(t, a)$, $\chi(t, a)$ are of class C^{iv} in the region

$$t_0 \leqq t \leqq t_1,\quad |a| \leqq k>0,$$

and for $a=0$ define the curve C, then the value of I becomes a function $I(a)$, which for $a=0$ reduces to the value of I taken along C. Hence the derivative of I with respect to a must vanish for $a=0$.

---

*Bliss has developed the theory for the integral $\int f(x, y, \tau)\sqrt{x_t^2+y_t^2}\,dt$. See "A New Form of the Simplest Problem of the Calculus of Variations," *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 405.

†For definitions of terms and notations used in this paper see Bolza, "Vorlesungen über Variationsrechnung."

This derivative is found to be

$$\frac{dI}{da}=\int_{t_0}^{t_1}\left(f_x x_a+f_y y_a+f_z z_a+f_\tau \tau_a+f_\sigma \sigma_a + f\frac{x_t x_{ta}+y_t y_{ta}+z_t z_{ta}}{x_t^2+y_t^2+z_t^2}\right)\sqrt{x_t^2+y_t^2+z_t^2}\,dt. \tag{3}$$

If the values of τ_a, σ_a are obtained from (2) and substituted in (3), after some reductions

$$\frac{dI}{da}=\int_{s_0}^{s_1}(f_x x_a+f_y y_a+f_z z_a+p x_{sa}+q y_{sa}+r z_{sa})\,ds, \tag{4}$$

where

$$\left.\begin{aligned} p&=f\cos\sigma\cos\tau-f_\tau\frac{\sin\tau}{\cos\sigma}-f_\sigma\sin\sigma\cos\tau,\\ q&=f\cos\sigma\sin\tau+f_\tau\frac{\cos\tau}{\cos\sigma}-f_\sigma\sin\sigma\sin\tau,\\ r&=f\sin\sigma+f_\sigma\cos\sigma, \end{aligned}\right\} \tag{5}$$

and where t has been replaced by s, the Euclidean length of arc.

The usual integration by parts gives

$$\frac{dI}{da}=\int_{s_0}^{s_1}(P x_a+Q y_a+R z_a)\,ds+[p x_a+q y_a+r z_a]_{s_0}^{s_1}, \tag{6}$$

in which

$$P=f_x-\frac{dp}{ds},\quad Q=f_y-\frac{dq}{ds},\quad R=f_z-\frac{dr}{dz}. \tag{7}$$

It follows at once that *the Euler equations are*

$$P=0,\quad Q=0,\quad R=0. \tag{8}$$

These equations are not independent, however, for the relation

$$P\cos\sigma\cos\tau+Q\cos\sigma\sin\tau+R\sin\sigma\equiv0 \tag{9}$$

is readily verified.

When the differentiation indicated in (7) is carried out, equations (8) become

$$a_i\frac{d\tau}{ds}+b_i\frac{d\sigma}{ds}+c_i=0,\qquad i=1, 2, 3, \tag{10}$$

where the *a's*, *b's* and *c's* are defined as follows:

$$a_1=f\cos\sigma\sin\tau+f_\tau\sin\sigma\tan\sigma\cos\tau-f_\sigma\sin\sigma\sin\tau+f_{\tau\tau}\frac{\sin\tau}{\cos\sigma}+f_{\tau\sigma}\sin\sigma\cos\tau,$$

$$b_1=f\sin\sigma\cos\tau+f_\tau\frac{\sin\sigma\sin\tau}{\cos^2\sigma}+f_{\tau\sigma}\frac{\sin\tau}{\cos\sigma}+f_{\sigma\sigma}\sin\sigma\cos\tau,$$

$$\begin{aligned} c_1=f_x&(1-\cos^2\sigma\cos^2\tau)-f_y\cos^2\sigma\sin\tau\cos\tau-f_z\sin\sigma\cos\sigma\cos\tau\\ &+f_{x\tau}\sin\tau\cos\tau+f_{y\tau}\sin^2\tau+f_{z\tau}\tan\sigma\sin\tau\\ &+f_{x\sigma}\sin\sigma\cos\sigma+f_{y\sigma}\sin\sigma\cos\sigma\sin\tau\cos\tau+f_{z\sigma}\sin^2\sigma\cos\tau, \end{aligned}$$

$$
\begin{aligned}
a_2 &= -f\cos\sigma\cos\tau + f_\tau \sin\sigma\tan\sigma\sin\tau + f_\sigma \sin\sigma\cos\tau - f_{\tau\tau}\frac{\cos\tau}{\cos\sigma} + f_{\tau\sigma}\sin\sigma\sin\tau,\\
b_2 &= f\sin\sigma\sin\tau - f_\tau \frac{\sin\sigma\cos\tau}{\cos^2\sigma} - f_{\tau\sigma}\frac{\cos\tau}{\cos\sigma} + f_{\sigma\sigma}\sin\sigma\sin\tau,\\
c_2 &= -f_x\cos^2\sigma\sin\tau\cos\tau + f_y(1-\cos^2\sigma\sin^2\tau) - f_z\sin\sigma\cos\sigma\sin\tau\\
&\quad - f_{x\tau}\cos^2\tau - f_{y\tau}\sin\tau\cos\tau - f_{z\tau}\tan\sigma\cos\tau\\
&\quad + f_{x\sigma}\sin\sigma\cos\sigma\sin\tau\cos\tau + f_{y\sigma}\sin\sigma\cos\sigma\sin^2\tau + f_{z\sigma}\sin^2\sigma\sin\tau,\\
a_3 &= -f_\tau\sin\sigma - f_{\tau\sigma}\cos\sigma,\\
b_3 &= -f\cos\sigma - f_{\sigma\sigma}\cos\sigma,\\
c_3 &= -f_x\sin\sigma\cos\sigma\cos\tau - f_y\sin\sigma\cos\sigma\sin\tau + f_z\cos^2\sigma\\
&\quad - f_{x\sigma}\cos^2\sigma\cos\tau - f_{y\sigma}\cos^2\sigma\sin\tau - f_{z\sigma}\sin\sigma\cos\sigma.
\end{aligned}
$$

The three equations (10) are linear and non-homogeneous in $\frac{d\tau}{ds}$, $\frac{d\sigma}{ds}$, and are consistent because of the identity (9). If any two-rowed determinant of the matrix

$$
\begin{Vmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{Vmatrix} \tag{11}
$$

is different from zero, equations (10) can be solved for $\frac{d\tau}{ds}$, $\frac{d\sigma}{ds}$. The values of these determinants are found to be

$$
\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = f_1\cos\tau, \quad \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} = f_1\sin\tau, \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = f_1\tan\sigma, \tag{12}
$$

where

$$
f_1 = (f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau})(f + f_{\sigma\sigma}) - (f_\tau\tan\sigma + f_{\tau\sigma})^2. \tag{13}
$$

If $f_1 \neq 0$, not all the determinants in (12) can be zero, and (10) can be solved, giving

$$
\frac{d\tau}{ds} = A(x, y, z, \tau, \sigma), \quad \frac{d\sigma}{ds} = B(x, y, z, \tau, \sigma). \tag{14}
$$

From (2),

$$
\frac{dx}{ds} = \cos\sigma\cos\tau, \quad \frac{dy}{ds} = \cos\sigma\sin\tau, \quad \frac{dz}{ds} = \sin\sigma. \tag{15}
$$

By reason of known existence theorems for differential equations, it may be stated that *in the region R, through the point x_0, y_0, z_0, and in the direction defined by τ_0, σ_0, one and only one extremal can be drawn. The equations of these extremals are found by integrating* (14) *and* (15). *They have the form*

$$
\left.
\begin{aligned}
x &= \phi(s, x_0, y_0, z_0, \tau_0, \sigma_0),\\
y &= \psi(s, x_0, y_0, z_0, \tau_0, \sigma_0),\\
z &= \chi(s, x_0, y_0, z_0, \tau_0, \sigma_0).
\end{aligned}
\right\} \tag{16}
$$

It may further be stated that the following initial conditions are satisfied:

$$\left.\begin{aligned}\phi(0, x_0, y_0, z_0, \tau_0, \sigma_0) &= x_0, & \phi_s(0, x_0, y_0, z_0, \tau_0, \sigma_0) &= \cos\sigma_0 \cos\tau_0, \\ \psi(0, x_0, y_0, z_0, \tau_0, \sigma_0) &= y_0, & \psi_s(0, x_0, y_0, z_0, \tau_0, \sigma_0) &= \cos\sigma_0 \sin\tau_0, \\ \chi(0, x_0, y_0, z_0, \tau_0, \sigma_0) &= z_0, & \chi_s(0, x_0, y_0, z_0, \tau_0, \sigma_0) &= \sin\sigma_0.\end{aligned}\right\} \quad (17)$$

§ 2. *The Jacobi Necessary Condition.*

If in equations (16) the constants x_0, y_0, z_0 are made equal to the coordinates of the point 0, and the constants τ_0, σ_0 are replaced by functions $\tau(u, v)$, $\sigma(u, v)$, which for $u=u_0$, $v=v_0$ reduce to the angles defining the direction of C at the point 0, these equations take the form

$$x=\phi(s, u, v), \quad y=\psi(s, u, v), \quad z=\chi(s, u, v). \qquad (18)$$

Equations (18) represent the extremal C_{01} (*i. e.*, the extremal from the point 0 to the point 1) for $u=u_0$, $v=v_0$ and satisfy the identities

$$x_0=\phi(0, u, v), \quad y_0=\psi(0, u, v), \quad z_0=\chi(0, u, v), \qquad (19)$$

which express the fact that all of the extremals pass through the point 0 for $s=0$.

The functional determinant of equations (18) is

$$\Delta(s, u, v) \equiv \begin{vmatrix} \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{vmatrix},$$

and it can be shown in the usual way that if an extremal C_{01} makes the integral I either a maximum or a minimum, it can not have on it a point 2 at which the determinant Δ vanishes, *i. e.*, a point conjugate to the initial point 0, provided that at least one of the three-rowed determinants of the matrix

$$\begin{Vmatrix} \Delta_s & \Delta_u & \Delta_v \\ \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{Vmatrix}$$

is different from zero at the point 2.

§ 3. *The Hilbert Invariant Integral, the Weierstrass E-function, and the Necessary Condition of Weierstrass.*

An extremal arc C_{01} that does not have on it a point 2 conjugate to the initial point 0, can be imbedded in a two-parameter family of extremals through a point 0′ in the neighborhood of 0 and in the order 0′01, whose determinant Δ is different from zero at every point of C_{01}. The family of extremals thus forms a field about the extremal C, which is given by the parameter values $u=u_0$, $v=v_0$. Through each point of the region

$$|u-u_0|\leqq\delta, \quad |v-v_0|\leqq\delta, \tag{20}$$

provided δ, which is a positive constant, is sufficiently small, there passes one, and but one, extremal of the set. Furthermore, δ can be so restricted that in the region (20) Δ is different from zero, and the equations

$$x=\phi(s, u, v), \quad y=\psi(s, u, v), \quad z=\chi(s, u, v) \tag{21}$$

have unique solutions

$$s=s(x, y, z), \quad u=u(x, y, z), \quad v=v(x, y, z) \tag{22}$$

of class C'. Consider a curve

$$x=\bar{x}(a), \quad y=\bar{y}(a), \quad z=\bar{z}(a) \tag{$\bar{C}$}$$

that lies entirely within the region (20). Equations (22) and ($\bar{C}$) determine u and v as functions of a. Substitution of these functions in (21) gives a one-parameter family of extremals

$$x=\phi(s, a), \quad y=\psi(s, a), \quad z=\chi(s, a), \tag{23}$$

all of which are of course members of the two-parameter family (21).

The value of the integral I taken along an extremal of the set (23) from the point 0, where $s=0$ to the point 4, where the extremal intersects the curve $\bar{C}$ is obviously a function of a. By the method used in deriving the Euler equations it is found that

$$\frac{d}{da}I(C_{0'4})=f\,\frac{\partial s}{\partial a}\bigg|^4+[p\phi_a+q\psi_a+r\chi_a]_{0'}^4, \tag{24}$$

p, q, r being defined by (5). The terms in the brackets all vanish at the point $0'$, for all of the extremals pass through that point, and consequently ϕ_a, ψ_a, χ_a are zero there. At the point 4,

$$\phi_s\frac{\partial s}{\partial a}+\phi_a=\bar{x}_a, \quad \psi_s\frac{\partial s}{\partial a}+\psi_a=\bar{y}_a, \quad \chi_s\frac{\partial s}{\partial a}+\chi_a=\bar{z}_a,$$

and, furthermore, ϕ, ψ, χ, τ, σ of (23) are equal, respectively, to ϕ, ψ, χ, τ, σ of (21) where the arguments s, u, v of the latter are expressed as functions of a by means of equations (22) and ($\bar{C}$). Since

$$p\phi_s+q\psi_s+r\chi_s=f,$$

equation (24) may be written

$$\frac{d}{da}I(C_{0'4})=[p\bar{x}_a+q\bar{y}_a+r\bar{z}_a]\,|^4,$$

the arguments $\phi(s, a)$, $\psi(s, a)$, $\chi(s, a)$, $\tau(s, a)$, $\sigma(s, a)$ of f, f_τ, f_σ in (24) having been replaced by $\phi(s, u, v)$, $\psi(s, u, v)$, $\chi(s, u, v)$, $\tau(s, u, v)$, $\sigma(s, u, v)$ considered as functions of a.

Let a_4 be the value of a that gives the extremal $C_{0'4}$ and a_5 the value of a that gives the extremal $C_{0'5}$, where 5 as well as 4 is a point of the curve $\overline{C}$. Then

$$\begin{aligned}
I(C_{0'5})-I(C_{0'4}) &= \int_{a_4}^{a_5}(p\overline{x}_a+q\overline{y}_a+r\overline{z}_a)\,da \\
&= \int_{a_4}^{a_5}\Big[\Big(f\cos\sigma\cos\tau-f_\tau\frac{\sin\tau}{\cos\sigma}-f_\sigma\sin\sigma\cos\tau\Big)\cos\overline{\sigma}\cos\overline{\tau} \\
&\qquad+\Big(f\cos\sigma\sin\tau+f_\tau\frac{\cos\tau}{\cos\sigma}-f_\sigma\sin\sigma\sin\tau\Big)\cos\overline{\sigma}\sin\overline{\tau} \\
&\qquad+(f\sin\sigma+f_\sigma\cos\sigma)\sin\overline{\sigma}\Big]\sqrt{\overline{x}_a^2+\overline{y}_a^2+\overline{z}_a^2}\,da \\
&= \int_{a_4}^{a_5}\Big\{f[\cos\sigma\cos\overline{\sigma}\cos(\overline{\tau}-\tau)+\sin\sigma\sin\overline{\sigma}]+f_\tau\frac{\cos\overline{\sigma}}{\cos\sigma}\sin(\overline{\tau}-\tau) \\
&\qquad-f_\sigma[\sin\sigma\cos\overline{\sigma}\cos(\overline{\tau}-\tau)-\cos\sigma\sin\overline{\sigma}]\Big\}\sqrt{\overline{x}_a^2+\overline{y}_a^2+\overline{z}_a^2}\,da,
\end{aligned}$$

where $\overline{\tau}$, $\overline{\sigma}$ are the angles defining the direction of the curve $\overline{C}$.

Since $I(C_{0'5})-I(C_{0'4})$ does not depend on the form of the curve $\overline{C}$, but merely on the points 4 and 5, it is seen that the integral

$$\begin{aligned}
I^* = \int\Big\{f[\cos\sigma\cos\overline{\sigma}\cos(\overline{\tau}-\tau)+\sin\sigma\sin\overline{\sigma}]+f_\tau\frac{\cos\overline{\sigma}}{\cos\sigma}\sin(\overline{\tau}-\tau) \\
-f_\sigma[\sin\sigma\cos\overline{\sigma}\cos(\overline{\tau}-\tau)-\cos\sigma\sin\overline{\sigma}]\Big\}\sqrt{\overline{x}_a^2+\overline{y}_a^2+\overline{z}_a^2}\,da,
\end{aligned}$$

taken along a curve $\overline{C}$ in the field of extremals does not depend on the path of integration, but is a function of the end points only. It is Hilbert's invariant integral and along an extremal reduces to the integral I, as is seen at once if $\overline{\tau}$, $\overline{\sigma}$ are set equal to τ, σ, respectively.

The Weierstrass E-function is the function

$$\begin{aligned}
e(x, y, z;\ \tau, \sigma;\ \overline{\tau}, \overline{\sigma}) &= f(x, y, z, \overline{\tau}, \overline{\sigma}) \\
&\quad-f[\cos\sigma\cos\overline{\sigma}\cos(\overline{\tau}-\tau)+\sin\sigma\sin\overline{\sigma}]-f_\tau\frac{\cos\overline{\sigma}}{\cos\sigma}\sin(\overline{\tau}-\tau) \\
&\quad+f_\sigma[\sin\sigma\cos\overline{\sigma}\cos(\overline{\tau}-\tau)-\cos\sigma\sin\overline{\sigma}]
\end{aligned}$$

occurring in the integrand of the difference

$$I(\overline{C})-I^*(\overline{C})=\int e\sqrt{\overline{x}_a^2+\overline{y}_a^2+\overline{z}_a^2}\,da.$$

Weierstrass's necessary condition follows at once: *If the arc C_{01} is a solution of the Euler equations that minimizes the integral I with respect to* *admissible curves* $\overline{C}_{01}$ *lying entirely within the field of extremals surrounding* C_{01}, *then the condition*

$$e(x, y, z;\ \tau, \sigma;\ \overline{\tau}, \overline{\sigma}) \geqq 0$$

must be satisfied at every point of C_{01} *and for every direction* $\overline{\tau}, \overline{\sigma}$. *If* C_{01} *maximizes* I, e *must be* $\leqq 0$.

§ 4. *The Necessary Condition of Legendre.*

In order to deduce the necessary condition of Legendre, the connection between the e-function of the last section and the function f_1 defined in § 1 must be established. To do this, expand by Taylor's series with a remainder, each term of e considered as a function of $\overline{\tau}$ and $\overline{\sigma}$. It is found that

$$\begin{aligned} f(x, y, z, \overline{\tau}, \overline{\sigma}) &= f + (\overline{\tau}-\tau) f_\tau + (\overline{\sigma}-\sigma) f_\sigma \\ &\quad + \tfrac{1}{2}[(\overline{\tau}-\tau)^2 f_{\tau\tau} + 2(\overline{\tau}-\tau)(\overline{\sigma}-\sigma) f_{\tau\sigma} + (\overline{\sigma}-\sigma)^2 f_{\sigma\sigma}] + h_1, \\ f[\cos\sigma \cos\overline{\sigma} \cos(\overline{\tau}-\tau) + \sin\sigma \sin\overline{\sigma}] &= f + \tfrac{1}{2}[-(\overline{\tau}-\tau)^2 f \cos^2\sigma - (\overline{\sigma}-\sigma)^2 f] + h_2 f, \\ f_\tau \frac{\cos\overline{\sigma}}{\cos\sigma} \sin(\overline{\tau}-\tau) &= (\overline{\tau}-\tau) f_\tau - (\overline{\tau}-\tau)(\overline{\sigma}-\sigma) f_\tau \tan\sigma + h_3 f_\tau, \\ f_\sigma [\sin\sigma \cos\overline{\sigma} \cos(\overline{\tau}-\tau) - \cos\sigma \sin\overline{\sigma}] &= -(\overline{\sigma}-\sigma) f_\sigma - \tfrac{1}{2}(\overline{\tau}-\tau)^2 f_\sigma \sin\sigma \cos\sigma + h_4 f_\sigma. \end{aligned}$$

The remainder h_1 is homogeneous of degree 3 in $\overline{\tau}-\tau$, $\overline{\sigma}-\sigma$, and contains third partial derivatives of f with mean value arguments τ^*, σ^*. Since it has been assumed that f is of class C^{iv} with respect to all of its arguments, these partial derivatives are finite, and h_1 vanishes as $\overline{\tau}-\tau$, $\overline{\sigma}-\sigma$ both approach zero. The expressions h_2, h_3, h_4 are also homogeneous of degree 3 in $\overline{\tau}-\tau$, $\overline{\sigma}-\sigma$, and, consequently, converge to zero with these two quantities. Thus,

$$\begin{aligned} e(x, y, z;\ \tau, \sigma;\ \overline{\tau}, \overline{\sigma}) &= \tfrac{1}{2}(\overline{\tau}-\tau)^2 (f \cos^2\sigma - f_\sigma \sin\sigma \cos\sigma + f_{\tau\tau}) \\ &\quad + (\overline{\tau}-\tau)(\overline{\sigma}-\sigma)(f_\tau \tan\sigma + f_{\tau\sigma}) + \tfrac{1}{2}(\overline{\sigma}-\sigma)^2 (f + f_{\sigma\sigma}) + h, \end{aligned}$$

where h is a finite expression that is homogeneous of degree 3 in $\overline{\tau}-\tau$, $\overline{\sigma}-\sigma$.

Consider now the quadratic form

$$K \equiv \xi^2 (f \cos^2\sigma - f_\sigma \sin\sigma \cos\sigma + f_{\tau\tau}) + 2\xi\eta (f_\tau \tan\sigma + f_{\tau\sigma}) + \eta^2 (f + f_{\sigma\sigma})$$

in the real variables ξ, η. If the arc C_{01} furnishes a minimum (either strong or weak) for the integral I, then K must be $\geqq 0$ at all points of the arc C_{01} and for all values of ξ and η.

For if ξ, η were a pair of values making K negative at some point of C_{01}, $\overline{\tau}$ and $\overline{\sigma}$ could be chosen so that $\overline{\tau}-\tau = \varepsilon\xi$, $\overline{\sigma}-\sigma = \varepsilon\eta$, and for sufficiently small values of ε the e-function would be negative. But it was seen in the preceding section that e must be $\geqq 0$ for a minimum. Similarly, for a maximum, K must be $\leqq 0$.

A necessary and sufficient condition that K be definite is that

$$(f_\tau \tan\sigma + f_{\tau\sigma})^2 - (f\cos^2\sigma - f_\sigma \sin\sigma\cos\sigma + f_{\tau\tau})(f + f_{\sigma\sigma}) < 0,$$

i. e., that $f_1 > 0$. Therefore, for a so-called regular problem (one in which $f_1 \neq 0$), f_1 must be positive if the extremal C_{01} furnishes either a minimum or a maximum for the integral I.

The relations

$$\left.\begin{aligned} &(f + f_{\sigma\sigma})K = f_1\xi^2 + [(f_\tau \tan\sigma + f_{\tau\sigma})\xi + (f + f_{\sigma\sigma})\eta]^2, \\ &(f\cos^2\sigma - f_\sigma \sin\sigma\cos\sigma + f_{\tau\tau})K \\ &\quad = [(f\cos^2\sigma - f_\sigma \sin\sigma\cos\sigma + f_{\tau\tau})\xi + (f_\tau \tan\sigma + f_{\tau\sigma})\eta]^2 + f_1\eta^2 \end{aligned}\right\} \quad (25)$$

are easily verified. If it is assumed that $f_1 > 0$, it follows that

$$K,\ f + f_{\sigma\sigma},\ f\cos^2\sigma - f_\sigma \sin\sigma\cos\sigma + f_{\tau\tau}$$

must all have the same sign. But if f_1 is positive, neither $f + f_{\sigma\sigma}$ nor $f\cos^2\sigma - f_\sigma \sin\sigma\cos\sigma + f_{\tau\tau}$ can be equal to zero. Consequently, they must be greater than zero for a minimum and less than zero for a maximum.

§ 5. *Sufficient Conditions for a Weak Extremum.*

The e-function can be expressed in forms similar to (25) of the preceding section:

$$\begin{aligned} &2(f + f_{\sigma\sigma})e = f_1(\bar\tau - \tau)^2 + [(f_\tau \tan\sigma + f_{\tau\sigma})(\bar\tau - \tau) + (f + f_{\sigma\sigma})(\bar\sigma - \sigma)]^2 + 2(f + f_{\sigma\sigma})h \\ &2(f\cos^2\sigma - f_\sigma \sin\sigma\cos\sigma + f_{\tau\tau})e \\ &\quad = [(f\cos^2\sigma - f_\sigma \sin\sigma\cos\sigma + f_{\tau\tau})(\bar\tau - \tau) + (f_\tau \tan\sigma + f_{\tau\sigma})(\bar\sigma - \sigma)]^2 \\ &\qquad\qquad + f_1(\bar\sigma - \sigma)^2 + 2(f\cos^2\sigma - f_\sigma \sin\sigma\cos\sigma + f_{\tau\tau})h \end{aligned}$$

where h, as previously stated, is homogeneous of degree 3 in $\bar\tau - \tau$, $\bar\sigma - \sigma$.

For a regular problem, $f + f_{\sigma\sigma}$ and $f\cos^2\sigma - f_\sigma \sin\sigma\cos\sigma + f_{\tau\tau}$ are either both positive or else both negative. Assume them positive, *i. e.*, assume that Legendre's necessary condition for a minimum is fulfilled. Then, at least if $|\bar\tau - \tau|$ and $|\bar\sigma - \sigma|$ are less than some positive constant γ, the e-function is positive except when $\bar\tau = \tau$, $\bar\sigma = \sigma$, in which case it vanishes.

It follows that if along the extremal arc C_{01} Jacobi's necessary condition for an extremum and Legendre's necessary condition for a minimum are satisfied, C_{01} will furnish at least a weak minimum for the integral I. A similar theorem of course holds for a weak maximum.

§ 6. *The Transversality Condition.*

It can easily be proved that if one end point of the extremal C is allowed to vary along a curve $\overline{C}$, then the condition

$$f[\cos\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)+\sin\sigma\sin\bar{\sigma}]+f_\tau\frac{\cos\bar{\sigma}}{\cos\sigma}\sin(\bar{\tau}-\tau)$$
$$-f_\sigma[\sin\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)-\cos\sigma\sin\bar{\sigma}]=0$$

must be satisfied at the point of intersection of C and $\overline{C}$.

When $f=1$ this condition reduces to the condition for perpendicularity as is to be expected.

When $\bar{\sigma}=\sigma=0$ it reduces to the condition as given by Bliss for the problem in two dimensions.

§ 7. *A Generalized Notion of Angle.*

Consider the two-parameter family of extremals (18) that pass through the point 0. On each of the extremals take an extremal arc such that the value of the integral I taken along the arc from the point 0 to the other end of the arc shall be equal to a constant ρ. The value of s determining the end-point of this arc is defined by the equation

$$\rho=\int_0^s f(\phi,\psi,\chi,\tau,\sigma)\sqrt{\phi_s^2+\psi_s^2+\chi_s^2}\,ds,\quad \phi_s^2+\psi_s^2+\chi_s^2=1. \tag{26}$$

The locus of the terminal points of these arcs is a so-called transversal surface. Since ρ may be considered as the generalized distance from the point 0 to this surface, the surface itself may be thought of as a generalized sphere whose center is the point 0 and whose radius is ρ.

Assuming that f is greater than zero, $\dfrac{\partial\rho}{\partial s}=f$ is different from zero at the point 0, and (26) can be solved for s in terms of ρ, τ_0, σ_0, and the solution will have continuous first derivatives with respect to all three arguments in a properly chosen region. Since

$$\frac{\partial s}{\partial\rho}=\frac{1}{f}>0$$

in the vicinity of the point 0, the value of s will increase from zero with ρ. The equations of the transversal surface are found by substituting in equations (16) the value of s obtained from (26). The parameters of the surface are τ_0 and σ_0. It is convenient to introduce here the following functions of the theory of surfaces:

$$E=x_{\tau_0}^2+y_{\tau_0}^2+z_{\tau_0}^2,\quad F=x_{\tau_0}x_{\sigma_0}+y_{\tau_0}y_{\sigma_0}+z_{\tau_0}z_{\sigma_0},\quad G=x_{\sigma_0}^2+y_{\sigma_0}^2+z_{\sigma_0}^2.$$

Consider any curve $\bar{C}$ on the transversal surface, and suppose that τ_0 and σ_0 are functions of a single parameter a. Then the value of the integral I, taken over this curve from the point $A_0(a=a_0)$ to the point $A_1(a=a_1)$ is

$$\lambda=\int_{a_0}^{a_1} f(\bar{x}, \bar{y}, \bar{z}, \bar{\tau}, \bar{\sigma})\sqrt{E\tau_0'^2+2F\tau_0'\sigma_0'+G\sigma_0'^2}\,da, \tag{27}$$

where $\bar{x}, \bar{y}, \bar{z}, \bar{\tau}, \bar{\sigma}$ refer to $\bar{C}$, and where accents denote differentiation with respect to a. The quantity λ may be thought of as the generalized length of the curve $\bar{C}$ from the point A_0 to the point A_1.

Let the angle through which the extremal OA_1 has passed in moving to its position from the position OA_0 be defined as the limit, as ρ approaches zero, of the ratio $\dfrac{\lambda}{\rho}$.

The process of calculating this limit is somewhat lengthy, but can be accomplished in a perfectly straightforward manner. By partial differentiation with respect to τ_0 and σ_0, from (26) are obtained the two equations:

$$0=f(\phi, \psi, \chi, \tau, \sigma)\frac{\partial s}{\partial\tau_0}+\int_0^s\frac{\partial}{\partial\tau_0}[f(\phi, \psi, \chi, \tau, \sigma)\sqrt{\phi_s^2+\psi_s^2+\chi_s^2}]\,ds,$$
$$0=f(\phi, \psi, \chi, \tau, \sigma)\frac{\partial s}{\partial\sigma_0}+\int_0^s\frac{\partial}{\partial\sigma_0}[f(\phi, \psi, \chi, \tau, \sigma)\sqrt{\phi_s^2+\psi_s^2+\chi_s^2}]\,ds,$$

which reduce, when integrated by parts, to

$$\left.\begin{aligned}0&=f\frac{\partial s}{\partial\tau_0}+p\phi_{\tau_0}+q\psi_{\tau_0}+r\chi_{\tau_0},\\ 0&=f\frac{\partial s}{\partial\sigma_0}+p\phi_{\sigma_0}+q\psi_{\sigma_0}+r\chi_{\sigma_0},\end{aligned}\right\} \tag{28}$$

Now

$$x_{\tau_0}=\phi_s\frac{\partial s}{\partial\tau_0}+\phi_{\tau_0}=\cos\sigma\cos\tau\frac{\partial s}{\partial\tau_0}+\phi_{\tau_0},$$

and there are similar expressions for y_{τ_0}, z_{τ_0}. By using the expression fo $\dfrac{\partial s}{\partial\tau_0}$ obtained from (28), and making some reductions, the values of x_{τ_0}, y_{τ_0}, z_τ are found to be

$$\begin{aligned}x_{\tau_0}=\frac{1}{f}\,[&\phi_{\tau_0}(f-f\cos^2\sigma\cos^2\tau+f_\tau\sin\tau\cos\tau+f_\sigma\sin\sigma\cos\sigma\cos^2\tau)\\ &+\psi_{\tau_0}(-f\cos^2\sigma\sin\tau\cos\tau-f_\tau\cos^2\tau+f_\sigma\sin\sigma\cos\sigma\sin\tau\cos\tau)\\ &+\chi_{\tau_0}(-f\sin\sigma\cos\sigma\cos\tau-f_\sigma\cos^2\sigma\cos\tau)]\\ =\frac{1}{f}\,[&q(\phi_{\tau_0}\cos\sigma\sin\tau-\psi_{\tau_0}\cos\sigma\cos\tau)-r(\chi_{\tau_0}\cos\sigma\cos\tau-\phi_{\tau_0}\sin\sigma)],\end{aligned}$$

$$\begin{aligned}
y_{\tau_0} &= \frac{1}{f}\,[\phi_{\tau_0}(-f\cos^2\sigma\sin^2\tau+f_\tau\sin^2\tau+f_\sigma\sin\sigma\cos\sigma\sin\tau\cos\tau)\\
&\qquad +\psi_{\tau_0}(f-f\cos^2\sigma\sin^2\tau-f_\tau\sin\tau\cos\tau+f_\sigma\sin\sigma\cos\sigma\sin^2\tau)\\
&\qquad +\chi_{\tau_0}(-f\sin\sigma\cos\sigma\sin\tau-f_\sigma\cos^2\sigma\sin\tau)]\\
&= \frac{1}{f}\,[r(\psi_{\tau_0}\sin\sigma-\chi_{\tau_0}\cos\sigma\sin\tau)-p(\phi_{\tau_0}\cos\sigma\sin\tau-\psi_{\tau_0}\cos\sigma\cos\tau)],\\
z_{\tau_0} &= \frac{1}{f}\,[\phi_{\tau_0}(-f\sin\sigma\cos\sigma\cos\tau+f_\tau\tan\sigma\sin\tau+f_\sigma\sin^2\sigma\cos\tau)\\
&\qquad +\psi_{\tau_0}(-f\sin\sigma\cos\sigma\sin\tau-f_\tau\tan\sigma\cos\tau+f_\sigma\sin^2\sigma\sin\tau)\\
&\qquad +\chi_{\tau_0}(f-f\sin^2\sigma-f_\sigma\sin\sigma\cos\sigma)]\\
&= \frac{1}{f}\,[p(\chi_{\tau_0}\cos\sigma\cos\tau-\phi_{\tau_0}\sin\sigma)-q(\psi_{\tau_0}\sin\sigma-\chi_{\tau_0}\cos\sigma\sin\tau)].
\end{aligned}$$

The values of x_{σ_0}, y_{σ_0}, z_{σ_0} are obtained by replacing the subscript τ_0 by σ_0 in the right-hand members of the last three equations.

Some limits which are needed in finding the value of the limit of $\dfrac{\lambda}{\rho}$ are now to be calculated. Since

$$\frac{ds}{d\rho}=\frac{1}{f(\phi,\psi,\chi,\tau,\sigma)},$$

it follows that

$$\lim_{\rho=0}\frac{s}{\rho}=\frac{1}{f(x_0,y_0,z_0,\tau_0,\sigma_0)}. \tag{29}$$

By the law of the mean

$$\begin{aligned}
\phi_{\tau_0}(s,\tau_0,\sigma_0) &= \phi_{\tau_0}(0,\tau_0,\sigma_0)+s\phi_{s\tau_0}(\theta s,\tau_0,\sigma_0), \quad 0<\theta<1,\\
&= s\phi_{s\tau_0}(\theta s,\tau_0,\sigma_0),
\end{aligned}$$

since from (17) $\phi_{\tau_0}(0,\tau_0,\sigma_0)$ is identically zero. Thus,

$$\lim_{\rho=0}\frac{1}{\rho}\phi_{\tau_0}(s,\tau_0,\sigma_0)=\lim_{\rho=0}\frac{1}{\rho}s\phi_{s\tau_0}(\theta s,\tau_0,\sigma_0)=-\frac{\cos\sigma_0\sin\tau_0}{f(x_0,y_0,z_0,\tau_0,\sigma_0)},$$

using (29) and (17). Similarly,

$$\begin{aligned}
&\lim_{\rho=0}\frac{1}{\rho}\psi_{\tau_0}(s,\tau_0,\sigma_0)=\frac{\cos\sigma_0\cos\tau_0}{f(x_0,y_0,z_0,\tau_0,\sigma_0)},\\
&\lim_{\rho=0}\frac{1}{\rho}\chi_{\tau_0}(s,\tau_0,\sigma_0)=0,\\
&\lim_{\rho=0}\frac{1}{\rho}\phi_{\sigma_0}(s,\tau_0,\sigma_0)=-\frac{\sin\sigma_0\cos\tau_0}{f(x_0,y_0,z_0,\tau_0,\sigma_0)},\\
&\lim_{\rho=0}\frac{1}{\rho}\psi_{\sigma_0}(s,\tau_0,\sigma_0)=-\frac{\sin\sigma_0\sin\tau_0}{f(x_0,y_0,z_0,\tau_0,\sigma_0)},\\
&\lim_{\rho=0}\frac{1}{\rho}\chi_{\sigma_0}(s,\tau_0,\sigma_0)=\frac{\cos\sigma_0}{f(x_0,y_0,z_0,\tau_0,\sigma_0)}.
\end{aligned}$$

From these limits may be computed

$$\lim_{\rho=0}\frac{1}{\rho}x_{\tau_0}=-\frac{1}{f^2}(q\cos^2\sigma_0+r\sin\sigma_0\cos\sigma_0\sin\tau_0)$$

$$=-\frac{1}{f^2}\cos\sigma_0(f\sin\tau_0+f_\tau\cos\tau_0),$$

$$\lim_{\rho=0}\frac{1}{\rho}y_{\tau_0}=\frac{1}{f^2}(p\cos^2\sigma_0+r\sin\sigma_0\cos\sigma_0\cos\tau_0)$$

$$=\frac{1}{f^2}\cos\sigma_0(f\cos\tau_0-f_\tau\sin\tau_0),$$

$$\lim_{\rho=0}\frac{1}{\rho}z_{\tau_0}=\frac{1}{f^2}(p\sin\sigma_0\cos\sigma_0\sin\tau_0-q\sin\sigma_0\cos\sigma_0\cos\tau_0)=-\frac{1}{f^2}f_\tau\sin\sigma_0,$$

$$\lim_{\rho=0}\frac{1}{\rho}x_{\sigma_0}=-\frac{1}{f^2}r\cos\tau_0=-\frac{1}{f^2}\cos\tau_0(f\sin\sigma_0+f_\sigma\cos\sigma_0),$$

$$\lim_{\rho=0}\frac{1}{\rho}y_{\sigma_0}=-\frac{1}{f^2}r\sin\tau_0=-\frac{1}{f^2}\sin\tau_0(f\sin\sigma_0+f_\sigma\cos\sigma_0),$$

$$\lim_{\rho=0}\frac{1}{\rho}z_{\sigma_0}=\frac{1}{f^2}(p\cos\tau_0+q\sin\tau_0)=\frac{1}{f^2}(f\cos\sigma_0-f_\sigma\sin\sigma_0),$$

from which it follows that

$$\lim_{\rho=0}\frac{E}{\rho^2}=\frac{1}{f^4}(f^2\cos^2\sigma_0+f_\tau^2),\quad \lim_{\rho=0}\frac{F}{\rho^2}=\frac{1}{f^4}f_\tau f_\sigma,\quad \lim_{\rho=0}\frac{G}{\rho^2}=\frac{1}{f^4}(f^2+f_\sigma^2).$$

Going back to (27), *it is found that*

$$\lim_{\rho=0}\frac{\lambda}{\rho}=\int_{a_0}^{a_1}\frac{f(x_0,y_0,z_0,\overline{\tau}_0,\overline{\sigma}_0)}{f(x_0,y_0,z_0,\tau_0,\sigma_0)}\sqrt{(f^2\cos^2\sigma_0+f_\tau^2)\tau_0'^2+2f_\tau f_\sigma\tau_0'\sigma_0'+(f^2+f_\sigma^2)\sigma_0'^2}\,da, \quad (30)$$

which is the generalized expression for angle.

In the case of Euclidean geometry, $f=1$, the extremals are straight lines, and the transversal surface is the sphere

$$x=\rho\cos\sigma_0\cos\tau_0,\quad y=\rho\cos\sigma_0\sin\tau_0,\quad z=\rho\sin\sigma_0,$$

for which $E=\rho^2\cos^2\sigma_0$, $F=0$, $G=\rho^2$. Since f_τ and f_σ both vanish, the expression (30) reduces to

$$\int_{a_0}^{a_1}\sqrt{\cos^2\sigma_0\tau_0'^2+\sigma_0'^2}\,da, \quad (31)$$

which is the ordinary definition of the angle through which the line OA has moved in passing from the position OA_0 to the position OA_1, the point A traversing any curve on the sphere.

A slightly less general definition of angle in Euclidean geometry is the value of (31) when the point A is restricted to move along a great circle of the sphere, *i. e.*, when the line moves from OA_0 to OA_1 keeping always in the same plane. It will be shown that a curve $\overline{C}$ on the transversal surface (or in fact on any surface) that minimizes the integral (27) with respect to neighboring curves on the surface, must satisfy Euler equations, which for $f=1$ reduce to the differential equations of geodesics. Since the geodesics of a sphere are great circles, the value of the integral in (30) taken over such a curve as $\overline{C}$, which may be called a generalized geodesic, obviously reduces for Euclidean geometry to the length of the circular arc A_0A_1 divided by the radius of the sphere, which is the definition of the plane angle that OA_1 makes with OA_0. For simplicity write

$$\lambda=\int_{a_0}^{a_1}\overline{f}\Gamma da,$$

where

$$\overline{f}=f(\overline{x}, \overline{y}, \overline{z}, \overline{\tau}, \overline{\sigma}), \quad \Gamma=\sqrt{E\tau_0'^2+2F\tau_0'\sigma_0'+G\sigma_0'^2}.$$

It is readily found that *the curve $\overline{C}$ mentioned above must satisfy the two equations*

$$\begin{aligned}\overline{x}_{\tau_0}&\left[\overline{f}_x\Gamma+\frac{d}{da}\left(\overline{f}_\tau\frac{\sin\overline{\tau}}{\cos\overline{\sigma}}+\overline{f}_\sigma\sin\overline{\sigma}\cos\overline{\tau}\right)\right]\\&+\overline{y}_{\tau_0}\left[\overline{f}_y\Gamma-\frac{d}{da}\left(\overline{f}_\tau\frac{\cos\overline{\tau}}{\cos\overline{\sigma}}-\overline{f}_\sigma\sin\overline{\sigma}\sin\overline{\tau}\right)\right]\\&+\overline{z}_{\tau_0}\left[\overline{f}_z\Gamma-\frac{d}{da}\overline{f}_\sigma\cos\overline{\sigma}\right]+\overline{f}\Gamma_{\tau_0}-\frac{d}{da}\overline{f}\Gamma_{\tau'_0}=0,\end{aligned}\tag{32}$$

$$\begin{aligned}\overline{x}_{\sigma_0}&\left[\overline{f}_x\Gamma+\frac{d}{da}\left(\overline{f}_\tau\frac{\sin\overline{\tau}}{\cos\overline{\sigma}}+\overline{f}_\sigma\sin\overline{\sigma}\cos\overline{\tau}\right)\right]\\&+\overline{y}_{\sigma_0}\left[\overline{f}_y\Gamma-\frac{d}{da}\left(\overline{f}_\tau\frac{\cos\overline{\tau}}{\cos\overline{\sigma}}-\overline{f}_\sigma\sin\overline{\sigma}\sin\overline{\tau}\right)\right]\\&+\overline{z}_{\sigma_0}\left[\overline{f}_z\Gamma-\frac{d}{da}\overline{f}_\sigma\cos\overline{\sigma}\right]+\overline{f}\Gamma_{\sigma_0}-\frac{d}{da}\overline{f}\Gamma_{\sigma'_0}=0,\end{aligned}\tag{33}$$

where $\overline{\tau}$ and $\overline{\sigma}$, the angles determining direction on the curve $\overline{C}$, are defined analytically by the equations

$$\cos\overline{\sigma}\cos\overline{\tau}=\frac{\overline{x}'}{\sqrt{\overline{x}'^2+\overline{y}'^2+\overline{z}'^2}}=\frac{\overline{x}'}{\Gamma},$$

$$\cos\overline{\sigma}\sin\overline{\tau}=\frac{\overline{y}'}{\Gamma}, \quad \sin\overline{\sigma}=\frac{\overline{z}'}{\Gamma}.$$

When $f=1$ equations (32) and (33) reduce to

$$\Gamma_{\tau_0}-\frac{d}{da}\Gamma_{\tau'_0}=0, \quad \Gamma_{\sigma_0}-\frac{d}{da}\Gamma_{\sigma'_0}=0.$$

These are the differential equations of geodesics on the surface whose parameters are τ_0, σ_0.

Multiply (32) by τ_0' and (33) by σ_0' and add, remembering that

$$\bar{x}_{\tau_0}\tau_0'+\bar{x}_{\sigma_0}\sigma_0'=\bar{x}'.$$

The result is the single equation

$$\begin{aligned}&\bar{x}'\left[\bar{f}_x\Gamma+\frac{d}{da}\left(\bar{f}_\tau\frac{\sin\bar{\tau}}{\cos\bar{\sigma}}+\bar{f}_\sigma\sin\bar{\sigma}\cos\bar{\tau}\right)\right]\\&+\bar{y}'\left[\bar{f}_y\Gamma-\frac{d}{da}\left(\bar{f}_\tau\frac{\cos\bar{\tau}}{\cos\bar{\sigma}}-\bar{f}_\sigma\sin\bar{\sigma}\sin\bar{\tau}\right)\right]+\bar{z}'\left[\bar{f}_z\Gamma-\frac{d}{da}\bar{f}_\sigma\cos\bar{\sigma}\right]\\&+\tau_0'\Gamma_{\tau_0}+\sigma_0'\Gamma_{\sigma_0}-\tau_0'\frac{d}{da}\bar{f}\Gamma_{\tau_0'}-\sigma_0'\frac{d}{da}\bar{f}\Gamma_{\sigma_0'}=0.\end{aligned}\qquad(34)$$

Since Γ satisfies the Weierstrass homogeneity condition, the last four terms of (34) may be written

$$(1-\bar{f})\,(\tau_0'\Gamma_{\tau_0}+\sigma_0'\Gamma_{\sigma_0})-\Gamma\frac{d}{da}\bar{f}=0.$$

Dividing (34) by Γ, which is assumed to be different from zero, the equation becomes

$$\begin{aligned}&\cos\bar{\sigma}\cos\bar{\tau}\left[\bar{f}_x\Gamma+\frac{d}{da}\left(\bar{f}_\tau\frac{\sin\bar{\tau}}{\cos\bar{\sigma}}+\bar{f}_\sigma\sin\bar{\sigma}\cos\bar{\tau}\right)\right]\\&+\cos\bar{\sigma}\sin\bar{\tau}\left[\bar{f}_y\Gamma-\frac{d}{da}\left(\bar{f}_\tau\frac{\cos\bar{\tau}}{\cos\bar{\sigma}}-\bar{f}_\sigma\sin\bar{\sigma}\sin\bar{\tau}\right)\right]\\&+\sin\bar{\sigma}\left[\bar{f}_z\Gamma-\frac{d}{da}\bar{f}_\sigma\cos\bar{\sigma}\right]+\frac{1-\bar{f}}{\Gamma}\,(\tau_0'\Gamma_{\tau_0}+\sigma_0'\Gamma_{\sigma_0'})-\frac{d}{da}\bar{f}=0.\end{aligned}$$

If a is replaced by s, the Euclidean length of arc, $\Gamma=1$, and the last equation reduces to

$$\begin{aligned}&\cos\bar{\sigma}\cos\bar{\tau}\left[\bar{f}_x+\frac{d}{ds}\left(\bar{f}_\tau\frac{\sin\bar{\tau}}{\cos\bar{\sigma}}+\bar{f}_\sigma\sin\bar{\sigma}\cos\bar{\tau}\right)\right]\\&+\cos\bar{\sigma}\sin\bar{\tau}\left[\bar{f}_y-\frac{d}{ds}\left(\bar{f}_\tau\frac{\cos\bar{\tau}}{\cos\bar{\sigma}}-\bar{f}_\sigma\sin\bar{\sigma}\sin\bar{\tau}\right)\right]\\&+\sin\bar{\sigma}\left[\bar{f}_z-\frac{d}{ds}\bar{f}_\sigma\cos\bar{\sigma}\right]-\frac{d}{ds}\bar{f}=0.\end{aligned}$$

§ 8. *A Generalization of the Definition of Solid Angle.*

The definition of the measure of a solid angle is the area cut out on a unit sphere by the cone which makes the solid angle (the vertex of the cone of course being at the center of the sphere), or what is the same thing, the ratio of the area cut out on any sphere to the square of the radius of that sphere. In the present section is given a generalized definition of solid angle, which for Euclidean geometry reduces to the ordinary definition.

If on the transversal surface considered in the preceding section there is a closed curve $\overline{C}$, the area $\overline{S}$ enclosed by $\overline{C}$ is

$$\overline{S}=\iint_S \sqrt{EG-F^2}\,d\tau_0 d\sigma_0,$$

where S is the image, in the τ_0, σ_0 plane of $\overline{S}$. A more general definition of of such area is obtained by setting

$$\overline{S}=\iint_S g(\tau_0, \sigma_0, \tau_0', \sigma_0')\sqrt{EG-F^2}\,d\tau_0 d\sigma_0.$$

Solid angle may then be defined by

$$\lim_{\rho=0}\frac{\overline{S}}{\rho^2}.$$

The values of

$$\lim_{\rho=0}\frac{E}{\rho^2},\quad \lim_{\rho=0}\frac{F}{\rho^2},\quad \lim_{\rho=0}\frac{G}{\rho^2}$$

have already been found; from these values can be obtained

$$\lim_{\rho=0}\frac{1}{\rho^4}(EG-F^2)=\frac{1}{f^8}(f^2\cos^2\sigma_0+f_\tau^2)(f^2+f_\sigma^2)-\frac{1}{f^8}f_\tau^2 f_\sigma^2,$$

from which it follows that

$$\lim_{\rho=0}\frac{\overline{S}}{\rho^2}=\iint_S \frac{g}{f^4}\sqrt{(f^2\cos^2\sigma_0+f_\tau^2)(f^2+f_\sigma^2)-f_\tau^2 f_\sigma^2}\,d\tau_0 d\sigma_0.$$

It is obvious that for Euclidean geometry, when $f=g=1$ and the transversal surface is a sphere, this last expression reduces to the usual definition of solid angle as given above.

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Derivation of the Complementary Theorem from the Riemann-Roch Theorem.

BY SAMUEL BEATTY.

In his book[*] and subsequent memoirs[†] on the "Theory of the Algebraic Functions of a Complex Variable," Dr. J. C. Fields gives several proofs of the complementary theorem, of which theorem the well-known Riemann-Roch theorem is a particular case. The purpose of the present paper is not to give an independent proof of the complementary theorem, but rather to derive the same from the Riemann-Roch theorem.

As a preliminary we shall state the complementary theorem in its most general form. We shall suppose that the fundamental equation is

$$f(z,u)=u^n+u^{n-1}f_1(z)+\dots+f_n(z)=0, \tag{1}$$

in which each $f_s(z)$ is a rational function of z. The only limitation on $f(z, u)$ is that it may not possess a repeated factor. The n expansions of u in the vicinity of a value $z=a_k$ group themselves into r_k cycles of orders

$$\nu_1^{(k)},\quad \nu_2^{(k)},\dots,\quad \nu_{r_k}^{(k)},$$

respectively. The expansions of a cycle of order $\nu_s^{(k)}$ proceed according to ascending integral powers of

$$(z-a_k)^{1/\nu_s^{(k)}}.$$

The n expansions of u in the vicinity of the value $z=\infty$ group themselves into r_∞ cycles of orders

$$\nu_1^{(\infty)},\quad \nu_2^{(\infty)},\dots,\quad \nu_{r_\infty}^{(\infty)},$$

respectively. The expansions of a cycle of order $\nu_s^{(\infty)}$ proceed according to ascending integral powers of

$$\left(\frac{1}{z}\right)^{1/\nu_s^{(\infty)}}.$$

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[*] Mayer and Muller, "Theory of the Algebraic Functions of a Complex Variable," Berlin (1916).

[†] One of the most recent and complete bears the title "On the Foundations of the Theory of Algebraic Functions of one Variable." *Phil. Trans. of the Royal Soc. of London*, Series A, Vol. CCXII, pp. 339–373.

Let us imagine we have a set of numbers, one to a cycle, in connection with all values of z. The nature of these numbers we require to be such that the number relating to a cycle of order $\nu_s^{(k)}$ in connection with a value $z=a_k$ is a multiple of $1/\nu_s^{(k)}$, while the number relating to a cycle of order $\nu_s^{(\infty)}$ in connection with the value $z=\infty$ is a multiple of $1/\nu_s^{(\infty)}$; moreover, only a finite number of such numbers may be different from zero. These numbers we agree to denote by $\tau_s^{(k)}$, $\tau_s^{(\infty)}$, respectively, and the set by (τ). Such a set (τ) is called a basis and the numbers $\tau_s^{(k)}$, $\tau_s^{(\infty)}$ are called elements of the basis. Those elements of (τ) relative to finite values of z and to the value $z=\infty$ constitute the partial bases $(\tau)'$ and $(\tau)^\infty$, respectively. The order of coincidence of $G(z, u)$, a rational function of (z, u), relative to a cycle of order $\nu_s^{(k)}$ in connection with a value $z=a_k$ is defined to be the least number appearing as a power of the element of expansion $z-a_k$ in the expression obtained on replacing u in $G(z, u)$ by any of the expansions of u relative to such cycle. The order of coincidence of $G(z, u)$ relative to a cycle in connection with the value $z=\infty$ is defined in a corresponding manner, where, of course, the element of expansion is $1/z$. Suppose now that $G(z, u)$ is a specific rational function of (z, u) not identically zero for any of the branches. Then its orders of coincidence form a basis (γ). The partial bases

$$\left(-\tau+\gamma-1+\frac{1}{\nu}\right)', \quad \left(-\tau+\gamma+1+\frac{1}{\nu}\right)^\infty$$

constitute a basis $(\bar{\tau})$ complementary adjoint to the basis (τ) to the level furnished by the function $G(z, u)$. A rational function of (z, u) is built on a basis (τ) if each of its orders of coincidence is equal to or greater than the corresponding element of the basis (τ). Denoting the general rational function of (z, u) built on a basis (τ) by $R_\tau(z, u)$ and the number of arbitrary constants appearing therein by N_τ, the complementary theorem consists in the formula

$$N_\tau+\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \tau_s^{(k)}\nu_s^{(k)} = N_{\bar{\tau}}+\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)}\nu_s^{(k)}. \tag{2}$$

Moreover, for the purpose of the present paper this formula will be regarded as affording a statement of the Riemann-Roch theorem, where for such purpose, however, equation (1) must be irreducible, the specific rational function of (z, u) used as a level in determining a basis $(\bar{\tau})$ complementary adjoint to (τ) must be $f'_u(z, u)$ and all the elements in one of the bases (τ), $(\bar{\tau})$ must be zero or negative. This statement of the Riemann-Roch theorem does not differ essentially from the usual one.

We now propose to derive the complementary theorem from the Riemann-Roch theorem. For the present we shall suppose that equation (1) is irreducible and that (τ), $(\bar{\tau})$ are complementary adjoint to the level furnished by the function $f'_u(z,u)$. It may happen that not both N_τ, $N_{\bar{\tau}}$ are zero; in that case we suppose that N_τ is different from zero. Then $R_\tau(z,u)$ does not vanish identically, and, consequently, particular values can be ascribed to the N_τ arbitrary constants appearing therein, in such a way as to insure that the resulting specific function $R(z,u)$ is not zero identically. But the orders of coincidence of $R(z,u)$ form a basis (ρ). The bases $(\tau-\rho)$, $(\bar{\tau}+\rho)$ are complementary adjoint to the level furnished by the function $f'_u(z,u)$ and all the elements in the former of these are zero or negative. The Riemann-Roch theorem is, therefore, applicable and hence we may write

$$N_{\tau-\rho}+\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} (\tau_s^{(k)}-\rho_s^{(k)})\nu_s^{(k)}=N_{\bar{\tau}+\rho}+\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} (\bar{\tau}_s^{(k)}+\rho_s^{(k)})\nu_s^{(k)}. \tag{3}$$

The general rational functions of (z,u) built on the bases $(\tau-\rho)$, $(\bar{\tau}+\rho)$ are evidently $R_\tau(z,u)/R(z,u)$, $R_{\bar{\tau}}(z,u)R(z,u)$, respectively, and consequently,

$$N_{\tau-\rho}=N_\tau, \quad N_{\bar{\tau}+\rho}=N_{\bar{\tau}}. \tag{4}$$

Moreover, the elements of the basis (ρ) are connected by the relation

$$\sum_k \sum_{s=1}^{r_k} \rho_s^{(k)}\nu_s^{(k)}=0. \tag{5}$$

On utilizing formulae (4), (5) to effect a simplification of formula (3), the result is the complementary formula (2).

Still supposing that equation (1) is irreducible and that (τ), $(\bar{\tau})$ are complementary adjoint to the level furnished by the function $f'_u(z,u)$ it remains to treat the case in which N_τ, $N_{\bar{\tau}}$ are both zero. Suppose that (t) is a basis each element of which is equal to or less than the corresponding element of the basis (τ) and suppose, moreover, that N_t is different from zero. Denote by $(\bar{t})$ the basis complementary adjoint to (t) to the level furnished by the function $f'_u(z,u)$. We may, therefore, utilize the result of the previous paragraph in its application to the bases (t), $(\bar{t})$ to arrive at the formula

$$N_t+\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} t_s^{(k)}\nu_s^{(k)}=N_{\bar{t}}+\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \bar{t}_s^{(k)}\nu_s^{(k)}. \tag{6}$$

But since in all cases

$$t_s^{(k)}+\bar{t}_s^{(k)}=\tau_s^{(k)}+\bar{\tau}_s^{(k)}, \tag{7}$$

and because each element $t_s^{(k)}$ is equal to or less than the corresponding element $\tau_s^{(k)}$, it follows that each element $\bar{t}_s^{(k)}$ is equal to or greater than the corresponding element $\bar{\tau}_s^{(k)}$ and, consequently,

$$N_{\bar{t}}=0. \tag{8}$$

By utilizing formulae (7), (8) we can give to formula (6) the form

$$\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} (\tau_s^{(k)}-\bar{\tau}_s^{(k)})\nu_s^{(k)}=\sum_k \sum_{s=1}^{r_k} (\tau_s^{(k)}-t_s^{(k)})\nu_s^{(k)}-N_t. \tag{9}$$

Now, the number of conditions to be applied to $R_t(z, u)$ to reduce it to $R_\tau(z, u)$ is certainly not more than

$$\sum_k \sum_{s=1}^{r_k} (\tau_s^{(k)}-t_s^{(k)})\nu_s^{(k)}$$

Since, however, the number of these conditions is N_t, it follows that the right side of (9) is not negative. That is,

$$\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \tau_s^{(k)}\nu_s^{(k)}-\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)}\nu_s^{(k)} \tag{10}$$

is not negative. By interchanging the rôles of the bases (τ), $(\bar{\tau})$, it follows in a similar manner that the expression in (10) is not positive. Therefore,

$$\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \tau_s^{(k)}\nu_s^{(k)}=\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)}\nu_s^{(k)}. \tag{11}$$

This is, however, the complementary formula for the case under discussion, since N_τ, $N_{\bar{\tau}}$ are both zero. This completes the derivation of the complementary theorem from the Riemann-Roch theorem for the case in which equation (1) is irreducible, and in which $(\bar{\tau})$ is complementary adjoint to (τ) to the level furnished by the function $f'_u(z, u)$.

We shall denote by $(\tau)-\dfrac{1}{\nu}$ a basis, the elements of which with a single exception are all equal to the corresponding elements of a basis (τ) and for the excepted cycle of order ν the element is $\dfrac{1}{\nu}$ less than the corresponding element of (τ). Substituting in the above sentence for the word less the word greater we have the definition of a basis denoted by $(\tau)+\dfrac{1}{\nu}$. If (τ), $(\bar{\tau})$ are complementary adjoint to the level furnished by a specific rational function of (z, u) not identically zero for any of the branches, it is evident that $(\tau)-\dfrac{1}{\nu}$, $(\bar{\tau})+\dfrac{1}{\nu}$ are complementary adjoint to the same level. If equation (1) is irreducible, and if the specific rational function of (z, u) used as level in determining a basis $(\bar{\tau})$ complementary adjoint to a given basis (τ) is $f'_u(z, u)$, then from the complementary formulae as already established relative to the complementary bases (τ), $(\bar\tau)$ and $(\tau)-\frac{1}{\nu}$, $(\bar\tau)+\frac{1}{\nu}$, it readily follows that

$$(N_{\tau-\frac{1}{\nu}}-N_\tau)+(N_{\bar\tau}-N_{\bar\tau+\frac{1}{\nu}})=1. \qquad (12)$$

This formula is equivalent to the statement that of the two functions $R_{\tau-\frac{1}{\nu}}(z,u)$, $R_{\bar\tau}(z,u)$ one and only one possesses relative to the single cycle of order ν the precise order of the basis on which it is built.

Still supposing that equation (1) is irreducible, if any specific rational function of (z,u) not identically zero is used as level in determining a basis $(\bar\tau)$ complementary adjoint to a given basis (τ) it is plain that formula (12) holds good, since the quotient of $R_{\bar\tau}(z,u)$ by the specific rational function used as level in determining $(\bar\tau)$, is the same whatever be the function employed as level, or in other words $N_{\bar\tau}$ is the same for all such bases $(\bar\tau)$.

Finally, we shall prove that formula (12) is true, should equation (1) happen to be reducible. In that case suppose that (τ), $(\bar\tau)$ are complementary adjoint to the level furnished by a specific rational function $G(z,u)$ not identically zero for any of the branches. Now,

$$(N_{\tau-\frac{1}{\nu}}-N_\tau)+(N_{\bar\tau}-N_{\bar\tau+\frac{1}{\nu}}) \qquad (13)$$

must possess one of the values 0, 1 or 2. Suppose that the single cycle of order ν for which $(\tau)-\frac{1}{\nu}$ differs from (τ) relates to the equation $f_i(z,u)=0$, which is any one of the irreducible equations comprised under equation (1). Denote by $R^{(i)}_\tau(z,u)$ the general rational function of (z,u) conditioned by that part of a basis (τ) relating to the irreducible equation $f_i(z,u)=0$. We may then employ formula (12) as already arrived at, to conclude that one of the functions $R^{(i)}_{\tau-\frac{1}{\nu}}(z,u)$, $R^{(i)}_{\bar\tau}(z,u)$ possesses for the single cycle of order ν the precise order of the basis on which it is built. Writing $f(z,u)$ in the form $f_i(z,u)Q_i(z,u)$ and denoting the reduced form, relative to $f_i(z,u)=0$ of $R^{(i)}_\tau(z,u)/Q_i(z,u)$ by $H^{(i)}_\tau(z,u)$, it is plain that $H^{(i)}_{\tau-\frac{1}{\nu}}(z,u)Q_i(z,u)$, $H^{(i)}_{\bar\tau}(z,u)Q_i(z,u)$ are built on the bases $(\tau)-\frac{1}{\nu}$, $(\bar\tau)$, respectively, relative to $f(z,u)=0$, and one of these functions possesses for the single cycle of order ν the precise order of the basis on which it is built. The same is, therefore, true for at least one of $R_{\tau-\frac{1}{\nu}}(z,u)$, $R_{\bar\tau}(z,u)$. In other words, expression (13) must equal 1 or 2. It is, however, not possible that it should possess the value 2 for then $R_{\tau-\frac{1}{\nu}}(z,u)R_{\bar\tau}(z,u)/G(z,u)$ would be a function with the sum of its residues equal to ν times an arbitrary constant. Consequently, expression (13) equals 1 in all cases.

Just in passing, the writer wishes to remark that this result may be employed even more conveniently than the complementary theorem, of which it is a corollary, to demonstrate the existence of Abelian integrals of the third kind; also if equation (1) is irreducible, formula (12) may be employed to prove that the general ϕ-function is either zero identically or possesses for any cycle the order of coincidence required by the ϕ-basis; finally, if equation (1) is irreducible, it follows readily from formula (12) that a rational function of (z, u) can be constructed with exactly one pole, when and only when the general ϕ-function is identically zero.

We now proceed to derive the general complementary theorem from the generalized theorem contained in formula (12). Before doing so we shall state formula (12) in the form

$$(N_{\tau-\frac{1}{\nu}}-N_{\bar{\tau}+\frac{1}{\nu}})=(N_{\tau}-N_{\bar{\tau}})+1. \tag{14}$$

The bases (τ), $(\bar{\tau})$ may be written in the forms

$$(\bar{\tau})=(\tau)-(h)+(g), \quad (\tau)=(\bar{\tau})+(h)-(g), \tag{15}$$

in which (h), (g) are bases, all the elements of which are zero or positive. By repeated application of formula (13) to the proper complementary bases and by addition of the results, we get the formulae

$$\left.\begin{aligned}(N_{\tau-h}-N_{\bar{\tau}+h})&=(N_{\tau}-N_{\bar{\tau}})+\sum_{k}\sum_{s=1}^{r_k}h_s^{(k)}\nu_s^{(k)},\\(N_{\bar{\tau}-g}-N_{\tau+g})&=(N_{\bar{\tau}}-N_{\tau})+\sum_{k}\sum_{s=1}^{r_k}g_s^{(k)}\nu_s^{(k)}.\end{aligned}\right\} \tag{16}$$

The left sides of these two equations are, however, equal. Hence,

$$2(N_{\tau}-N_{\bar{\tau}})=\sum_{k}\sum_{s=1}^{r_k}g_s^{(k)}\nu_s^{(k)}-\sum_{k}\sum_{s=1}^{r_k}h_s^{(k)}\nu_s^{(k)}. \tag{17}$$

Replacing the right side of equation (17) by its equivalent value

$$\sum_{k}\sum_{s=1}^{r_k}\bar{\tau}_s^{(k)}\nu_s^{(k)}-\sum_{k}\sum_{s=1}^{r_k}\tau_s^{(k)}\nu_s^{(k)},$$

and dividing by 2 throughout, the result is the complementary formula, arrived at under the most general conditions.

UNIVERSITY OF TORONTO, *December* 31, 1915.

On the Equivalence of Relations $K_{q_1q_2m}$.

By E. W. Chittenden.

Introduction.

T. H. Hildebrandt* has developed the theory of systems $(\mathfrak{Q}; K)$ where K is a relation $K_{q_1q_2m}$.† In terms of a relation K a relation L (limit) is defined. This leads, through the conditioning of the K relation, to a basis system $(\mathfrak{Q}; K^{1367})$ which is found to be an adequate substitute for the system $(\mathfrak{Q}; V)$ of Fréchet, where V denotes a voisinage.‡ In a system $(\mathfrak{Q}; K^{1367})$§ the relation K may be unsymmetric in the arguments q_1, q_2; and the identical sequence consisting of a single repeated element q may not have q as a limit. This constitutes a reduction in the hypothesis of Fréchet.

It is shown in the present paper (§ 6) that from any system $(\mathfrak{Q}; K)$ a system $(\mathfrak{Q}; \dot{\bar{K}})$ may be defined such that $\dot{\bar{K}}$ is symmetric and the identical sequence for q always has q for a limit. If K^{167}, every continuous function in the system $(\mathfrak{Q}; K)$ is continuous in the derived system $(\mathfrak{Q}; \dot{\bar{K}})$. The systems $(\mathfrak{Q}; K)$, $(\mathfrak{Q}; \dot{\bar{K}})$ are said to be equivalent with respect to continuity.

The form of equivalence holding between K and $\dot{\bar{K}}$ follows from the relation between L and $\dot{\bar{L}}$ derived from K and $\dot{\bar{K}}$, respectively. The general theory of equivalences of systems determined by limit relations is important, and is developed, together with some of its consequences, in §§ 1, 2. Two systems $\tilde{\Sigma}$, $\hat{\Sigma}$ containing a class $\mathfrak{Q}$ and defining limit in $\mathfrak{Q}$ may have one of three types of equivalence, denoted by (A), (B), (C). For example $(\mathfrak{Q}; K^{167})$, $(\mathfrak{Q}; \dot{\bar{K}}^{167})$ are equivalent (B). Each of the three types of equivalence implies its predecessor.

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*"A Contribution to the Foundations of Fréchet's Calcul Fonctionnel," American Journal of Mathematics, Vol. XXXIV (1912), pp. 237–290.

†This is a relation K_2, introduced by E. H. Moore, "Introduction to a Form of General Analysis." *The New Haven Mathematical Colloquium*, New Haven (1910), p. 126.

‡*Rendiconti del Circolo di Palermo*, Vol. XXII, pp. 1–64.

§Eight properties of a K relation are defined by Hildebrandt, *loc. cit.*, pp. 243–244. The presence of property i is denoted by K^i. The notation and results of Hildebrandt's memoirs are presupposed in the present paper.

In § 4 necessary and sufficient conditions for equivalence (B) or (C) of systems $(\mathfrak{Q}; \tilde{K}^1)$, $(\mathfrak{Q}; \hat{K}^1)$ are obtained. These conditions are applied in § 5 to show that a relation K^{167} (unsymmetric) is equivalent (C) to a relation $\bar{K}^{125}$ (symmetric), and also in § 6 to show that a relation K^{1867} is equivalent (B) to a relation $\dot{\bar{K}}^{12345}$, therefore equivalent (B) to a voisinage. Extensions of theorems of Fréchet obtained by Hildebrandt, which involve the hypotheses K^{1867}, are immediate consequences of this theory of equivalence.*

1. Denote by $S \equiv \{q_n\}$ an infinite sequence of elements of a class $\mathfrak{Q}$. We shall understand by the notation qLS that q is a limit of S. If S consists of a single repeated element q, S is the identical sequence Iq for q. We consider systems Σ containing a class $\mathfrak{Q}$ relative to which, for every element q and sequence S, it is determined, whether or not qLS, L being defined in some manner in the system Σ.

(A) *Two systems* $\tilde{\Sigma}$, $\hat{\Sigma}$ *are equivalent with respect to limiting element if the derived class* $\mathfrak{R}'$ *of any subclass* $\mathfrak{R}$ *of* $\mathfrak{Q}$ *is the same in* $\tilde{\Sigma}$ *as in* $\hat{\Sigma}$.

(B) $\tilde{\Sigma}$, $\hat{\Sigma}$ *are equivalent with respect to continuity if for every sequence* S *and element* q, *such that* Iq *is not a subsequence of* S, q *is a limit of* S *in either system if it is a limit of* S *in the other.*

(C) $\tilde{\Sigma}$; $\hat{\Sigma}$ *are equivalent with respect to limit of a sequence if every limit in one system is a limit in the other.*

The significance of the notation; $\tilde{\Sigma}$, $\hat{\Sigma}$ are equivalent (B) [equivalent (C)], is evident. Equivalence (B) implies equivalence (A) and is in turn implied by equivalence (C). While it is desired to call attention to the large body of theorems which equivalence (A) will carry over from a system $\tilde{\Sigma}$ to a system $\hat{\Sigma}$, nevertheless the principal results of this paper relate to equivalences (B) and (C).

If $\tilde{\Sigma}$, $\hat{\Sigma}$ are equivalent (B) and the respective limit relations $\tilde{L}$, $\hat{L}$ are of the type L^2† then $q\tilde{L}S$ implies $q\hat{L}S'$, if S' is any subsequence of S which does not contain Iq. A similar statement holds for $q\hat{L}S$.

Two sequences S', S'' are complementary subsequences of S if every element of S not in S' is in S''. A relation L is a relation L^7 if qLS', qLS'' imply qLS.

If $\tilde{\Sigma}$, $\hat{\Sigma}$ *are equivalent with respect to continuity and* $\tilde{L}^{2467}$, $\hat{L}^{2467}$; *then* $\tilde{L}$, $\hat{L}$ *are identical.*

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* For example, the theorems of Hildebrandt, *loc. cit.*, § 22, pp. 288–290.

† A relation L is of type L^2 in case qLS implies qLS', where S' is any subsequence of S. Properties 1–6 of a relation L are defined by Hildebrandt, *loc. cit.*, pp. 281–282.

We must show that, given $q\tilde{L}S$, $q\hat{L}S$ follows. If S does not contain Iq, $q\hat{L}S$ is implied by the definition of equivalence with respect to continuity. If, except for a finite set of elements, S coincides with Iq, we have from $\tilde{L}^2$, $q\tilde{L}Iq$, and from $\hat{L}^6$, $q\hat{L}Iq$. Hence from $\hat{L}^4$, $q\hat{L}S$. Otherwise S contains Iq and an infinite complementary sequence S' which does not contain Iq. Therefore $q\hat{L}Iq$, $q\hat{L}S'$; and it follows from $\hat{L}^7$ that $q\hat{L}S$. A repetition of the argument with $\tilde{L}$, $\hat{L}$ interchanged completes the proof.

If $\tilde{\Sigma}$, $\hat{\Sigma}$ *are equivalent with respect to continuity and* $\tilde{L}^2$, $\hat{L}^2$, *every function* μ continuous on $\mathfrak{Q}$ in either system is continuous in the other.*

We have to show, for example, that if μ is continuous on $\mathfrak{Q}$ in $\tilde{\Sigma}$ and $q\hat{L}S$, $(S=\{q_n\})$ then $L_n\mu_{q_n}=\mu_q$. If S differs from Iq only in a finite number of elements, the result is immediate. Otherwise S will contain S' complementary to Iq. From $\hat{L}^2$, $q\hat{L}S'$; and from equivalence $q\tilde{L}S'$. Therefore $L_n\mu_{q'_n}=\mu_q$, and we have immediately $L_n\mu_{q_n}=\mu_q$.

2. The following properties of classes are definable in terms of limiting element: compact, closed, interior, perfect, separable. Hence,

If $\tilde{\Sigma}$, $\hat{\Sigma}$ *are equivalent* (*A*) [*with respect to limiting element*] *the properties compact, closed, etc., of classes have the same significance in one system as in the other.*

That is, for example, if $\mathfrak{R}$ is compact in $\tilde{\Sigma}$ it is compact in $\hat{\Sigma}$.

If $\tilde{\Sigma}$, $\hat{\Sigma}$ *are equivalent* (*B*) [*with respect to continuity*] *and* $\tilde{L}^2$, $\hat{L}^2$ *then in addition to the preceding proposition every function continuous in one system is continuous in the other.*

If $\tilde{\Sigma}$, $\hat{\Sigma}$ *are equivalent* (*C*) [*with respect to limit of a sequence*] *and* $\tilde{L}^2$, $\hat{L}^2$ *we have the results above and the further result that* $\tilde{L}$ *and* $\hat{L}$ *are identical relations.*

3. In terms of a relation K we define a relation L (limit) as follows:† given a sequence $\{q_n\}$ and an element q such that for every integer m there exists an integer n_m such that $n=m_m$ implies

$$K_{q_nqm},$$

q is said to be a limit of the sequence $\{q_n\}$, and we write

$$q=\underset{n}{L}q_n.$$

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*The relation (*B*) between $\tilde{L}$ and $\hat{L}$ derives its name from the fact expressed in this proposition.

† Cf. Hildebrandt, *loc. cit.*, p. 249. This definition differs from that of Fréchet in terms of voisinage only in non-essentials.

The relation L so defined is a relation L^K (dependent on K), and is a relation L^{23457}.[*] Unless K is conditioned *limit* is not necessarily unique. It is evident that from different relations K, distinct relations L may be defined for a single class $\mathfrak{Q}$.

It will be noticed that in case the relation K is unsymmetric, the relations K_{q_nqm} and K_{qq_nm} may have different significance, and that if we replace K_{q_nqm} by K_{qq_nm} in the definition above a different relation L may result. A number of relations L may be defined for K unsymmetric which reduce to the same relation under the hypothesis of symmetry. The following example will serve to make clear the significance of these remarks. Let $\mathfrak{Q}$ denote the interval $0 \leqq q \leqq 1$. The relation $K_{q_1q_2m}$ holds if $q_1 \leqq q_2$ and $|q_1-q_2| = \frac{1}{2^m}$. Then $q = \underset{n}{L} q_n$ implies that the elements q_n all lie to the left of q, if n is sufficiently large; *i. e.*, q is a right-hand limit of the sequence $\{q_n\}$. The relation K_{qq_nm} may hold for every m and $n \geqq n_m$, but q will not be a limit of $\{q_n\}$ in this sense.

If we denote by $\underline{K}$ a relation such that

$$K_{q_1q_2m} \textit{ implies } \underline{K}_{q_2q_1m},$$

we call $\underline{K}$ the *conjugate* of K. The limit relation $\underline{L}$ will be the conjugate of the relation L defined in terms of K. In the example above the relations L, $\underline{L}$ are distinct, since $\underline{K}_{q_1q_2m}$ implies $q_1 \leqq q_2$, and therefore $q = \underset{n}{\underline{L}} q_n$ implies q is a left-hand limit.

4. Denote by $\phi(q, m)$ any integral-valued single-valued function of two arguments q, m such that for every q

$$\underset{m}{L} \phi(q, m) = \infty.$$

In terms of these functions ϕ we state the theorem:

THEOREM I. *A necessary and sufficient condition that two systems* $(\mathfrak{Q}; \hat{K}^1)$,[†] $(\mathfrak{Q}; \tilde{K}^1)$ *be equivalent with respect to limit of a sequence is that there exist a function* ϕ, *and for every element* q_2 *an* m_2, *such that for every* $m \geqq m_2$,

$$\hat{K}_{q_1q_2m} \textit{ implies } \tilde{K}_{q_1q_2\phi(q_2, m)}, \tag{a}$$

$$\tilde{K}_{q_1q_2m} \textit{ implies } \hat{K}_{q_1q_2\phi(q_2, m)}; \tag{b}$$

furthermore, if in each case q_1 *is supposed to be distinct from* q_2, *conditions* (a) *and* (b) *become necessary and sufficient for equivalence of* $(\mathfrak{Q}; \hat{K}^1)$ $(\mathfrak{Q}; \tilde{K}^1)$ *with respect to continuity.*

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[*] Cf. Hildebrandt, *loc. cit.*, p. 249. That L^7 is an immediate consequence of the definition of limit.

[†] A relation K is a relation K^1 if $K_{q_1q_2m}$ implies $K_{q_1q_2m'}$, where m' is any integer less than m.

A proof of the first part of the theorem follows. A slight modification of this proof serves to establish the second part of the theorem.

We will establish the sufficiency of the condition by showing that $\hat{L}$ and $\tilde{L}$ are identical relations. From the symmetrical nature of the situation it will be sufficient to show that if $q = \underset{n}{\hat{L}} q_n$ then $q = \underset{n}{\tilde{L}} q_n$.

By definition, from $q = \underset{n}{\hat{L}} q_n$ we have for every integer m_0 an integer n_{m_0} such that $n \geqq n_{m_0}$ implies

$$\hat{K}_{q_n q m_0}.$$

Condition (a) gives for $m_0 \geqq m_2$ (dependent on q)

$$\tilde{K}_{\bar{q}_n q \phi(q,\, m_0)}.$$

Since $\underset{m}{L} \phi(q, m) = \infty$ we have for every m an m_0 such that $\phi(q, m_0) \geqq m$. If we choose $n_m \geqq (n_m, n_{m_2})$ we have for every $n \geqq n_m$, because of $\tilde{K}^1$ and the preceding relation,

$$\tilde{K}_{q_n q m}.$$

Therefore $q = \underset{m}{\tilde{L}} q_n$, as was to be proved.

We complete the proof by showing that conditions (a) and (b) are necessary consequences of the definition of equivalence for systems $(\mathfrak{Q}; K)$. Denote by $\hat{\mathfrak{Q}}_{q_2 m}$ the class of all elements q_1 in the relation $\hat{K}_{q_1 q_2 m}$ but not in the relation $\hat{K}_{q_1 q_2 (m+1)}$. Denote by $\tilde{m}_{q_1}$ the greatest value of m for which the relation $\tilde{K}_{q_1 q_2 m}$ holds, in case such greatest value exists.

We now define $\tilde{\phi}(q_2, m)$ as follows: If among the values $\tilde{m}_{q_1}$ for elements q_1 of $\hat{\mathfrak{Q}}_{q_2 m}$ there is a least, take $\tilde{\phi}(q_2, m)$ equal to this $\tilde{m}_{q_1}$. If there is no least $\tilde{m}_{q_1}$ there is either no $\tilde{m}_{q_1}$ or there is a q_1 such that $\tilde{m}_{q_1} < 1$. If there is no $\tilde{m}_{q_1}$, take $\tilde{\phi}(q_2, m) = m$. In the remaining case $\tilde{\phi}(q_2, m) = \tilde{m}_{q_1} < 1$. It is desired to prove that for every q_2, $\underset{m}{L} \tilde{\phi}(q_2, m) = \infty$. Suppose there exists $\bar{m}$ such that for every m, $\phi(q_2, m) < m$. Consider first the values of m for which $\tilde{\phi}(q_2, m) = \tilde{m}_{q_1 m}$ for some q_{1m} in $\hat{\mathfrak{Q}}_{q_2 m}$. From $\hat{K}^1$ it follows that either every m is of this type, or else there is a least m for which there exists a q_{1m} in $\hat{\mathfrak{Q}}_{q_2 m}$ which possesses an $\tilde{m}_{q_1}$. In the first case $q_2 = \underset{m}{L} q_{1m}$. From the equivalence assumed in the hypothesis, $q_2 = \underset{m}{L} q_{1m}$. This contradicts the assumption that all the $m_{q_1 m}$ are less than $\bar{m}$. In the second case there exists an m_0 such that for every $m > m_0$, $\tilde{\phi}(q_2, m) = m$, and therefore $\underset{m}{L} \phi(q_2, m) = \infty$. The assumption $\tilde{\phi}(q_2, m) < \bar{m}$ is therefore contradicted in every case.

We define a number $\tilde{m}_2$ as follows: Either $\tilde{\phi}(q_2, m) = \tilde{m}_{q_1m}$ for every m, or there exists m_0 such that for $m' \geqq m_0$, $\tilde{\phi}(q_2, m') = m'$. In the first case take $\tilde{m}_2$ such that $\tilde{\phi}(q_2, \tilde{m}) \geqq 1$; in the second case take $\tilde{m}_2 = m_0$. Then it is only necessary to take account of $\tilde{K}^1$, and the definitions of $\tilde{m}_2$, $\tilde{m}_{q_1}$, $\tilde{\phi}(q_2, m)$ to see that for every $m \geqq m_2$ condition (a) is satisfied.

By means of a similar choice of $\hat{m}_2$, $\hat{\phi}(q_2, m)$ we may satisfy condition (b): Choosing $m_2 \geqq (\tilde{m}_2, \hat{m}_2)$ and $\phi(q_2, m)$ equal to the lesser of $\tilde{\phi}(q_2, m)$, $\hat{\phi}(q_2, m)$ we have m_2 and ϕ (dependent on q) as required. This completes the proof of the first part of Theorem I.

The hypothesis of equivalence with respect to limit of sequence of K^1 and $\underline{K}^1$ (conjugate of K) therefore leads to the existence of the function ϕ of Theorem I. We show in § 5 that if K^1 and $\underline{K}^1$ are equivalent (C) they are equivalent (C) to a symmetric relation $\bar{K}^1$ derived from K^1. In view of the fact that L and $\underline{L}$ may be distinct for unsymmetric relations K, while their identity implies equivalence (C) to a symmetric K it would seem desirable to confine attention to the symmetric relations in applications of K relations to the study of classes and functions. Especially when the resulting simplification of the treatment is considered.

There may exist a function ϕ, effective in Theorem I whose values are independent of q_2. We thus obtain a corollary to this theorem:

COROLLARY: *A sufficient condition that two systems* $(\mathfrak{Q}; \hat{K}^1)$ $(\mathfrak{Q}; \tilde{K}^1)$ *be equivalent* (C) *is that there exists a function* ϕ, *such that for every* m *and* q_1, q_2

$$\hat{K}_{q_1q_2m} \text{ implies } \tilde{K}_{q_1q_2\phi(m)}, \tag{a}$$

$$\tilde{K}_{q_1q_2m} \text{ implies } \hat{K}_{q_1q_2\phi(m)}. \tag{b}$$

That these conditions are not necessary will be seen if we suppose $\mathfrak{Q}$ represents the open interval $0 \leqq q \leqq 1$ and define $\hat{K}, \tilde{K}$ as follows:

For $m < 1$, every pair of elements q_1q_2 is in the relations $\hat{K}_{q_1q_2m}$, $\tilde{K}_{q_1q_2m}$. For $m \geqq 1$, we take

$$\hat{K}_{q_1q_2m} \equiv |q_1 - q_2| \leqq \frac{1}{2^m}.^{*}$$

If $(k-1)/k = k/(k+1)$ then

$$\tilde{K}_{q_1q_2m} \equiv |q_1 - q_2| \leqq \frac{1}{(k+1)^m}.$$

From the definitions $\hat{K}, \tilde{K}$ are symmetric. They are easily seen to be equivalent (C), but the hypothesis of existence of a function ϕ, leads to a contradiction.

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[*] That is, the relation $K_{q_1q_2m}$ is defined by this inequality for every pair of elements q_1, q_2.

5. Given an unsymmetric relation K we may define a symmetric relation $\overline{K}$ as follows:

A pair of elements q_1, q_2 are in the relation $\overline{K}_{q_1q_2m}$ if and only if they are in one of the relations $K_{q_1q_2m}$, $K_{q_2q_1m}$.

It is easy to see that if K^i $(i=1, 2, 3, 4)$ then $\overline{K}^i$ also.

Theorem II. K^{167} *implies* $\overline{K}^{125(678)}$.*

The properties (6), (7), (8) are each equivalent to 5 because of $\overline{K}^2$. We have to show that there exists a function ϕ such that

$$\overline{K}_{q_1q_2m},\ \overline{K}_{q_2q_3m}\ \textit{imply}\ \overline{K}_{q_1q_3\phi(m)}.$$

Now $\overline{K}_{q_1q_2m}$, $\overline{K}_{q_2q_3m}$, by definition of $\overline{K}$, imply that one of the following four pairs of relations is holding:

$$K_{q_1q_2m},\quad K_{q_2q_3m};$$
$$K_{q_1q_2m},\quad K_{q_3q_2m};$$
$$K_{q_2q_1m},\quad K_{q_2q_3m};$$
$$K_{q_2q_1m},\quad K_{q_3q_2m}.$$

If we choose a function ϕ effective in each of the four instances 5, 6, 7, 8, as is possible when K^1, and recall that K^{167} implies K^{58},† we obtain

$$\overline{K}_{q_1q_3\phi(m)}.$$

This, because of the definition of $\overline{K}$, implies

$$\overline{K}_{q_1q_2\phi(m)}.$$

The following proposition is now evident:

K^{167} *implies that there exists a function* ϕ *such that* $\overline{K}_{q_1q_2m}$ *implies* $K_{q_1q_2\phi(m)}$ *and* $K_{q_2q_1\phi(m)}$.

We have shown that from an unsymmetric relation K^1 a symmetric relation $\overline{K}^1$ may always be defined. The converse is not true. Suppose that for every m and q_1, q_2 the relation $\overline{K}_{q_1q_2m}$ holds. $\overline{K}$ is symmetric. It is evident that a relation K^1 can be made unsymmetric only by adding new relations $K_{q_1q_2m}$ to those already established. But this is impossible in the case cited.

Assuming that for at least one pair $\bar{q}_1$, $\bar{q}_2$ there is an m_0 such that for $m \geqq m_0$ the relation $\overline{K}_{\bar{q}_1\bar{q}_2m_1}$ does not hold, we define K unsymmetric by assuming that the relation $\overline{K}_{q_1q_2m}$ implies $K_{q_1q_2m}$ and in addition $K_{\bar{q}_1\bar{q}_2(m_0+1)}$, but not $K_{\bar{q}_2\bar{q}_1(m+1)}$. Then $\overline{K}^i$ implies K^i where i denotes any of the properties (1), (2), (3), (4), (5), (6), (7), (8).

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* For definitions of the properties 1–8 of a K relation see Hildebrandt, *loc. cit.*, pp. 244–246.

† Hildebrandt, *loc. cit.*, p. 246.

In § 4 it was remarked that if K^1 and $\underline{K}^i$ (conjugate) were equivalent (C) they would be together equivalent (C) to a symmetric relation K. This relation is in fact $\overline{K}$. Since K, $\underline{K}$ are equivalent (C) we have (Theorem I) for every q_2 an m_2 and a function ϕ, such that for every $m \geqq m_2$

$$K_{q_1q_2m} \textit{ implies } \underline{K}_{q_1q_2\phi(q_2,m)} \textit{ which implies } K_{q_2q_1\phi(q_2,m)}, \tag{a}$$

$$\underline{K}_{q_1q_2m} \textit{ implies } K_{q_1q_2\phi(q_2,m)} \textit{ which implies } \underline{K}_{q_2q_1\phi(q_2,m)}. \tag{b}$$

To demonstrate the equivalence (C) of K and $\overline{K}$, we choose m_2 so that (a) and (b) are satisfied, and $\phi(q_2, m) = (\text{lesser of } m, \phi(q_2, m))$. Then the relations:

$$K_{q_1q_2m} \textit{ implies } \overline{K}_{q_1q_2\phi(q_2,m)},$$
$$\overline{K}_{q_1q_2m} \textit{ implies } K_{q_1q_2\phi(q_2,m)},$$

are easily seen to hold for all $m \geqq m_2$. The equivalence (C) of $\overline{K}$ and $\underline{K}$ is similarly established.

THEOREM III. *Any system* $(\mathfrak{Q}; K^{167})$ *is equivalent to a system* $(\mathfrak{Q}; K^{125})$.

This theorem is a consequence of Theorems I and II. To demonstrate the application of Theorem I we must show that conditions (a), (b) are fulfilled. It is sufficient to remark that $\bar{\phi}_m \leqq (m, \phi_m)$, where ϕ is effective in the proposition of § 5, and $m_2 = 1$, are effective in this instance.

In view of this theorem and § 2, it follows that the theorems of Hildebrandt,* which involve the hypothesis K^{167} are not more general than the corresponding theorems with hypothesis K^{125}, that is the hypothesis of an unsymmetrical K does not in this instance lead to more general results.

6. From K we obtain $\dot{K}^4$ as follows:

$$K_{q_1q_2m} \textit{ implies } \dot{K}_{q_1q_2m}(m); \tag{a}$$

$$\textit{for every } q \textit{ and } m \textit{ we have } \dot{K}_{qqm}. \tag{b}$$

Then $\dot{K}_{q_1q_2m}$ implies either $K_{q_1q_2m}$ or $q_1 = q_2$. It is easy to see that K^i implies $\dot{K}^i$ where i is any of the properties (1), (2), ..., (8). Furthermore, K^1 and $\dot{K}^1$ are equivalent (B). In fact $\phi(q_1', m) = m$ is effective in conditions (a), (b) of Theorem I.

THEOREM IV. *Any system* $(\mathfrak{Q}; K^{1367})$ *is equivalent* (B) *to a system* $(\mathfrak{Q}; K^{12345})$.

From Theorem III, K^{167} is equivalent (C) to $\overline{K}^{125}$. But $\overline{K}^{1235}$ is equivalent (B) to $\dot{\overline{K}}^{12345}$. Hence K^{1367} is equivalent (B) to $\dot{\overline{K}}^{12345}$.

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* *Loc. cit.*, pp. 273–290.

Hildebrandt has shown* that the concepts K^{12345} and voisinage (V) are equivalent. Therefore, we have:

THEOREM V. *Any system* $(\mathfrak{Q}; K^{1367})$ *is equivalent* (B) *to a system* $(\mathfrak{Q}; V)$ *of Fréchet, and furthermore, any system* $(\mathfrak{Q}; K^{13467})$ *is equivalent* (C) *to a system* $(\mathfrak{Q}; V)$.

As an application of this theorem we call attention to the theorem:†

In a system $(\mathfrak{Q}; K^{1367})$ *a necessary and sufficient condition that every continuous function on* $\mathfrak{R}$ *be bounded on* $\mathfrak{R}$ *is that* $\mathfrak{R}$ *be extremal,*

which may be obtained from Hahn's ‡ extension of a theorem of Fréchet in the following manner. The condition is necessary in a system $(\mathfrak{Q}; V)$. A system $(\mathfrak{Q}; K^{1367})$ is equivalent (B) to a system $(\mathfrak{Q}; V)$. Therefore every function continuous on $\mathfrak{R}$ in $(\mathfrak{Q}; V)$ is continuous on $\mathfrak{R}$ in $(\mathfrak{Q}; K)$. If every function continuous on $\mathfrak{R}$ in $(\mathfrak{Q}; K)$ is bounded on $\mathfrak{R}$ then $\mathfrak{R}$ is extremal in $(\mathfrak{Q}; V)$. From § 2 $\mathfrak{R}$ is extremal in $(\mathfrak{Q}; K)$, which was to be proved. The condition is sufficient since $\mathfrak{R}$ extremal in a system $(\mathfrak{Q}; K)$ is extremal in any system equivalent (B) to $(\mathfrak{Q}; K)$, that is in $(\mathfrak{Q}; V)$.

URBANA, ILLINOIS, *October* 14, 1916.

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* *Loc. cit.*, p. 248.

† Cf. Hildebrandt, *loc. cit.*, pp. 288–290.

‡ Cf. *Monatshefte für Math. u. Physik*, Vol. XIX, p. 251 ff.

Some Properties of Certain Finite Algebras.[*]

By Edward Kircher.

The object of this paper is to study some of the properties of a finite algebra $\mathfrak{A}$ whose elements combine by addition and multiplication, subject to the commutative, associative, and distributive laws. The elements also form a group when combined by addition, so that subtraction is always possible in $\mathfrak{A}$, but division, the inverse of multiplication, is not always possible and when possible is not necessarily unique. The unit element of the additive group we denote by 0. We also assume the existence of a *unit* U in $\mathfrak{A}$, *i. e.*, an element such that the equation $UX=0$ in $\mathfrak{A}$ admits only the solution $X=0$. Division by a unit is always possible in $\mathfrak{A}$ and is uniquely determined.[†] Again $\mathfrak{A}$ contains a sub-algebra of elements simply isomorphic to the set of integers $0, 1, \ldots, m-1$, taken modulo m.[‡] The elements in $\mathfrak{A}$ corresponding to 0 and 1 are denoted by these symbols. Papers closely related to the subject in hand have been written by Vandiver,[†] Dickson,[§] and Fraenkel.[||] The last of these papers has many results analogous to those of the following pages, but restricts itself to algebras $\mathfrak{A}$ all of whose elements satisfy the unique factorization law.

I. *The Finite Algebra $\mathfrak{A}$ As a System of Residue Classes.*

In a previous paper[¶] the writer discussed the group properties of the residue classes belonging to a modular system $\mathfrak{M}=(\mathfrak{m}_n, \ldots, \mathfrak{m}_1, \mathfrak{m})$, where $\mathfrak{m}$ is an ideal of the algebraic field Ω of degree k, while

$$\mathfrak{m}_i=(\psi_1^{(i)}, \ldots, \psi_j^{(i)}, \ldots, \psi_{j_i}^{(i)}),$$

where each $\psi_j^{(i)}$ is a rational integral function of x_i with coefficients that are in turn rational integral functions of $x_1, x_2, \ldots, x_{i-1}$, with coefficients that are integers in Ω. We assume the number of residue classes to be finite, which is always true if the following conditions are fulfilled:

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[*] Presented to the American Mathematical Society, December 27, 1915.

[†] H. S. Vandiver, *Transactions of the American Mathematical Society*, Vol. XIII (1912), pp. 293–304.

[‡] L. E. Dickson, "Linear Groups," p. 9.

[§] L. E. Dickson, *Transactions of the American Mathematical Society*, Vol. VI (1905), pp. 344–348.

[||] A. Fraenkel, "Ueber die Teiler der Null und die Zerlegung von Ringen," *Crelle's Journal*, Vol. CXLV (1915), pp. 139–175.

[¶] Kircher, "Group Properties of the Residue Classes of Certain Kronecker Modular Systems," etc., *Transactions of the American Mathematical Society*, Vol. XVI (1915), pp. 413–434. This reference will be cited as K in the following pages.

(a) Every $\psi_j^{(i)}$ is of finite degree in x_i;

(b) Every $\mathfrak{m}_i$ contains at least one $\psi_j^{(i)}$ whose highest power of x_i has a coefficient α relatively prime to the modular system

$$(\mathfrak{m}_{i-1}, \mathfrak{m}_{i-2}, \ldots, \mathfrak{m}_1, \mathfrak{m}), \text{ i. e., } (\alpha, \mathfrak{m}_{i-1}, \mathfrak{m}_{i-2}, \ldots, \mathfrak{m}_1, \mathfrak{m}) = (1).[*]$$

(c) For all values of i we have $j_i > 0$ and finite.

The residue classes are formed by all rational integral functions in $x_1, \ldots, x_n$, with coefficients that are integers in Ω, taken modulo $\mathfrak{M}$. Such a set of residue classes evidently forms a finite algebra of the type under consideration. We shall now show that every finite algebra $\mathfrak{A}$ satisfying the conditions laid down can be represented by the residue classes of a modular system $\mathfrak{M}$. The existence and proof of this correspondence was first brought to the writer's attention by H. S. Vandiver.

We know that $\mathfrak{A}$ contains a sub-algebra of integral marks which we may denote by $0, 1, \ldots, m-1$. If there exists another element x_1 of $\mathfrak{A}$ not contained in the above set, it follows that $\mathfrak{A}$ contains the marks

$$\alpha_0 x_1^r + \alpha_1 x_1^{r-1} + \ldots + \alpha_r,$$

where the α's range over $0, 1, 2, \ldots, m-1$. Since $\mathfrak{A}$ is finite it follows that in the set $x_1, x_1^2, \ldots, x_1^k, \ldots$ there must exist an element x_1^k such that there exists a relation of the type $x_1^k = \beta_1 x_1^{k-1} + \beta_2 x_1^{k-2} + \ldots + \beta_k$, which can be written in the form

$$x_1^k + \gamma_1 x_1^{k-1} + \gamma_2 x_1^{k-2} + \ldots + \gamma_k = 0, \qquad \beta_i = -\gamma_i, \tag{I}$$

where the γ's are integral marks. Let k be the smallest integer for which a power of x_1 will satisfy a relation of this type. Then all polynomials in x_1 are included in the set $\beta_1 x_1^{k-1} + \beta_2 x_1^{k-2} + \ldots + \beta_k$, the β's being integral marks. This does not preclude the existence of relations of lower degree between the powers of x_1 provided the coefficient of the highest power of x_1 in this relation is not prime to m. Such a relation is of the form

$$\delta_1 x_1^{k-t} + \delta_2 x_1^{k-t-1} + \ldots + \delta_{k-t} = 0. \tag{II}$$

For brevity denote relation (I) by $\psi_1^{(1)}$, and the various relations (II) by $\psi_2^{(1)}, \ldots, \psi_j^{(1)}, \ldots, \psi_{j_1}^{(1)}$. If there exists in $\mathfrak{A}$ another element x_2 that cannot be written as a polynomial in x_1 alone, we can show in a similar manner that all elements of $\mathfrak{A}$ that can be obtained from x_1, x_2, and the integral marks, are of the form $\varepsilon_1 x_2^t + \varepsilon_2 x_2^{t-1} + \ldots + \varepsilon_t$, where the ε's range over all distinct elements of the form $\beta_1 x_1^{k-1} + \beta_2 x_1^{k-2} + \ldots + \beta_k$. Similarly, there exists a relation

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[*] K, see theorem, p. 425.

$\psi_1^{(2)}=x_2^m+\sigma_1x_2^{m-1}+\ldots+\sigma_m=0$, and possibly some relations of lower degree in x_2 which we may designate by $\psi_2^{(2)},\ldots,\psi_{j_2}^{(2)}$. The coefficients of the highest powers of x_2 in these last relations are not relatively prime to the modular system $(\psi_1^{(1)},\ldots,\psi_j^{(1)},\ldots,\psi_{j_1}^{(1)},m)$. This process may be repeated until all elements of $\mathfrak{A}$ have been exhausted. By referring to a theorem previously proved[*] we see that the totality of relations $\psi_j^{(i)}=0$ so obtained together with $m=0$ defines a modular system whose residue classes correspond to the elements of $\mathfrak{A}$. We therefore have the

THEOREM: *The elements of the finite algebra* $\mathfrak{A}$ *may be represented by the residue classes of a modular system* $\mathfrak{M}=(\mathfrak{m}_n,\mathfrak{m}_{n-1},\ldots,\mathfrak{m}_i,\ldots,\mathfrak{m}_1,\mathfrak{m})$, *where* m *is a rational integer and* $\mathfrak{m}_i=(\psi_1^{(i)},\ldots,\psi_j^{(i)},\ldots,\psi_{j_i}^{(i)})$, *where each* $\psi_j^{(i)}$ *is a rational integral function of* $x_1,x_2,\ldots,x_i$ *with coefficients that are rational integers, at least one* $\psi_j^{(i)}$ *in each* $\mathfrak{m}_i$ *having a term of form* $x_i^{a_i}$ *representing the highest power of* x_i *in that* $\psi_j^{(i)}$.

We are now able to study the properties of the abstract elements of our algebra by studying the concrete residue classes of a modular system representing that algebra. This fact will often prove to be of help in the following pages.

For the necessary definitions and properties of modular systems we refer the reader to the reference K previously cited.[†] There he will find the definition of such terms as: $\mathfrak{M}$ contains the modular system $\mathfrak{M}'$, $\mathfrak{M}$ is divisible by $\mathfrak{M}'$, $\mathfrak{M}$ is an irreducible, an absolute prime, or a simple system. In Fraenkel's paper the term $\mathfrak{M}$ contains $\mathfrak{M}'$ is the same as $\mathfrak{M}$ is divisible by $\mathfrak{M}'$.

Let the elements of $\mathfrak{A}$ be represented by the residue classes of a modular system $\mathfrak{M}$ in the domain of integrity of $(1,x_1\ldots,x_n)$. If δ and ε are any two polynomials in this domain and C_δ and C_ε are the residue classes containing them, then the operations of addition, subtraction, and multiplication in $\mathfrak{A}$ are evidently defined by the relations:

$$C_\delta+C_\varepsilon=C_{\delta+\varepsilon},\qquad C_\delta-C_\varepsilon=C_{\delta-\varepsilon},\qquad C_\delta C_\varepsilon=C_{\delta\varepsilon}.$$

In general, the letters A, B, C, etc., denote elements of $\mathfrak{A}$, but units are often denoted by U. The significance of 1 and 0 has already been explained. We say that A is *divisible* by B if there exists an element C in $\mathfrak{A}$ such that $A=BC$. From the definition of a unit it follows that the product of a U into each element of $\mathfrak{A}$ gives back all elements of $\mathfrak{A}$, and therefore, by a theorem previously proved,[‡] we have the modular system $(U,\mathfrak{M})=(1)$, *i. e.*, a unit of $\mathfrak{A}$ is *relatively prime* to $\mathfrak{M}$. Two elements of $\mathfrak{A}$ that differ from another by a unit factor only are said to be *equivalent* and for purposes of factorization will be regarded as essentially the same. We write $A\equiv0$, mod $\mathfrak{M}'$, if every polynomial ξ in the residue class A of the system $\mathfrak{M}$ satisfies the relation $\xi \equiv 0$, mod $\mathfrak{M}'$. In particular, if $A \equiv 0$, mod $\mathfrak{M}$, we have $A = 0$ in $\mathfrak{A}$. This also defines the meaning of $A \equiv B$, mod $\mathfrak{M}'$. Since $(U, \mathfrak{M}) = (1)$ it follows that $\mathfrak{M}$ contains no system $\mathfrak{M}'$ such that $U \equiv 0$, mod $\mathfrak{M}'$, where U is any unit of $\mathfrak{A}$. When $\mathfrak{M}$ is an absolute prime modular system the algebra $\mathfrak{A}$ reduces to a Galois field as studied by E. H. Moore and Dickson.*

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[*] K, p. 425.

[†] K, pp. 415–418.

[‡] K, pp. 431–432.

II. *The Standard Forms of Elements in* $\mathfrak{A}$.

It has previously been proved† that we have:

THEOREM: *If* $\mathfrak{M} = \prod_{i=s}^{i=1} \mathfrak{Q}_i$, *where the* $\mathfrak{Q}_i$ *are the simple factors of* $\mathfrak{M}$, *and if* f_i, $i = 1, 2, \ldots, s$, *are any polynomials in* $x_1, x_2, \ldots, x_n$, *with rational integral coefficients, then there exist polynomials* f *such that* $f \equiv f_i$, *mod* $\mathfrak{Q}_i$, $i = 1, 2, \ldots, s$, *and these polynomials are all congruent each to each modulo* $\mathfrak{M}$.

The different polynomials f of this theorem evidently form a residue class of $\mathfrak{M}$ and represent an element A of $\mathfrak{A}$. If $\mathfrak{M} = \mathfrak{Q}_1 \mathfrak{Q}_2 \ldots \mathfrak{Q}_i \ldots \mathfrak{Q}_s$ represents $\mathfrak{M}$ factored into its simple modular factors we shall say that A_i is a *simple* element of type i in $\mathfrak{A}$ if every polynomial ξ in the residue class representing A_i, mod $\mathfrak{M}$, satisfies the relations $\xi \equiv 1$, mod $\mathfrak{Q}_j$, $j = 1, 2, \ldots, s$, $i \neq j$, while no restriction is made concerning the congruence taken modulo $\mathfrak{Q}_i$. If A is any element of $\mathfrak{A}$ it follows at once from the proof used in the third section of the reference K that there exists one and only one simple element A_i in $\mathfrak{A}$ satisfying the relations

$$A_i \equiv A, \text{ mod } \mathfrak{Q}_i, \quad A_i \equiv 1, \text{ mod } \mathfrak{Q}_j, \quad j = 1, 2, \ldots, s;\ i \neq j.$$

Hence the product $A_1 A_2 \ldots A_s$ is congruent to A modulo every simple factor of $\mathfrak{M}$ and therefore modulo $\mathfrak{M}$. Hence $A = A_1 A_2 \ldots A_s$ in $\mathfrak{A}$. From the theory of residue classes we also see that this factorization into simple elements is uniquely determined. Hence we have:

THEOREM: *Every element of a finite algebra* $\mathfrak{A}$ *can be uniquely represented as a product of simple elements.*

We shall represent the factorization of A into simple factors by the notation $A = A_1 A_2 \ldots A_s$. If A is a simple element A_i of type i we may write $1_1 1_2 \ldots 1_{i-1} A_i 1_{i+1} \ldots 1_s$, where $1_i = 1$ for all values of i. By 0_i we understand an element in $\mathfrak{A}$ that is congruent to 0, mod $\mathfrak{Q}_i$, and congruent to 1, mod $\mathfrak{Q}_j$, $j = 1, 2, \ldots, s$, $i \neq j$. Although $1_i = 1$, $0_i \neq 0$. Evidently we have

$$1 = 1_1 1_2 \ldots 1_s \text{ and } 0 = 0_1 0_2 \ldots 0_s.‡$$

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* E. H. Moore, *Bulletin New York Mathematical Society*, Vol. III (1893), p. 75; L. E. Dickson, "Linear Groups," Teubner, 1901.

† K, p. 433.

‡ The results of this section should be compared with Hensel, "Zahlentheorie," pp. 86–92.

We shall define two non-unit elements A and B as *relatively prime* in $\mathfrak{A}$ if there exist two other elements C and D such that $AC+BD=1$. Suppose $A=A_1A_2\ldots A_i\ldots A_s$ when written as a product of simple factors. Choose $C=C_1C_2\ldots C_i\ldots C_s$ so that $C_i=0_i$ whenever $(A_i,\mathfrak{O}_i)\neq(1)$, and $A_iC_i=1$ whenever $(A_i,\mathfrak{O}_i)=(1)$. It has previously been proved that in the latter case the relation $A_iC_i=1$ is always possible.[*] Evidently we have $A_i0_i=0_i$. Choose $D=D_1D_2\ldots D_i\ldots D_s$ so that $D_i=0_i$ whenever $A_iC_i=1$. This includes all cases where $(B_i,\mathfrak{O}_i)\neq(1)$, for otherwise the residue classes corresponding to both A and B are congruent to 0 modulo the absolute prime modular system contained in $\mathfrak{O}_i$, and this would preclude any relation of the form $AC+BD=1$ in $\mathfrak{A}$. For all other values of i let $B_iD_i=1$. We see at once that when C and D are so chosen we have $AC+BD\equiv1$, mod $\mathfrak{O}_i$, $i=1, 2, \ldots, s$, one term always being congruent to 1, the other to 0. Hence, we see from the first theorem of this section that $AC+BD=1$. Hence:

THEOREM: *When two elements A and B of $\mathfrak{A}$ are relatively prime there exist two elements C and D such that $AC+BD=1$, where C always satisfies one of the relations $(C,\mathfrak{O}_i)=(1)$ or $(C,\mathfrak{O}_i)=\mathfrak{O}_i$, $i=1, 2, \ldots, s$, and the same is true of D.*

The expression $A_1\ldots A_i\ldots A_s$ is the standard multiplicative form of A. Fraenkel obtained an analogous result but his proof is no longer valid for our more general case.[†] We shall now proceed to obtain a standard additive form. By a *component* element of type i, written A'_i, we understand an element of $\mathfrak{A}$ such that $A'_i\equiv0$, mod $\mathfrak{O}_j$, $j=1, 2, \ldots, s$, $i\neq j$, with no restriction upon the case of $\mathfrak{O}_i$. The standard additive form of an element A in $\mathfrak{A}$ is given by the sum of its components, *i. e.*, $A=A'_1+A'_2+\ldots+A'_i+\ldots+A'_s$. A component of type i written in factor form gives the expression

$$A'_i=0_10_2\ldots0_{i-1}A_i0_{i+1}\ldots0_s.$$

Hence $A\equiv A'_i$, mod $\mathfrak{O}_i$, $i=1, 2, \ldots, s$, and our standard additive form is seen to be unique. Hence:

THEOREM: *Every element in $\mathfrak{A}$ is uniquely determined by the sum of its components.*

We see that $A_i\equiv A'_i$, mod $\mathfrak{O}_i$, but $A_i\equiv1$, and $A'_i\equiv0$, mod $\mathfrak{O}_j$, $j=1, 2, \ldots, s$, $i\neq j$. Again it is evident that the *i-th* component of $A+B$ is $A'_i+B'_i$. In the product AB we have $A'_iB'_j=0$, $i\neq j$, for this product is congruent to zero modulo every $\mathfrak{O}_i$. Therefore we have:

THEOREM: *The i-th component of the sum or product of two elements of $\mathfrak{A}$ is equal to the sum or product of the i-th components of the two elements.*

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[*] K, pp. 423–424.

[†] Fraenkel, *loc. cit.*, p. 166. See also Hensel, "Zahlentheorie," p. 89.

From the preceding theorems we see that if

$$A=A_1A_2\ldots .A_i\ldots .A_s=A'_1+\ldots .+A'_i+\ldots .+A'_s,$$

and

$$B=B_1B_2\ldots .B_i\ldots .B_s=B'_1+\ldots .+B'_i+\ldots .+B'_s,$$

then

$$AB=A_1B_1\ldots .A_iB_i\ldots .A_sB_s=A'_1B'_1+\ldots .+A'_iB'_i+\ldots .+A'_sB'_s.$$

From these considerations it follows that if we write the elements of $\mathfrak{A}$ as sums of their components we have a finite linear algebra with s units E_i, where $E_i=0_1\ldots .0_{i-1}1_i0_{i+1}\ldots .0_s$, such that every element of $\mathfrak{A}$ is of the form $K_1E_1+\ldots .+K_iE_i+\ldots .+K_sE_s$. Here K_i ranges over a set of values simply isomorphic with the set of elements of the algebra $\mathfrak{A}_i$ represented by the residue classes of the modular system $\mathfrak{Q}_i$.* In performing the operations of addition and multiplication we can use the components of type i in place of the elements K_i for all values of i, for these two sets of elements are simply isomorphic. Enough has been said to show that $\mathfrak{A}$ is of the type studied by Dickson,† being equal to the sum of s simple algebras having one unit each, the coefficients of these algebras running over different ranges.

III. *Other Properties of* $\mathfrak{A}$.

We shall not take up the study of the multiplicative groups formed by the elements of $\mathfrak{A}$ since these were treated in the reference cited to which the reader is referred. Turning to the additive groups we see that in many respects they correspond to the modules in the Dedekind theory. In order to obtain a sufficient condition that a set of elements in $\mathfrak{A}$ form an additive group it is merely necessary to take all elements A of $\mathfrak{A}$ that are congruent to zero modulo any system $\mathfrak{M}'$ contained in $\mathfrak{M}$. That this condition is not necessary we see from the group formed by 0 and 1 modulo $(x^2, 2)$. We have, however, the following

THEOREM: *A necessary and sufficient condition that a set of elements in* $\mathfrak{A}$ *form an additive group* G *is that all components of type* i, $i=1, 2, \ldots ., s$, *found among the elements of the set form a group* G_i *and that there exists no element of* $\mathfrak{A}$ *not in the set whose* i*-th component is in* G_i *for all values of* i. *The group* G *is equal to the direct product of the groups* $G_1, G_2, \ldots ., G_s$.

To prove that the condition is necessary take the group G' formed by all elements of $\mathfrak{A}$. We know all elements of $\mathfrak{A}$ are obtained if in the expression

$$A=A'_1+A'_2+\ldots .+A'_i+\ldots .+A'_s$$

each of the components A'_i, $i=1, 2, \ldots ., s$, ranges unrestrictedly over all values of its set, independent of the others. From this it necessarily follows that G' is the direct product of the s additive groups H_i each of which is composed of all the components of the type designated by the subscript. Since any additive group G of $\mathfrak{A}$ is a subgroup of G' it follows from the theory of groups that G is the direct product of s groups G_i, each G_i being that subgroup of its corresponding H_i that is formed by all components of the type indicated by the subscripts that are found in the elements of G. It is also clear that there exists no element in $\mathfrak{A}$ but not in G all of whose components are in the various G_i, for in forming G as the direct product of the G_i we evidently get an element in G that has exactly this same set of components. This establishes the necessary condition.

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* Fraenkel, *loc. cit.*, p. 175.

† L. E. Dickson, *Transactions of the American Mathematical Society*, Vol. VI (1905), pp. 344 ff.

To obtain the sufficient condition let us recall that if A and B are two elements of $\mathfrak{A}$ in the set under consideration, and if A'_i and B'_i are their components of type i, then $A+B$ has $A'_i+B'_i$ as its component of type i. Since A'_i and B'_i are in G_i, $i=1, 2, \ldots, s$, the same holds for $A'_i+B'_i$. Therefore it follows from the conditions assumed that $A+B$ must belong to the set. That this set forms a group follows at once for when we add one member of the set to all elements of the set we get back all elements. This proves the theorem.

The multiplication group G formed by the units of $\mathfrak{A}$* furnishes us with an analogue to Dirichlet's general theorem on the units of an algebraic realm, a set of independent generators of the abelian group G corresponding to the linearly independent units of Dirichlet's theorem. To each U corresponds its inverse (or reciprocal) U^{-1} such that $UU^{-1}=1$. Hence the reciprocal of a unit is also a unit. Non-units do not have reciprocals for they are divisors of zero.

We now proceed to define a *prime* element in $\mathfrak{A}$. We say that A is prime in $\mathfrak{A}$ if it cannot be factored into the product of two non-units neither of which is equivalent to A. This is the definition found in the theory of algebraic numbers with an added condition. A non-unit that is not a prime can always be factored into two elements neither of which is a unit. Unless both of these factors are primes this process can be repeated. We shall now show that after a finite number of such steps we can always represent such an element of $\mathfrak{A}$ as a product of primes or powers of primes. Suppose the element has been factored into its simple factors which we now proceed to consider separately. It should be remembered that in the following equivalent elements are regarded as essentially the same. Suppose we proceed to factor the simple element A_i into primes, where A_i naturally is a non-unit, for units are not factored. To do this we must first determine the effect of the above-mentioned additional condition given in the definition of a prime. We see that an element X_i in $\mathfrak{A}$ is defined as a prime even when it factors into two non-units, provided at least one factor is equivalent to the element. If U_iX_i is the equivalent factor and $X_i=D_iU_iX_i$, write $D_iU_i=B_i$ and proceed to study the equation $X_i=B_iX_i$, where B_i is a non-unit. Throughout we restrict ourselves to simple elements of type *i*. If we take the simple modular factor $\mathfrak{O}_i$ of $\mathfrak{M}$ associated with our simple elements of type *i* and write down a descending sequence of modular systems as explained on pages 422 and 423 of the reference *K* previously cited, we see that the only possible solution for X_i is 0_i. For, suppose that $X_i\equiv0$, mod $\mathfrak{O}_{i,\,\tau}$, but not modulo $\mathfrak{O}_{i,\,\tau-1}$. Since B_i is a non-unit it therefore follows that $X_iB_i=X_i$ is congruent to zero modulo $\mathfrak{O}_{i,\,\tau-1}$, which leads to a contradiction except when $X_i=0_i$, where we may take $\mathfrak{O}_{i,\,0}=\mathfrak{O}_{i,\,1}=\mathfrak{O}_i$. If $B_i\neq 0_i$ it follows from the reference just cited that there exists a power of B_i, $B_i^\epsilon\neq0_i$, such that $B_iB_i^\epsilon=0_i$. Hence in this case 0_i cannot be a prime by definition. This case always occurs except when $\mathfrak{O}_i$ is an absolute prime modular system, the only case in which there exists no non-unit element of type *i* outside of 0_i. Hence in this case we have $0_i=0_i^\alpha$, $\alpha=1, 2, \ldots,$ etc., but no other factorization. Therefore our definition of a prime includes

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* K, pp. 423–424. See also Vandiver, *loc. cit.*, p. 294.

(a) all elements that cannot be factored into two non-unit elements,

(b) the element 0_i whenever the corresponding modular system $\mathfrak{O}_i$ is an absolute prime system.

Let us now proceed to factor A_i and suppose for the present that $A_i\neq0_i$. If a finite number of steps is not sufficient to factor this element into a product of primes and their powers it follows that since $\mathfrak{A}$ contains but a finite number of simple elements of type *i*, that at least one factor must occur to an arbitrarily high power. Denote this factor by C_i. Since $(C_i, \mathfrak{O}_i)\neq(1)$, there exists by the reference of the preceding paragraph a definite power α of C_i such that $(C_i^\alpha, \mathfrak{O}_i)=\mathfrak{O}_i$, which contradicts the fact that C_i^α is a factor of A_i while $A_i\neq0_i$. Hence a finite number of factorizations is sufficient for this case. If $C_i=0_i$ it follows from the preceding paragraph that we can either write 0_i as the product of two non-units neither of them equal to 0_i, and then proceed as before, or we have 0_i equal to a prime by definition. Hence every element of $\mathfrak{A}$ can be written as the product of prime factors. But this factorization is not unique as we see from the example

$$x_2^2\equiv x_2x_2\equiv(x_1+x_2)^2, \quad \text{mod } (x_1^3, x_2^2, 2),$$

where we can easily verify that both x_2 and x_1+x_2 are primes in the algebra defined by $(x_1^3, x_2^2, 2)$. In this respect our algebra differs essentially from the type studied by Fraenkel. Two primes of $\mathfrak{A}$ need not necessarily be relatively prime, a fact that might be expected from the failure of the unique factorization law. In fact this presupposes the condition $AC+BD=1$ which cannot hold in the above example since $A=x_2$ and $B=x_1+x_2$ are both congruent to 0, mod $(x_2, x_1, 2)$, so that the left-hand side of $AC+BD=1$ is congruent to 0, mod $(x_2, x_1, 2)$, while this cannot be true of the right-hand side. It furthermore follows that no two primes of an algebra defined by a simple modular system can be relatively prime. Hence, we have the following

THEOREM: *In the finite algebra corresponding to the modular system* $\mathfrak{M} = \mathfrak{Q}_1\mathfrak{Q}_2 \ldots . \mathfrak{Q}_s$ *there exists s and no more than s primes that are relatively prime each to each.*

Any prime in $\mathfrak{A}$ is equivalent to a prime that is also simple. The first α powers of this latter element are distinct elements of $\mathfrak{A}$, but all later ones are equal to the α-th power which is of the form $1_1 \ldots . 1_{i-1} 0_i 1_{i+1} \ldots . 1_s$. The first prime when raised to powers has its first α powers essentially different, but all following ones are of the form $U_1 \ldots . U_{i-1} 0_i U_{i+1} \ldots . U_s$, where the various U_j run over subgroups of the group of units. Hence, we get:

THEOREM: *The first α powers of a simple prime in $\mathfrak{A}$ are essentially different, those following are identical with the α-th power which equals 0_i. If a prime is not simple its first α powers are essentially different, while all following powers form a repeating cycle of equivalent elements.*

The algebra $\mathfrak{A}$ also contains sub-algebras. Such can be obtained by taking all elements of $\mathfrak{A}$ that have certain components, say the last $s-t$, equal to 0, *i. e.*, in product form these elements can be written $A_1 \ldots . A_t 0_{t+1} 0_{t+2} \ldots . 0_s$. The 0 of $\mathfrak{A}$ is always in such a sub-algebra, but this is never true of any unit of $\mathfrak{A}$. We may, however, let $1_1 \ldots . 1_t 0_{t+1} \ldots . 0_s$ take the place of 1, and let the units of the sub-algebra be those elements whose first t simple factors are units of $\mathfrak{A}$. This sub-algebra is simply isomorphic to the algebra $\mathfrak{A}'$ corresponding to $\mathfrak{M}' = \mathfrak{Q}_1\mathfrak{Q}_2 \ldots . \mathfrak{Q}_t$.* Comparing this with a necessary and sufficient condition that a set of elements in $\mathfrak{A}$ form a multiplicative group † we have:

THEOREM: *There exists a (1, 1) correspondence between 2^λ sub-algebras of $\mathfrak{A}$ and the 2^λ multiplicative groups in $\mathfrak{A}$ such that the group corresponding to a given sub-algebra is composed of its units.*

In closing we may ask if it is possible to restore the unique factorization law by means of ideals as in the Dedekind theory. This is not the case. Closer study of the subject shows that in such a theory the analogue of an ideal would be the set of all elements of $\mathfrak{A}$ that are congruent to 0 modulo one of the s absolute prime modular systems that $\mathfrak{M}$ contains. In the example already given, namely $\mathfrak{M} = (x_2^3, x_1^2, 2)$, this would give but one (so-called) ideal, and this would be of no avail in solving the factorization of $x_2^2 \equiv x_2 x_2 \equiv (x_1 + x_2)^2$, mod $\mathfrak{M}$, uniquely. We can, however, obtain in this manner a number of results analogous to theorems in the theory of algebraic numbers.

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* See Hensel, "Zahlentheorie," pp. 78 ff.

† See K, p. 428.

The Primitive Groups of Class 15.

BY W. A. MANNING.

There are but three primitive groups of class $3p$, p an odd prime, in which occur substitutions of order p and degree $3p$.* All are of class 15. One is of degree 16 and order 80. The other two are the triply transitive group of degree 17 and order 4080, and its subgroup of order 240. Corresponding to every whole number k greater than 3 there exist four simply transitive primitive groups of class $3k$. The alternating group of degree $k+2$ permutes the $\frac{1}{2}(k+1)(k+2)$ binary products ab, ac, ..., according to a simply transitive primitive group of class $3k$ $(k>3)$.† The three groups ‡

$$(aa' \cdot bb' \cdot \ldots\ldots \cdot kk')(ab \ldots\ldots j)\ \text{pos}\ (a'b' \ldots\ldots j')\ \text{pos},$$
$$(aa' \cdot bb' \cdot \ldots\ldots \cdot kk')\{(ab \ldots\ldots j)\ \text{all}\ (a'b' \ldots\ldots j')\ \text{all}\}\text{pos},$$
$$(ab'ba' \cdot cc' \cdot dd' \cdot \ldots\ldots \cdot kk')\{(ab \ldots\ldots j)\ \text{all}\ (a'b' \ldots\ldots j')\ \text{all}\}\text{pos},$$

are maximal subgroups of

$$(aa' \cdot bb' \cdot \ldots\ldots \cdot kk')(ab \ldots\ldots k)\ \text{pos}\ (a'b' \ldots\ldots k')\ \text{pos},$$
$$(aa' \cdot bb' \cdot \ldots\ldots \cdot kk')\{(ab \ldots\ldots k)\ \text{all}\ (a'b' \ldots\ldots k')\ \text{all}\}\text{pos},$$
$$(ab'ba' \cdot cc' \cdot dd' \cdot \ldots\ldots \cdot kk')\{(ab \ldots\ldots k)\ \text{all}\ (a'b' \ldots\ldots k')\ \text{all}\}\text{pos},$$

respectively. The latter are therefore simply isomorphic to primitive groups of degree k^2, and of order $\frac{1}{2}(k!)^2$, $(k!)^2$, $(k!)^2$, respectively. They are simply transitive and of class $3k$. *When k is* 5, *these three and the alternating group of degree* 7 *written on twenty-one letters are the only primitive groups of class* 15 *that contain no substitution of order* 5 *and degree* 15.

Those primitive groups of class 15 which contain a substitution of order 5 and degree 15 are known. There remain to be determined those in which the substitutions of degree 15 are all of the type

$$s_1 = a_1a_2a_3 \,.\, b_1b_2b_3 \,.\, c_1c_2c_3 \,.\, d_1d_2d_3 \,.\, e_1e_2e_3 \,.$$

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* *Transactions of the American Mathematical Society*, Vol. VI (1905), p. 42.

† C. Jordan, *Comptes Rendus*, Vol. LXXV (1872), p. 1754; AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII (1910), p. 256, where the error in the formula for the class is corrected.

‡ Cayley, *Quarterly Journal of Mathematics*, Vol. XXV (1891), p. 71, where this notation is explained. For the three groups, see Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXI (1899), p. 299, Theorems III and IV.

Since there exists but one group of order 15, the cyclic group, the only other substitution on the same fifteen letters as s_1 is its square. Among all the substitutions in one of our groups G that are similar to s_1, let those substitutions be selected which displace a minimum number of letters new to s_1, s_1^2 excluded. First, it is easy to see that this minimum can not exceed five, the number of cycles in s_1. For if such a substitution s_2 had more than five letters new to s_1, two of them would be found adjacent in one of its cycles. The transform $s_2^2s_1s_2$ would then displace a smaller number of letters new to s_1 than the assigned minimum, and this is possible only if $s_2^2s_1s_2$ displaces no letters new to s_1, that is, is s_1 or s_1^2. But s_2, a substitution of odd order, can not transform s_1 into its inverse. Then s_1 is invariant in the subgroup H generated by the complete set of conjugates under G to which it belongs, and since one of these conjugates is invariant in H, all are invariant in H and H is Abelian. But H is not a regular group and hence, being Abelian, is not transitive, and can not be invariant in the primitive group G. *Then a substitution* s_2 *(not* s_1^2*) that displaces in its cycles the least possible number of letters new to* s_1, *and is similar to* s_1, *displaces at most five such new letters.*

If s_2 has two of these new letters in one cycle, $s_2^2s_1s_2$ leaves one of them fixed, and hence is s_1. Now since s_2 is commutative with s_1, it displaces exactly three new letters, and they form one of the five cycles of s_2. If s_2 does not permute cycles of s_1, either $s_1s_2^2$ or s_1s_2 is of degree less than 15, and obviously not identity. Then s_2 permutes cyclically three cycles of s_1. If s_2 has not this form, it has at most one new letter to a cycle. Let the group $\{s_1, s_2\}$ when not Abelian be called D. Of every substitution of D it may be remarked that it either permutes the letters new to s_1 only among themselves, or else replaces all of them by letters of s_1. Were this not true a transform of s_1 could at once be found that would displace at least one new letter, and at the same time fewer new letters than s_2. Any substitution of D that replaces one new letter by another new letter transforms s_1 into s_1 or s_1^2. A transitive constituent of D that involves more than one of the letters new to s_1 is imprimitive, and one of its systems of imprimitivity is composed of its new letters. If s_2 is not similar to s_1 in the letters of each of its transitive constituents, $t_1=s_2^2s_1s_2$ is similar to s_1 in those letters and may be used for s_2 in case $\{s_1, t_1\}$ has the same transitive constituents as $\{s_1, s_2\}$. But if t_1 is not similar to s_1 in the new (and smaller) transitive sets we may use $t_2=t_1^2s_1t_1$, $t_3=t_2^2s_1t_2$, or a final transform which is similar to s_1 in the various constituents of the group generated by it and s_1. This apparently fails when t_1, t_2, or one of these transforms is commutative with s_1, but then we may use this Abelian group of degree 18. *Hence G contains a non-Abelian group D generated by two similar substitutions s_1 and s_2 of degree* 15, *which are similar in the letters of each constituent, or an Abelian group* $\{s_1, s_2\}$ *of order* 9 *and degree* 18. In no case do s_1 and s_2 have more than one common cycle, for as many as two common cycles cause the product $s_1 s_2^2$ to displace fewer than fifteen letters. It is evident that the number of transitive constituents in D does not exceed 5. If there are exactly five constituents, D is either a simple isomorphism between five tetraedral groups or is a (1, 4) isomorphism between a cyclic group of degree and order 3, and an intransitive tetraedral group of degree 16, which is itself a simple isomorphism between four alternating groups of degree 4. Hence, if s_2 displaces less than four letters new to s_1, it connects cycles of s_1.

If D is of degree 16 or 17 it is transitive or is a simple isomorphism between transitive constituents. If D is of degree 17, 18, or 19, it is intransitive. For if D is transitive, the new letters in s_2 form a system of imprimitivity of D, which is possible only when there are three of them permuted according to a transitive group of degree 6. Since the two generators s_1 and s_2 are of degree 15, neither can do more than permute cyclically three of the six systems. Then they permute different systems and the group in the systems is intransitive, and D is intransitive.

If D is a transitive group of degree 16, it is of order 48 and has an invariant subgroup of degree and order 16. This group can not be contained in a larger group of degree 16, with the present limitations as to the class of G, nor is it contained in any primitive group of degree 17, 18, 19, or 20, if we may rely upon the accuracy of the lists of the primitive groups of these degrees.* If D is a subgroup of a doubly transitive group of degree 21, it must have five systems of imprimitivity of four letters each with a given letter, the new letter x, say, in common.† Now s_1 permutes among themselves the letters of the systems of imprimitivity to which x belongs, and so far as s_1 is concerned x may be a part of as many systems of four letters each as s_1 has cycles, but since s_2 must replace every system involving x by another system, the two letters of s_1 in a cycle of s_2 with x can neither of them belong to any system of which x is a part. Hence D can not be five-fold imprimitive. It can not have systems of imprimitivity of two or eight letters because of its two generators. Then the primitive groups under consideration do not contain a transitive subgroup of degree 16 and order 48. If D is intransitive the constituent involving the new letter x is of degree $3k+1$ and order $3(3k+1)$,

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* See AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXV (1913), p. 229, for references.

† *Transactions of the American Mathematical Society*, Vol. VII (1906), p. 500.

$k=1, 2, 3, 4$, but a non-regular constituent of degree 7, 10, or 13 is clearly out of the question.[*] A tetraedral group may be written as a transitive group on twelve and on six letters. But if D is set up as a simple isomorphism between three tetraedral groups of the degrees 4, 6 and 6, respectively, its class is 12.

There remains the simple isomorphism between two tetraedral groups of the degrees 4 and 12. This is not a subgroup of a primitive group of degree less than 21, of the type we are discussing. There exists in G a substitution s_8 similar to s_1 which directly unites the two transitive constituents of D, and has, if it is assumed that no substitution similar to s_1 having this property displaces fewer letters new to D, in no cycle more than one letter new to D.[†] Then the group $H=\{D, s_8\}$ is transitive. Its degree does not exceed 21. Every substitution in it replaces a letter of the transitive constituent of degree 12 in D by the same or another letter of that constituent, so that any substitution of H that replaces a letter $y_1, \ldots.$, new to D, by a letter y, certainly fails to juxtapose letters of the two constituents of D. In particular, the subgroup of H that leaves one letter fixed is intransitive. Within this subgroup, which we may call H_1, the subgroup (F) generated by all the substitutions of order 3 and degree 15 is invariant. F is positive and all its constituents are positive. It is known that H is not of degree 17 or 19, nor is H an imprimitive group of degree 16, for then its order is of necessity 48, and this case has been examined. If H is of degree 18, and if the two letters y_1, y_2 do not constitute a system of imprimitivity, the subgroup F is of degree 17, and has two transitive constituents, which may be of the degrees indicated by the partitions 13, 4 and 12, 5. The presence in G of a substitution of order 13 is impossible. Since by hypothesis G contains no substitution of order 5 and degree 15, the partition 12, 5 is not admissible. But H can not have systems of imprimitivity of two letters each because it is generated by substitutions of degree 15 and order 3. If H is of degree 20, systems of imprimitivity of two or of ten letters each are impossible. The five letters left fixed by s_1 can not form a system because s_2 displaces one of them. Hence s_1 does not permute systems of five letters, and it is clear also that s_1 can not permute the letters of each such system among themselves. Hence H is an imprimitive group with five systems of four letters each, which are permuted according to the alternating group of degree 5. Now D, because of its constituent of degree 12, certainly permutes systems, in fact can not fix a system to which a letter y belongs unless the four letters y_1, y_2, y_3, y_4 are one of the systems of imprimitivity of H. That is, there is only one system of imprimitivity in H involving a given letter y_1, so that H, if it is a subgroup of one of our primitive groups, is to be found in a doubly transitive group of degree 21. Now in H_1, the subgroup of H that leaves one letter (y_1) fixed, the letters y_2, y_3, y_4 can be permuted only among themselves, so that any subgroup of H_1 that is similar to D displaces the same letters as D and has the same two sets of letters in its two constituents. If it is assumed distinct from D, the group it and D together generate must coincide with D since one constituent of D is already alternating. Then D is unique in H_1, and the largest subgroup of H in which D is invariant has a transitive constituent of degree 4.[*] In the doubly transitive group of degree 21, since D is one of only twenty-one conjugates, the largest subgroup in which D is invariant has a doubly transitive constituent of degree 5. But the group of isomorphisms of the tetraedral group is the symmetric-4 group. Then the required doubly transitive group of degree 21 does not exist. Let H be of degree 21. It can not have systems of imprimitivity of seven letters each since no one of its three generators s_1, s_2, s_3 can permute such systems. No letter of D can belong to the same system of three letters with letters y. Hence H is primitive. The intransitive subgroup H_1, fixing one of the five letters y, has no constituent of degree 2,[†] nor has it a constituent of degree 3.[‡] In H_1 there is not a constituent of degree 4 composed of letters y, for then D would be invariant in H_1 and H_1 would have a transitive constituent of degree 12. It is obvious that *in any simply transitive primitive group, the degree of one transitive constituent of a subgroup leaving one letter fixed must exceed that of any transitive constituent of a subgroup leaving two or more letters fixed.* It is also true that *the degree of the subgroup that leaves one letter of a simply transitive primitive group fixed is equal to the degree of any one of its subgroups that is generated by a complete set of its similar substitutions.* F, for example, is of degree 20. Then the possible degrees of the constituents of F are given by the partitions 15, 5 and 16, 4; 13, 7 and 14, 6 being rejected immediately. If the partition is 15, 5, the second constituent is icosaedral, and F is a $(3, 1)$ or a simple isomorphism between its constituents. If F has an invariant head of order 3, the latter cannot be transformed by substitutions of H into any other subgroup of F because every other subgroup of order 3 in F belongs to a set of at least ten subgroups conjugate under the substitutions of H_1. This invariant head of order 3 can not be invariant in another subgroup leaving one letter fixed, and therefore is conjugate under substitutions of H to another subgroup of order 3 in H_1. But a given subgroup of degree 15 and order 3 is invariant in a subgroup of H that has, in the six letters left fixed by that subgroup, transitive constituents of degree proportional to the number of subgroups in the different sets of conjugates of H_1 which in H are united as part of a single set.* If the isomorphism is simple, D is one of five conjugates and by the same theorem should admit an isomorphism of order 5. Then there remains only the partition 16, 4. Obviously F is a (16, 1) isomorphism between its two constituents, the second of which is alternating. D is a subgroup leaving fixed one of the letters of the larger constituent, so that there are four such subgroups, all conjugate in H_1. Then there is no primitive group of degree 21 as required.

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[*] *Transactions of the American Mathematical Society*, Vol. XVI (1915), p. 139.

[†] *Transactions of the American Mathematical Society*, Vol. XII (1911), p. 375.

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[*] *Bulletin of the American Mathematical Society*, 2d series, Vol. XIII (1906), p. 20, Theorem I.

[†] Miller, *Proceedings of the London Mathematical Society*, Vol. XXVIII (1897), p. 536.

[‡] Bennett, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV (1912), p. 7.

Let the group D be of degree 17. It is not transitive, nor has it five transitive constituents. Because its degree exceeds its class by only two units, D is a simple isomorphism between its constituents. There is no constituent of degree 3. If there is one constituent of degree 4, D is tetraedral, and there must be one, and hence two, constituents of degree 6, which is absurd when D is of degree 17. There can not be a constituent of degree 7. A constituent of degree 8 must be of such a nature that it can be represented as a transitive group on nine letters. This constituent of degree 9 can not be regular, it can not be of class 7, nor can it be of class 8, for then the two generators of order 3 and degree 9 would be found in a regular self-conjugate subgroup of order 9. But its class certainly exceeds 6. The constituent of lowest degree in D can not be of degree greater than 8, so that finally D is not of degree 17.

If D is assumed to be a transitive group of degree 18 it has six systems of imprimitivity of three letters each. Since s_1 (or s_2) can not permute more than three of these systems, D is certainly not transitive. It has been shown above that if s_2 has two letters new to s_1 adjacent in one of its cycles, D is of order 9 with one constituent a transitive group of order 9. It remains to be seen what groups D, if any, are present in G in which the three new letters x occur in three different cycles of s_2. If D has one constituent of degree 3 (a cycle common to s_1 and s_2), there is no other constituent of degree 3, and there corresponds to identity of this constituent a subgroup of order 3 and degree 15 in the other letters, so that D is Abelian of order 9, contrary to the assumption that s_2 distributes its new letters. Then D is a simple isomorphism between its various transitive constituents. If D has one constituent of degree 4, that is, the alternating group of degree 4, there must be a constituent of higher degree, which can only be of degree 6, and then two more constituents of degree 4 each. As a group D, this is entirely possible. If it be assumed that the lowest degree of any constituent is 6, there is just one other constituent, of degree 12 and generated by two substitutions of order 3 and degree 9, and whose order is in consequence 36 or a multiple of 36, and not divisible by 5 because the twelve letters fall into four systems of imprimitivity of three letters each. It follows that the constituent of degree 6 is imprimitive. Since it is generated by two substitutions of order 3 and degree 6 it can not have systems of imprimitivity of three letters each. But the imprimitive groups of degree 6 and order 36 or 72 have systems of three letters each.* No constituent can be of degree 7 (10), when D is of degree less than 21 (20). A constituent of degree 8 requires the presence in D of a constituent of degree 10. This closes the search for possible groups D of degree 18, and we now take up the detailed study of the two groups of the orders 9 and 12.

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* *Bulletin of the American Mathematical Society, loc. cit.*

The next step forward will consist in showing that no one of our groups G contains the Abelian group of order 9 and degree 18 as a subgroup. No confusion need arise if we agree to call this group "D." There are then in D three transitive constituents of degree 3 and one of degree 9. There exists in G, among the substitutions similar to s_1, a substitution s_3 which connects the largest set of D with some other of the sets of D, and which has at most one new letter y in any cycle. Suppose that $H = \{D, s_3\}$ is transitive. If H is of degree 18, it is imprimitive, and has no systems of two, six, or nine letters because of the nature of its generators s_1, s_2, s_3. If H has systems of three letters, the letters x_1, x_2, x_3 of s_2, not in s_1, form a system, and any other system involving x_1 includes also x_2 and x_3. But H can not be a subgroup of a doubly transitive group of degree 19. Nor is H of degree 19. Let H be of degree 20. It is imprimitive. Since the two new letters y_1, y_2 do not by themselves constitute a system of imprimitivity, we can take from H a substitution s that fixes one of the letters y_1, y_2 and displaces the other. The group $\{D, s\}$ separates the letters of the constituents of degree 3 in D from those of the transitive constituent of degree 9. Its order is not divisible by either 5 or 7. Its degree may admit the two partitions, 9, 4, 3, 3 and 9, 4, 6. A constituent of degree 6 is simply isomorphic to the constituent of degree 13. Then there are not two transitive constituents in $\{D, s\}$ of degree 3 each. When the partition is 9, 4, 6, the constituent of degree 4 is alternating, and in (1, 3) isomorphism to the constituent of degree 15. The transitive constituents of degrees 9 and 6 are then each of order 36 and are in simple isomorphism. Now in any group of degree 6 and order 36, the subgroup of order 9 is invariant, so that the tetraedral quotient group is absurd. Let H be of degree 21. If H has seven systems of imprimitivity of three letters each, the group in the systems is alternating and involves a substitution of order 5, which in the letters of H is of degree 15. Then H is primitive. The following partitions of the degree of F seem possible: 9, 4, 4, 3, 9, 8, 3, 10, 10. When H_1 has a constituent of degree 9, a transform of F, because it includes D, has a transitive constituent in the same letters. If F has two transitive constituents of degree 10, the first is multiply but not quadruply transitive, and since it is positive, it is one of the groups of class 8, of order 360 or 720. The second constituent, also positive, is not primitive because its class is not greater than 6. But the second constituent can not have systems of two or of five letters. Let H be of degree 22. It is primitive. The possible partitions of the degree of F are, after obvious exclusions, 9, 3, 3, 3, 3, 9, 4, 4, 4, 9, 6, 3, 3, 9, 8, 4, 9, 6, 6, 9, 9, 3, 9, 12, 10, 6, 5, 12, 3, 3, 3, 12, 6, 3, 12, 9. In the first five cases H_1 has a transitive constituent of degree 9, and can not be maximal. In the next case H_1 has a transitive constituent of degree 3.* Now consider F when its transitive constituents are of the degrees 10, 6, 5. The constituents of degree 10 and of degree 6 are multiply transitive. The constituent of degree 5 is alternating, and the constituent of degree 6, being a simple group, is in simple isomorphism to it. Then F is a simple isomorphism between three icosaedral groups. But all the substitutions of order 3 in such an intransitive group are of degree 18. For it has been noted before in this paper that the icosaedral group when written as a transitive group on ten letters is of class 8, and when on six letters is of class 4. It is now possible to state that 5 does not divide the order of F. Any transitive constituent of degree 12 is imprimitive. In the last three partitions, the three letters y must form a system of imprimitivity, for otherwise a certain subgroup $\{D, s\}$ of F would have a transitive constituent of degree 10 or 11. The partition 12, 3, 3, 3 is impossible. Likewise 12, 6, 3 is impossible because the constituents of degree 12 and 6 must be simply isomorphic, while 108 does not divide the order of any group of degree 6. If the constituent of degree 9 (when the partition is 12, 9) is primitive its class is 6, and in consequence its order is 216, a number not divisible by 108. Then the second constituent has systems of imprimitivity of three letters each. These systems are permuted by neither s_1 nor s_2, nor by any of their conjugates in F. Now F has at least one substitution similar to s_1 that adds the new system y_1, y_2, y_3 to the first constituent, using three or more cycles for that purpose, and which consequently does not unite the other sets of D. Such a group was shown to be impossible in the discussion of the partitions 12, 3, 3, 3 and 12, 6, 3. Then the first constituent of F is of degree 9, and the second of degree 12. Now the subgroup F' of H_1' that fixes a second letter y, and which also includes D, has a transitive constituent of degree 9. Since the partition 12, 9 has been shown to be impossible, this constituent of degree 9 in F' is on the letters of the largest transitive constituent of D. Then H_1 is not maximal. Let H be of degree 23.* In this case H_1 is transitive and F has two transitive constituents of degree 11. The first being at least triply transitive is of order 7920 and class 8. But the class of the second constituent does not exceed 6.

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*Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXI (1899), p. 287.

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* Bennett, *loc. cit.*

The group $H'=\{D, s_3\}$ is intransitive. It will be convenient to have before us the actual substitutions of the group D. They are:

$$1,$$
$$s_1s_2=a_1b_2c_3 \,.\, a_2b_3c_1 \,.\, a_3b_1c_2 \,.\, d_1d_3d_2 \,.\, e_1e_2e_3 \,.\, x_1x_2x_3,$$
$$s_1^2s_2^2=a_1c_3b_2 \,.\, a_2c_1b_3 \,.\, a_3c_2b_1 \,.\, d_1d_2d_3 \,.\, e_1e_3e_2 \,.\, x_1x_3x_2,$$
$$s_1=a_1a_2a_3 \,.\, b_1b_2b_3 \,.\, c_1c_2c_3 \,.\, d_1d_2d_3 \,.\, e_1e_2e_3,$$
$$s_1^2s_2=a_1b_3c_2 \,.\, a_2b_1c_3 \,.\, a_3b_2c_1 \,.\, e_1e_3e_2 \,.\, x_1x_2x_3,$$
$$s_2^2=a_1c_1b_1 \,.\, a_2c_2b_2 \,.\, a_3c_3b_3 \,.\, d_1d_3d_2 \,.\, x_1x_3x_2,$$
$$s_1^2=a_1a_3a_2 \,.\, b_1b_3b_2 \,.\, c_1c_3c_2 \,.\, d_1d_3d_2 \,.\, e_1e_3e_2,$$
$$s_2=a_1b_1c_1 \,.\, a_2b_2c_2 \,.\, a_3b_3c_3 \,.\, d_1d_2d_3 \,.\, x_1x_2x_3,$$
$$s_1s_2^2=a_1c_2b_3 \,.\, a_2c_3b_1 \,.\, a_3c_1b_2 \,.\, e_1e_2e_3 \,.\, x_1x_3x_2.$$

This may be transformed into $\{s_1, a_1b_1c_1 . a_2b_2c_2 . a_3b_3c_3 . d_1d_3d_2 . x_1x_2x_3\}$ by means of $b_1c_1 . b_2c_2 . b_3c_3 . x_2x_3$. The group D is also invariant under all the substitutions of the group

$$\{d_1e_1 . d_2e_2 . d_3e_3 . b_1c_2 . b_2c_3 . b_3c_1 . x_2x_3,\quad d_1x_1 . d_2x_2 . d_3x_3 . a_2b_3 . b_2c_3 . c_2a_3 . e_2e_3,\quad d_1d_2d_3\}.$$

Hence we are at liberty to write $s_3=(a_1d_1—)\ldots\ldots$ One of the substitutions of degree 15 in $s_3^2Ds_3$ has the property of connecting the set $a_1, \ldots,$ of D and another set. Hence for s_3 we may choose a substitution similar in its transitive constituents to one of the substitutions of D. Whenever the transitive constituent $a_1, a_2, \ldots,$ of H' is of degree less than 18, every substitution of H' replaces some one of the nine letters $a_1, a_2, \ldots,$ of D by one of those letters, so that it is evident that a substitution of H' which replaces a y by a y (new letters to D) does not have a letter of the set $a_1, a_2, \ldots,$ of D in any cycle with one of the remaining letters $d_1, \ldots,$ of D. Partitions of the degree of H' not entailing substitutions of order 5 and degree 15, of order 7 and degree 14, and so forth, are 12, 3, 3, 12, 6, 12, 4, 3, 16, 3, 12, 4, 4, 12, 8, 16, 4, 14, 7, 15, 6, 18, 3, 16, 3, 3, 14, 8, 15, 7, 16, 6, 18, 4, 16, 4, 3, 15, 8, 16, 7, 20, 3. Consider the last case: 20, 3. The constituent of degree 20 is not primitive, nor has it systems of imprimitivity of degree 2 or 10. Only s_3 permutes systems to which the letters y belong. Then there are not systems of four letters each. If s_3 does not already fix x_1, x_2, x_3, transform D and s_3 by the substitution $e_1x_1 . e_2x_3 . e_3x_2 . b_1a_3 . b_2c_3 . a_2c_1 . d_2d_3$ under which D is invariant. Then s_3 fixes x_1, x_2, x_3. Then s_2 does not permute systems, nor consequently, does s_1 if we look at $d_1d_2d_3$, but it does if we look at $a_1b_1c_1$. The partition 20, 3 is impossible. If the partition is 18, 4 or 18, 3 the larger constituent is imprimitive and can only have systems of three letters, which in turn are permuted according to a primitive group of class 3. Here we encounter substitutions of order 5 and degree 15. A transitive constituent of degree 16 in H' involves one or four new letters y. Its order is at least 144. Since G can not contain a subgroup of degree 16 of order greater than 16, the partitions 16, 3 and 16, 4 are impossible. Let the partition be 16, 3, 3. H' is a (16, 1) isomorphism between a transitive group of order 144 and a direct product of order 9. The first constituent is not primitive. It has no systems of two or eight letters, hence its four systems of imprimitivity are permuted according to the alternating group of degree 4. The intransitive head of order 12 is also of order 12 in each transitive constituent, hence is a simple isomorphism between four tetraedral constituents. Now the transitive constituents of degree 3 are in the letters $e_1e_2e_3$, and $x_1x_2x_3$. The substitution $s_1^2s_2$ is in the intransitive head of the constituent of degree 16. It is of degree 9 in those letters whereas it should be of degree 12. Let H' have constituents on sixteen and on six letters. Since the only group of degree 6 whose order is divisible by 144 is the symmetric group, H' is not a simple isomorphism. The constituent of degree 6 is alternating because it is generated by three cycles of degree 3. The constituent of degree 16 is primitive and therefore is of order 5760. There are two such groups, but they are both of class 12 while our constituent in H', because of $s_1^2s_2$, is of class less than 12. If the partition is 16, 4, 3, and if we are not to have a subgroup of degree 16 with the factor 3 in its order, the intransitive constituent of degree 7 is a direct product of order 36. The large constituent is of order 144, 288, or 576, as the isomorphism is (4, 1), (8, 1) or (16, 1). This alone shows that the first constituent is imprimitive when of order 144 or 288. Suppose it is primitive of order 576. Its class should be 12, but $s_1^2s_2$ is of degree 9 in the letters of this constituent. This first constituent because of $s_1^2s_2$ can not have systems of imprimitivity of two letters each. Systems of four letters each are left unchanged by $s_1^2s_2$ of degree 9 in the leading constituent, and these four systems are permuted according to the tetraedral group. The intransitive head consisting of all the substitutions of the constituent in question that do not permute systems is then of order 12, 24, or 48. A transitive constituent of this head is of degree 4, and is alternating or symmetric. If the order of the head is 12 or 24 it is a simple isomorphism, and all its substitutions of order 3 are of degree 12, a condition violated by $s_1^2s_2$. If the order is 48 the symmetric group is not involved. There are only two ways by which we can get the order 48 from four alternating groups of degree 4. In one way an intransitive constituent of degree 12 is formed as a simple isomorphism, and that is put in a (4, 4) isomorphism to a fourth alternating constituent. Another way is to put in (4,4) isomorphism two similar intransitive tetraedral groups on eight letters. But in any event the substitutions of order 3 in the head are of degree 12. The constituent of degree 7 is on the letters $e_1e_2e_3x_1x_2x_3y_5$, and is an alternating group. The alternating group of degree 7 can not be written as a transitive group of degree 16. Then it should be a (16, 1) isomorphism between a primitive group of degree 16 and order 40320 and the alternating group of order 2520. But the former group is of class 12 and does not contain its part of $s_1^2s_2$. Let H' have a transitive constituent of fifteen letters. Its order is divisible by 135. No non-alternating primitive group of degree 15 has the factor 27 in its order. Then a constituent of degree 15 has five systems of imprimitivity of three letters each, permuted according to an icosaedral group. The smaller transitive constituent, containing a substitution of order 5, is an alternating group or the icosaedral group written on six letters. None of the orders of these groups are divisible by 27, and the last not even by 9. A (3,1) isomorphism between the imprimitive group of degree 15 with its icosaedral quotient group and one of these three alternating groups is impossible. Let H' have a leading constituent of degree 14. It is a simple isomorphism. The constituent of degree 14 is not primitive. It must have seven systems of two letters each. But $s_1^2s_2$ can not respect such systems if $s_3=a_1d_1—\ldots.$. The partition 12, 3, 3 demands an absurdity, a simple isomorphism between an intransitive constituent of degree 6 and order 9, and a transitive constituent of degree 12. Let 12, 6 be the next partition for study. The second constituent is imprimitive because 5 can not divide the order of H' in this case. But the group generated by $(e_1e_2e_3)$, $(x_1x_2x_3)$ and $(e_1x_1—)\ldots.$, the last of degree 3 or 6 in these letters, is necessarily alternating. Let the partition of the degree of H' be 12, 4, 3. The constituent of degree 7 can not be of order 12, for then the leading constituent is the regular tetraedral group and all the substitutions of order 3 in H' are of degree 18. Let the intransitive constituent be of order 36, a direct product. It has an invariant subgroup of order 12 and degree 16 which is a simple isomorphism. This subgroup has been fully considered. Let the partition be 12, 4, 4. The group H' is not a simple isomorphism, nor does it involve the direct product of two tetraedral groups. The intransitive constituent of degree 8 is therefore of order 48, and this is the order of H', a number not divisible by 9. The remaining partition is 12, 8. The order of the second constituent of this simple isomorphism is 72 or a multiple of 72. Since it is of class 3, it is not primitive: $\{e_1e_2e_3, x_1x_2x_3\}$ does not respect systems of two letters, and s_3 can not permute systems of four letters.

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* Burnside, "Theory of Groups," 2d edition (1911), p. 341.

Let D be the group of degree 18 and order 12 for present consideration. There exists a substitution s_3 similar to s_1 that replaces a letter of the transitive constituent of degree 6 by a letter of another constituent and that has no two new letters, new to D, in one cycle. Let H be transitive if possible. If H is of degree less than 21, it is imprimitive. If H is of degree 18, there is no system of imprimitivity of degree 9, 6, or 2. In case there are systems of three letters, the four letters of a constituent of degree 4 in D belong to different systems, while the constituent of degree 6 has systems of two but not of three letters, so that D can not permute systems of three letters when H is of degree 18. Nor could it do so if H were an imprimitive group of degree 21. If H is imprimitive of degree 20, no systems are of two or ten letters. Systems of five letters are also readily seen to be impossible. For if s_1 permutes systems, y_1, y_2, x_1, x_2, x_3, constitute a system, not respected by s_2. Similarly s_2 can not permute systems. Finally H can not have systems of four letters each, for the reason that no letter of D can belong to a system of four letters with y_1. If H is imprimitive of degree 21, it was noticed above that there can not be systems of three letters, and systems of seven letters are obviously impossible. If H is primitive of degree 21, we can find a substitution s, similar to s_1, in H which fixes one of the letters y_1, y_2, y_3, and displaces at least one of them. This substitution s can not, by hypothesis, connect the constituent of degree 6 of D with another constituent of D. The order of $\{D, s\}$ is not divisible by 5 (the degrees of the transitive constituents of $\{D, s\}$ are not all divisible by 5), 7, or 13. Then if $\{D, s\}$ is of degree 19, the transitive constituents in it are of the degrees indicated by the partition 6, 9, 4, which is an impossibility because the constituent of degree 4 is of order 12, and the intransitive constituent of degree 15 can not be in multiple isomorphism to it. If $\{D, s\}$ is of degree 20, s adds two letters to the constituent of degree 6 of D.

The degrees of the constituents can not be given by 8, 4, 4, 4, because the order of the intransitive constituents of degree 8 is 12, 48, or 144, while that of the other intransitive constituent in simple isomorphism to it is a multiple of 32. If the partition is 8, 8, 4, $\{D, s\}$ is not a simple isomorphism between its three constituents, since one is alternating of degree 4, but the two constituents of degree 8 are in simple isomorphism to each other. The constituent of degree 8 that involves two new letters permutes these two new letters as a system of imprimitivity, one of four, which go according to the alternating group of degree 4, because the constituent in D admits no other systems than those of two letters, and permutes them in a cycle of order 3. The head involves substitutions of degree 4 (as in D), and hence the class of G is lowered. If the degrees of the transitive constituents of $\{D, s\}$ are 8, 12, the above remarks still apply to the first constituent, though the class of G' is not necessarily lowered by the presence of substitutions of degree 4 in the head of the constituent of degree 8. The other constituent, in simple isomorphism to the one of degree 8, has the factor 9 in its order because of the substitutions of order 3 and degree 9 in it. This, the smaller constituent, whose order is twelve times a power of 2, does not permit. If H, transitive, is of degree 22, it is primitive, and we may again study the group $\{D, s\}$, confining ourselves to the degree 21, since the preceding remarks in regard to the degrees 19 and 20 apply to this new $\{D, s\}$ without change. To be certain that s may be found without three of the new letters y in one cycle, recall that H, being primitive, has in it a substitution t which replaces a y by a y, and replaces a letter of D by a y. Then $t^{-1}Dt$ has no constituent of degree less than 4, fixes one (and only one) y, and gives us the substitution s similar to s_1 with the three letters y in three cycles. The subgroup F, invariant in H_1, keeps the letters of the constituent of degree 6 of D and the other letters of D in separate constituents. F is of degree 21. If F has a constituent of degree 9, made by adding three letters y to the constituent of degree 6 in D, that constituent is primitive, and s may be chosen so that the degree of $\{D, s\}$ is 19 or 20. The possible partitions of the degree of F, which do not obviously lead to substitutions of order 11, of order 7, of order 5 and degree 15, are 6, 5, 5, 5, 6, 10, 5, 6, 9, 6, 6, 15, 8, 9, 4. We note that in F, and hence also in the other three subgroups that are conjugate to F and leave one of the four new letters y fixed, a transitive constituent that contains the six letters of the largest transitive constituent of D is of degree 6 or 8, thus compelling the presence of a transitive constituent of degree 6 on the same letters in two subgroups, F and one of its transforms. This is impossible when H_1 has a transitive constituent in these six letters, as must be the case if the partitions are 6, 5, 5, 5, 6, 10, 5, or 6, 15. If the partition is 6, 9, 6, the third constituent is alternating, and G contains a substitution of degree 15 and order 5. If one constituent of F is of degree 4, its order is 12, and the group is not a simple isomorphism between its three transitive constituents, but the constituents of degree 8 and degree 9 are simply isomorphic. The former is imprimitive and its order is not divisible by 9. We conclude that H is not a transitive group of degree 22. If H is of degree 23 it is doubly transitive. The subgroup F of H_1 must have its transitive constituents of equal degree, but the degree of one of them can range only from 6 to 10. Hence H is not of degree 23.

The group $H' = \{D, s_8\}$ is intransitive. Since s_8 replaces one of the six letters of the largest transitive constituent of D by a letter of that constituent, the constituent of degree 6 in $s_8^2 D s_8$ or $s_8 D s_8^2$ has letters of the constituent of degree 6 and of one of the constituents of degree 4 of D in a cycle of one of its generators of order 3. Then it may be assumed that H' has the three generators of any one of its constituents similar substitutions. The possible partitions of the degree of H' are 10, 5, 5, 10, 10, 12, 4, 4, 12, 8, 15, 5, 12, 9, 16, 4, 16, 5, 12, 5, 5, 12, 10, 18, 4, 18, 5. A transitive constituent of degree 12 is not doubly transitive because there is not another constituent of so high a degree as 11. Then it is an imprimitive group generated by three similar substitutions of order 3 and degree 9, whence it follows that it does not admit systems of two, four or six letters. Its systems of three letters go according to the tetraedral group. Then the order of this constituent is divisible by 144, but not by 5, a remark which disposes of the partitions 12, 5, 5 and 12, 10, and which requires that the intransitive constituent on eight letters in the case 12, 4, 4 be a direct product. But the direct product of two alternating groups of degree 4 can not be written as a transitive group of degree 12. When the partition is 12, 8, the second constituent, generated by two substitutions of degree 6 and order 3, has four systems of imprimitivity of two letters each so that its order is not divisible by 9. But in this case the two constituents are in simple isomorphism. A constituent of degree 9 generated by similar substitutions of degree 6 and order 3 is primitive, and being positive and not triply transitive (to avoid a substitution of order 7), is the group of order 216. Again the two constituents are in simple isomorphism. An absurdity has been reached: 144 does not divide 216. Consider the partitions 18, 4 and 18, 5. The larger constituent is not primitive, and since it is generated by four cycles of s_1, s_2, and s_8, it has nine systems of imprimitivity permuted according to a primitive group. The substitutions of order 2 in D are of degree 4 in these systems, for they do not interchange systems of two letters in the constituent of D of degree 6, and in the constituents of D of degree 4 no two letters of the same constituent can belong to the same system of imprimitivity. Now the group in the nine systems is not alternating, and there is no primitive group of class 4 of degree greater than 8. If the partition is 15, 5, the larger constituent, with the generators s_1, s_2, s_3, can not be imprimitive. If it is primitive it is simply transitive (to avoid a substitution of order 7) and of class greater than 9. There is but one group satisfying these conditions, and it is of order 360, simply isomorphic to the alternating group of degree 6, a simple group. In the case of a transitive constituent of degree 16, systems of imprimitivity of two letters are possible so far as D is concerned. But the two new letters in s_3 can not form part of a larger system of four or eight letters. The systems of two letters are therefore permuted according to a primitive group of degree 8, and again we encounter a substitution of order 7. Let H' have two transitive constituents of degree 10. The second constituent of degree 10 is generated by three substitutions of order 3 and degree 6. If the two new letters of s_3 do not form a system of imprimitivity, H' has a subgroup $\{D, s\}$ (s is similar to s_1 and is determined in the usual way) of degree 19. Among the small number of partitions of which the subgroup $\{D, s\}$ admits only one claims attention: 6, 9, 4. Here the intransitive constituent of $\{D, s\}$ of degree 15 must be in (3, 1) isomorphism to the alternating constituent of degree 4. But the constituent of degree 6 can not have an invariant subgroup of order 3, because of the substitutions of D in those six letters. Then the second constituent of H' has five systems of imprimitivity and its order is sixty times a power of 2. The other constituent is a positive primitive group (see its generators $s_1, \ldots.$) of order 360 or 720, multiples of 9. The primitive group of degree 10 and order 60 was passed over because it is not simply isomorphic to an imprimitive group of degree 10. But one partition is left: 10, 5, 5. In this case the intransitive constituent of degree 10 can only be a simple isomorphism between two icosaedral groups, because the direct product of two icosaedral groups can not be represented as a transitive group on ten letters. Hence the transitive constituent of degree 10 is also of order 60, a simple isomorphism between three icosaedral groups of the degrees 10, 5, and 5.

Because H' contains a substitution of order 5 and degree 20, the highest degree of any primitive group of which it is a subgroup is 25. Let now the substitution s_4 be chosen subject to the same conditions as was s_3. Let it be further assumed that $H = \{H', s_4\}$ is transitive. H is not a primitive group of degree 20. Since the transitive constituent of degree 10 in H' is primitive, systems of imprimitivity can only be of two or ten letters each. But such systems can not be permuted by any of the generating substitutions s_1, s_2, s_3, s_4. Let H be of degree 21. These generators show that it is not imprimitive. The subgroup F, generated by all the substitutions $s_1, \ldots,$ in H_1, the subgroup of H that leaves one letter fixed, is by hypothesis intransitive. Now F can not have a constituent of degree 15 unless it includes a substitution similar to s_1 that connects the transitive constituent of degree 10 of H' with another constituent of H' and displaces fewer new letters than s_4. Suppose that F has two constituents, both of degree 10. The icosaedral group of degree 10 (of class 8) is not contained in a larger group of the same degree and class,* nor in a positive group of degree 10 and class 6. Then F coincides with H'. The largest primitive group of degree 10 in which this icosaedral group on ten letters is invariant is of order 120, isomorphic to the symmetric group on five letters. Then the order of H is 1260 or 2520. But consider for a moment the possible order of the largest subgroup of H in which a subgroup of order 7 (and degree 21) is invariant. This subgroup can contain no substitution of order 3 or 2 capable of transforming a substitution of order 7 into a power without permutation of cycles, for all the subgroups of order 3 in H_1 are of degree 15, not 18, and subgroups of order 2 are of degree 16. This is evident when H_1 is of order 60, but perhaps requires explanation if the order of H_1 is 120. In the latter case one constituent of H_1 is, as we have seen, the primitive group of order 120 of class 6; it has fifteen substitutions of order 2 and degree 8, and ten substitutions of order 2 and degree 6, the latter corresponding to the ten transpositions of five letters. The other constituent of H_1 is then not intransitive (the class must be maintained) but is the imprimitive representation of the symmetric-5 on ten letters. This imprimitive constituent has two systems of five letters each, and also five systems of two letters each. The subgroup that leaves one letter fixed, leaves two fixed, and is the alternating group in the four systems. Then the imprimitive constituent has fifteen substitutions of order 2 and degree 8, and ten substitutions of order 2 and degree 10. In H_1, therefore, all the substitutions of order 2 are of degree 16. No substitution of H can transpose two cycles of a substitution of order 7 and leave one cycle unchanged, the class of H being 15. Hence the order of the largest subgroup of H in which a substitution of order 7 is invariant is 7 or 21. Of the four quotients 1260/7, 2520/7, 1260/21, 2520/21, only 120 is congruent to unity, modulo 7. Then H_1 is of order 120, and there are in H 120 conjugate subgroups of order 7, each of which is invariant in a group of order 21. In H there are one hundred and twenty-six conjugate subgroups of order 5. An invariant subgroup (K) of H is transitive, hence 7 is a divisor of its order, which can only be 840. But it is now clear that K must have in it the one hundred and twenty-six subgroups of order 5, giving to K more than eight hundred and forty operators. Then H is a simple group. It is well known that there is only one simple group of order 2520, the alternating group of degree 7. It has a transitive representation of class 15 on twenty-one letters, most readily set up as the group of the permutations of the twenty-one binary products $ab, ac, \ldots,$ of seven letters. It is not contained in a larger primitive group of the same degree and of class 15. Since all its substitutions of degree 15 are conjugate it is not maximal in a transitive group of degree 22.* Let H be of degree 22. Its generators show that it is primitive. The subgroup F is of degree 21, and has the same primitive constituent, the alternating-5 group on ten letters, as H'. Since F is a simple isomorphism this is impossible. Let H be of degree 23. It is doubly transitive, and F, invariant in the transitive subgroup H_1, has two transitive constituents of degree 11 which are interchanged by half the substitutions of H_1. The first constituent, in which the primitive group of degree 10 and order 60 is maximal, is quite possible, but we get into difficulties with the second constituent, simply isomorphic to it, which is of class 6 or less because of the subgroup D. Next, let H be of degree 24. Because of the primitive constituent of degree 10 in H', and the generators $s_1, \ldots,$ of H, H is not imprimitive. The only divisions of the letters of F into transitive sets that do not introduce substitutions of order 5 and degree less than 20, of order 7 and degree less than 21, of order 11 and degree less than 22, and cycles of thirteen letters are 11, 12, 12, 11, 12, 6, 5. The 12, 11 partition is impossible because of the substitutions of degree 6 in the second constituent. The non-alternating group of degree 11 when represented on twelve letters is multiply transitive and can not have substitutions of degree 6 in it, as the partition 11, 12 requires. In the case of the partition 12, 6, 5 the first constituent being primitive involves a substitution of order 11. Then H is not of degree 24. Finally the primitive group G that includes H is of degree 25. The order of G is not divisible by 7 or 11. Then all tentative partitions of the degree of F are seen at a glance to be impossible.

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* AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII (1910), p. 254.

The group $H'' = \{H', s_4\}$ may be intransitive. If s_4 is not similar to s_1 in the letters of each of the constituents of H'' (as s_3 was similar to s_1) another substitution from $s_4^2 H' s_4$ will serve for s_4 and will be similar to s_1 in the two sets of letters. For all subgroups of order 3 are conjugate in H', and *a fortiori* similar in the various sets of letters. Now the degree of the constituent of H'' that involves the primitive group of degree 10 and order 60 is at least 15, and at most 19. It is therefore primitive and of class 12. Since the second constituent is alternating of degree 5 or 6, we have to seek those positive primitive groups of class 12 and of degree 15 and 16 that can be isomorphic to one of these two alternating groups. If the isomorphism is simple, the larger constituent is the simply transitive primitive group of degree 15 and order 360. If the isomorphism is not simple it is a (16, 1) isomorphism between a primitive group of degree 16 and order 960 or 5760, and one of the given alternating groups. If the constituent of degree 16 is multiply transitive, every substitution of its subgroup that leaves one letter fixed replaces a letter of the transitive constituent of degree 10 of H' by a letter of that constituent, and one substitution at least replaces a letter of that constituent by a letter of a second constituent of H'. The transform of H' by this substitution contains a substitution similar to s_1 that has the properties of s_4 with fewer new letters. Then the constituent of degree 16 can only be the simply transitive group of degree 16 and order 960 (it is G_7 in Miller's list of the primitive groups of degree 16*). Then H'' is of degree 21 only. All the subgroups of degree 15 in H'' are conjugate. Now a substitution s_5 is chosen (after the manner of the choosing of s_4) which with H'' generates a transitive group H. Because of the fact that the degree of this group can not exceed 25, and because of the presence of the primitive constituent of degree greater than 14 in H'', H is primitive. If its degree is 21, it may be handled by means of the arguments used when s_1, s_2, s_3, s_4 was assumed to be a primitive group of degree 21, H' being in F and thus determining the simple group of order 2520, if any. If H is of degree 22, F coincides with H''. In H'', that is, in H_1, all the subgroups of degree 15 form just one set of conjugates, while one of them leaves fixed seven letters which therefore form a transitive set in the largest subgroup of H in which that subgroup is invariant, which latter group, however, can contain no substitution of order 7. Let H be of degree 23, a doubly transitive group. Then F should have two transitive constituents of degree 11. Let H be of degree 24. F has two distinct sets of letters. A constituent of degree 15, isomorphic to the alternating group of degree 8, is doubly transitive and a constituent of degree 16, isomorphic to the alternating group of degree 7, is triply transitive. To show the non-existence of the simply transitive primitive group H, we may call attention to the following general theorem, proof of which is appended:

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* *Bulletin of the American Mathematical Society, loc. cit.*

If the degree of a transitive constituent of the subgroup leaving one letter fixed in a simply transitive primitive group exceeds by two (or more) units the

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*Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XX (1898), p. 231.

degree of any other transitive constituent of that subgroup, then the constituent of highest degree is a simply transitive group.[*]

It may be proved in few words. Let G_1 fix the letter x of the simply transitive primitive group G of order g and degree n, and let $a_1, a_2, \ldots, a_m$ be the letters of the transitive constituent of highest degree in G_1, supposed, if possible, to be multiply transitive.

The order of the subgroup of G_1 that fixes a_1 and a_2 is $g/nm(m-1)$, while the order of any subgroup of G_1 that fixes two letters, one or both of which do not belong to the set $a_1, a_2, \ldots, a_m$ is greater than $g/nm(m-1)$. Now consider the subgroup G' of G that fixes a_1. Since the order of the subgroup of G that fixes a_1, x, and a_i, $i=2, 3, \ldots, m$, is exactly $g/nm(m-1)$, x and a_i must be in transitive constituents of the subgroup that fixes a_1 of degree greater than $m-2$; but there is in the subgroup that fixes a_1 only one such constituent. Then either G_1 is not a maximal subgroup of G or the transitive constituent of highest degree in G_1 is but simply transitive. The theorem is proved.

Let H displace twenty-five letters. Its order is not divisible by 7. Since the larger constituent of F (and therefore of H_1) is simply transitive, the other constituent is an alternating group and necessarily introduces the factor 7 into the order of H.

From now on we shall be concerned only with primitive groups G of class 15 in which no two substitutions of degree 15 have so many as twelve letters in common.

The group D of degree 19 is not transitive. It has no transitive constituent of degree 7, 11, or 15. Then it has one constituent of degree 3, represented by a cycle common to s_1 and s_2. If D has a constituent of degree 4, it is tetraedral, and D is a simple isomorphism between two intransitive constituents of degree 12 and degree 7. It is of order 12 and has five transitive constituents. If D has a transitive constituent of degree 8, a second transitive constituent of degree 8 is in simple isomorphism to the first. Two substitutions of degree 6 and order 3 that generate an imprimitive group of degree 8, such as we evidently have here, limit the group to systems of imprimitivity of two letters. It can not have systems of four letters. The group in the systems is tetraedral. The head is not the identity and is not of higher order than 2. There is one and only one group satisfying these conditions, $\{135 . 246, 147 . 238\}$ with the head $(12 . 34 . 56 . 78)$.[†]

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[*] Cf. Jordan, *Bulletin de la société mathématique de France*, Vol. I (1873), p. 198.

[†] Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXI (1899), see pp. 323 and 332.

Let us study this group D of order 24. It has no subgroup of order 12. There is of course in G a substitution s_8 similar to s_1 that unites two constituents of D and has at most one new letter to a cycle. One of the transforms by s_8 of the eight substitutions of order 3 in D certainly unites two transitive constituents of D and may be used for s_8. We are thus assured of the advantage of having s_8 similar to s_1 and s_2 in the constituents of $\{D, s_8\}$ whenever it is not transitive.

Let $H = \{D, s_8\}$ be transitive. It is not of degree 19. If s_8 displaces just one letter new to D, it displaces three or four of the letters of D that are fixed by s_1. H is imprimitive and no systems are possible except of four letters, and one such system consists of the four letters of s_8 that are new to s_1. Three of these letters do not form a cycle of s_8, for then the existence of a substitution similar to s_1 with not more than two letters new to s_1 could be deduced. Since s_8 bears this same relation to any one of the four subgroups of order 3 in D, it would need to be of degree 16 at least. Let H be of degree 21. It is primitive. In F the letters are partitioned thus: 9, 8, 3, 8, 8, 4. The first places an impossible constituent of degree 3 in H_1, and in the second case H_1 is not maximal because D is in two subgroups F and F', and thus determines the letters of one or two constituents of H_1 and H'_1. If H is of degree 22 it is primitive. The partitions of the degree of F are 9, 9, 3 and 9, 8, 4. Here there is no constituent of degree 3 in H_1. The constituent of degree 8 is of order 24 and should be simply isomorphic to the one of degree 9. H is not of degree 23. Let H be of degree 24. Systems of imprimitivity of two, six, eight, or twelve letters are obviously impossible. No system containing y_1 can contain any letter of D other than the letters of the constituent of degree 3 in D. Hence H has no systems of three letters each, and of systems of four letters each, two systems are $y_1e_1e_2e_3$ and $y_2y_3y_4y_5$, which is absurd because s_8 does not permute these two systems. Then H is primitive. If F has a constituent of degree 3 or 4 in the letters y, new to D, the partitions of its degree are 8, 8, 4, 3 and 9, 8, 3, 3. The first is impossible, as then H_1 would have a constituent of degree 3. As above, the constituent of degree 9 should be simply isomorphic to the constituent of degree 8 of order 24. The same remarks apply to the partitions 10, 10, 3, 12, 8, 3, 9, 8, 6 and 10, 8, 5. There is one other partition, 8, 8, 7, but H_1 is not maximal when the partition is 8, 8, 7.

Then $H' = \{D, s_8\}$ is not transitive. The possible partitions are 16, 3, 16, 4, 12, 8, 18, 3, 12, 9, 18, 4, 14, 8, 12, 10, 20, 3, 14, 9, 20, 4, 14, 10. Here we have borne in mind that s_8 is similar to s_1 in each of the two constituents. A constituent of degree 14 is not primitive, but since it is generated by three substitutions of order 3 and degree 9 it can have no systems of imprimitivity. Likewise a constituent of degree 12 is imprimitive, but does not respect systems of imprimitivity of three letters each (y_1 can not be in such a system), while it too has three generators of order 3 and degree 9. In the remaining partitions the larger constituent is generated by three substitutions of degree 12 and order 3. Assume that a constituent of degree 18 or 20 has systems of imprimitivity of two letters each. Each transposition of the substitution (t) of order 2 of D is a system. Transform t into t', a substitution that involves one or two transpositions new to t. The product tt' is not identity and is of degree 8 at most. Then the partitions 18, 3 and 18, 4 are impossible, and a transitive constituent of degree 20 has five systems of imprimitivity permuted according to a primitive group. A system of imprimitivity not composed of the four letters new to D is built from a system of two letters from one, and a system of two letters from another constituent of D. Then the group in the systems is alternating. Now s_8 leaves fixed the eight letters of two systems of the constituent of degree 20, so that t', the transform by s_8 of the substitution t of order 2 of D has four transpositions in common with t. Hence the product tt', not identity, is of degree 12 at most. If the leading constituent is of degree 16, t is invariant in H'. This is clear when we consider that the constituent in question, of order three times a power of 2, is not primitive and must accept the systems of two letters of the two constituents of D, even when it has systems of four letters each. If H' is of order 48 it has no substitution of order 3 not in D. There remains then only one partition: 16, 4. The eight systems of two letters each are not permuted according to a primitive group. Then systems of four letters are permuted according to the tetraedral group. The four letters of D that are fixed by s_1 form a system of imprimitivity. Similarly each subgroup of order 3 of D fixes the four letters of a system of the constituent of degree 16. Then s_8 must displace at least three letters of D new to s_1, three letters of D not in s_2, and so on, that is to say, s_8 displaces letters of four distinct systems of four letters each in four cycles, which, as a matter of fact, permute only three systems.

It can be shown that if $D' = \{s_1, s'\}$ (where s' is a substitution of degree 15 that displaces four letters new to s_1) has fewer than five transitive constituents, it contains (or is) the subgroup of the type D just discussed, generated by two substitutions s_1 and s_2 similar in each transitive constituent. If D' does not include the latter subgroup, it does, however, contain the subgroup D of order 12 that has five constituents. Among the substitutions of D' that are of degree 15 and apart from those which unite no two cycles of s_1, let t be a substitution that unites a minimum number of cycles of s_1. Both t^2s_1t and ts_1t^2 are similar to s_1 in the several transitive constituents of $\{s_1, t\}$, and $\{s_1, t^2s_1t\}$ or $\{s_1, ts_1t^2\}$ has the same constituents as $\{s_1, t\}$ unless each has just five constituents, in which case t displaces the six letters of two cycles of s_1. But then $s_1^2ts_1$ is not t or t^2, and displaces at most three letters new to t.

Hence G contains no subgroup D of degree 19 with fewer than five constituents.

Let D be the group of order 12 and degree 19. One transitive constituent is of order 3 on the letters $a_1a_2a_3$, and each of the others is an alternating group of degree 4. We may write $s_2=(b_1b_2b_4)\ldots.$, so that $s_1s_2=(b_1b_4)(b_2b_3)\ldots.$, a substitution of degree 16. Now if $s_2=(c_1c_3c_4)\ldots.$, $s_1s_2=(c_1c_2c_4)\ldots.$ Therefore,

$$s_2=a_1a_3a_2\,.\,b_1b_2b_4\,.\,c_1c_2c_4\,.\,d_1d_2d_4\,.\,e_1e_2e_4\,.$$

There exists in G a substitution s_3, similar to s_1, which replaces a letter of the set a by a letter of the set b, and by means of appropriate transformations we may write uniquely $s_3=(a_1b_1—)\ldots.$. Write

$$t=s_1^2s_2s_1=a_1a_3a_2\,.\,b_2b_3b_4\,.\,c_2c_3c_4\,.\,d_2d_3d_4\,.\,e_2e_3e_4\,.$$

Our substitution s_3 introduces a minimum number of new letters to D. Assume that s_3 displaces the three letters a_1, a_2, a_3, and in three different cycles. Then $t^2s_3t=(a_3b_1—)\ldots.$, displaces at most four letters new to s_3, and yet connects two cycles of s_3. Then s_3 certainly replaces an a by an a. The transform of any one of the eight substitutions of order 3 in D replaces an a by b_1, and can not displace fewer letters new to D than s_3 itself. Hence new letters y in s_3 can follow only a_2 or a_3. This permits s_3 at most one new letter y. Of course, since s_3 unites the cycles $a_1a_2a_3$ and $b_1b_2b_3$ of s_1, it must displace one new letter y. But now, on running over the substitutions of order 3 in D, it is evident that s_3 must displace $yb_4c_4d_4e_4b_2c_2d_2e_2b_3c_3d_3e_3$, and two of the letters $c_1d_1e_1$ in addition to a_1 and b_1, which is absurd. Hence this strong condition, as holding good in all that follows:

No two substitutions of degree 15 *in* G *have more than ten letters in common.*

Let $D=\{s_1, s_2\}$ be of degree 20. First assume it transitive. The five new letters in s_2 constitute one of four systems of imprimitivity permuted according to a tetraedral group. D has no systems of two, four, or ten letters. A system containing one new letter x_1 must contain just one other new letter x_2, or else all the five letters $x_1x_2x_3x_4x_5$. Hence there is one and only one system of imprimitivity to which a given letter x_1 may belong. Then D is to be looked for as a subgroup of a doubly transitive group G of degree 21. All the substitutions of degree 15 in D permute three of the four systems, and leave fixed each of the letters of one system. The doubly transitive group G contains a substitution s of degree 15 involving a letter y new to D. But s must displace, besides y, four letters from each of the four systems of imprimitivity of D. This is absurd.

Then D of degree 20 is intransitive. Constituents of degree 4 are alternating groups and transitive groups of higher degree are imprimitive groups generated by two similar substitutions of order 3 and whose systems are permuted according to an alternating group. A constituent of degree 8 is the group $\{135.246,\ 147.238\}$ of order 24. The order of a transitive constituent of degree 12 is divisible by 9. Hence the partitions 8, 4, 4, 4, 12, 4, 4 and 12, 8 are out of the question. There are three other possible partitions of the degree of D: 4, 4, 4, 4, 4, 8, 8, 4, 16, 4. A transitive constituent of degree 16 is of order 48 or more. The subgroup that corresponds to identity of the other constituent contains no substitution of order 3, and its order is therefore a power of 2. This leading constituent has substitutions of degree 12 and order 3, and since its order is not divisible by 9, it has no substitution of degree 15. Therefore D, whether the partition is 4, 4, 4, 4, 4, 8, 8, 4, or 16, 4, has exactly four conjugate subgroups of order 3. Each of them leaves fixed five letters which are displaced by the other three. We may now remark that G contains no substitution of degree 15 (not a substitution of D) that displaces fewer than two letters new to D. A fact of which considerable use will be made in what follows is this: if a substitution (t) similar to s_1 displaces or fixes the three letters of one cycle of a substitution (s) similar to s_1, s^2ts displaces at most four letters new to t and is therefore t itself, that is, s and t are commutative. The transitive group H generated by $D, s_8, \ldots,$ can not be of degree 22. For such a group is primitive and its subgroup F contains substitutions of degree 15 displacing one letter new to D. Nor can H be of degree 23.

Now assume that D is that group which has a transitive constituent of degree 16. If H is of degree 24 it is primitive, and since H_1 can have no constituent of degree 16, only one partition of the degree of F need be proposed, 18, 5. A constituent of degree 18 is imprimitive. But D does not respect systems of three letters, as the substitutions of order 5 require. Let H be of degree 25, and let it first be assumed that H is primitive. Of all the possible partitions of the degree of F only 18, 6 and 20, 4 seem to require more than a glance. In the second partition H_1 has an invariant subgroup of degree 20 that corresponds to the identical substitution in the constituent of degree 4, an impossible combination.* Since a constituent of degree 6 is alternating, there are as above substitutions of order 5 and degree 15 in the imprimitive constituent of degree 18. Suppose the group H to be imprimitive. A system of five letters can be chosen in but one way if it contains a given letter, and hence H is not invariant in a primitive group of degree 25, nor need we pass the degree 26 in a search for a doubly transitive group of which H is a subgroup. If H is included in a primitive group of degree 25, the totality of substitutions of degree 15 in that group generate a primitive group which must be simply transitive and whose non-existence can be shown by the arguments used when H was assumed to be primitive. The only remaining hypothesis is that H is a subgroup of a doubly transitive group G whose subgroup G_1 is imprimitive. The invariant subgroup F of G_1, generated by all substitutions of degree 15 is imprimitive, having five systems of imprimitivity of five letters each permuted according to the icosaedral group. Its order is 1200 or a multiple thereof. The head of F is of order 20 at least, and from the nature of the group D, which leaves fixed each of the five letters of the system formed of the letters of F that are new to D, the head is not a simple isomorphism between its five constituents. If the head contained a substitution of order 3, each constituent would be an alternating or symmetric group. Alternating constituents, if the class of G is to be maintained, can only stand in simple isomorphism to each other. Then the head contains no substitution of order 3. All the subgroups of degree 15 in G_1 are conjugate under the substitutions of G_1, which implies that the largest subgroup of G in which a given subgroup of degree 15 is invariant has a transitive constituent of degree 11. But G (of degree 26) can have in it no substitution of order 11. Thus the partition 16, 4 of the degree of D is shown to be impossible.

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* Bennett, *loc. cit.*

The group D has either five constituents of degree 4, or two transitive constituents of degree 8 and one of degree 4. Now assume that D is the latter group, of order 24. Let s_3, with D, generate a transitive group. The first degree to study is 24. If $H=\{D, s_3\}$ is imprimitive, the four letters new to D form a system; and the four letters of the small constituent of D form a system, by which s_3 or s_3^2 replaces the former system (of new letters). Then s_3 displaces the three letters of a cycle of s_1 and is not commutative with s_1, hence displaces at most four letters new to s_1. Then H is primitive. The partitions of 23, corresponding to the subgroup F are 8, 8, 4, 3, 8, 8, 7, 9, 8, 6, 10, 8, 5, 10, 9, 4, 9, 9, 5. The first puts a constituent of degree 3 in H_1, and the last two involve a substitution of order 5 and degree 15, as do the second and third which have an alternating constituent. If the partition is 10, 8, 5, the constituent of degree 8 must include a substitution of order 5, and therefore affords a substitution of order 7 and degree 14. Let H be of degree 25.

Assume H to be primitive. The partitions of F are 8, 8, 4, 4, 8, 8, 8, 8, 8, 5, 3, 9, 8, 4, 3, 9, 9, 6, 10, 8, 6, 10, 9, 5, 10, 10, 4, 12, 8, 4. The partitions 8, 8, 5, 3, 9, 8, 4, 3, and 10, 10, 4 impose a constituent of three or four letters on H_1 under impossible conditions; a constituent of degree 6 contains a substitution of order 5, thus disposing of 9, 9, 6; while the substitution of order 5 in the constituent of degree 8 (partition 10, 8, 6), and in the constituent of degree 9 (partition 10, 9, 5), leads to a substitution of order 7 in these two cases. A transitive constituent of degree 12 is imprimitive and the six systems of two letters each are permuted according to the alternating group of degree 6. This consideration reveals a substitution of order 5 and degree 15 in F. There remain two partitions, 8, 8, 4, 4, and 8, 8, 8. No constituent of degree 8 is primitive. The first two transitive constituents of F admit only those systems of imprimitivity of two letters each, which are determined by the subgroup D, that is, by constituents of D such as $\{123 \,.\, 456, 157 \,.\, 428\}$. Such a constituent contains no substitution of order 3 not of degree 6. A generator of F, not in D, can not fix all the letters of one of the constituents of D, for then that substitution would be commutative with s_1 and s_2. Then the generating substitutions of degree 15 of F have just four cycles involving letters of these first two constituents, and the remaining constituent (constituents) is (are) generated by cycles of three letters. Then the third constituent is not a transitive group of degree 8. It is the direct product of two alternating groups of degree 4. A transitive constituent of degree 8, being positive, is of order 24 or 96. Either of these groups may be generated by its substitutions of order 3, so that the intransitive constituent of F of degree 16 could have the factor 9 in its order only if it were the direct product of its two transitive constituents, which clearly is not possible. Let it now be assumed that H, of degree 25, is imprimitive. The group in the systems is the alternating group on five letters. There is only one possible system that contains one of the letters fixed by D, the system constituted of these five new letters. Then at once we know that neither H nor a larger imprimitive group of the same degree which includes H is invariant in a primitive group of degree 25, and that a primitive group of degree n that includes H is $n-24$ times transitive. A primitive group of degree 25, generated by substitutions of degree 15 would be disposed of by the arguments given above. We are only interested in the possible existence of a doubly transitive group G of degree 26. Consider the subgroup F of G_1, generated by substitutions of degree 15. The five systems of imprimitivity of F are permuted according to an icosaedral group, and if the order of F is divisible by 9, there is a substitution of order 3 in the intransitive head that permutes no systems. One of these substitutions of order 3 in the head is of degree 15, connects two transitive constituents of D and has at most three letters new to D, because a system of imprimitivity of F can be made by two letters from each transitive constituent of degree 8, and one letter from the constituent of degree 4 or by the five letters new to D. Then a subgroup of order 3 is a Sylow subgroup of G_1, which implies a substitution of order 11 in G.

The subgroup $H'=\{D, s_8\}$ is intransitive. We recall that s_8 has at least two letters new to D. Among the eight substitutions of order 3 in D, a given letter a_1 of a constituent of degree 8 is in the same cycle with six of the seven remaining letters of that constituent. Therefore the transform by s_8 of at least one of them will, like s_8, connect two constituents of D in one of its cycles, and may be used for s_8. Hence it may be assumed that s_8 is similar in each of the constituents of H' to one of the substitutions of order 3 in D. The plausible partitions are 18, 4, 18, 5, 20, 4, 20, 5, 10, 12, 8, 14, 9, 14, 8, 15, 10, 14, 9, 15, 10, 15. The partitions involving constituents of degree 14, 18 and 20 yield to the arguments of page 301. A constituent of degree 12 is not primitive, but must admit four systems of imprimitivity of three letters each, from which it follows that a constituent of degree 10 is not simply isomorphic to it. The remaining partitions, 8, 15, 9, 15, 10, 15 should also be simple isomorphisms between the two constituents. Let a constituent of degree 15 be primitive. It is not multiply transitive. Then being of class 8 it is the primitive group of degree 15 and order 720, simply isomorphic to the symmetric group of that order. But this symmetric group can not be written transitively on eight, nine, or ten letters. Then a constituent of degree 15 is imprimitive, and since it is generated by substitutions of degree 9 and order 3, it has five systems of three letters each permuted according to the alternating group of degree 5. Since the second constituents are alternating groups, the partitions 8, 15 and 9, 15 are impossible. A constituent of degree 10 is imprimitive and has systems of two letters permuted according to the alternating group of order 60. Its order is not divisible by 9.

Now consider any two substitutions of degree 15, s_1 and s. If s displaces more than five letters new to s_1 and is not commutative with s_1, s^2s_1s displaces exactly five letters new to s_1 and, as has just been shown, does not connect two cycles of s_1. Then

$$s=(a_1x_1—)\,(b_1x_2—)\,(c_1x_3—)\,(d_1x_4—)\,(e_1x_5x_6),$$

say. If $s=(b_1x_2a_2)\ldots.$, $s^2s_1s=(x_1b_1b)\ldots.$, but s can not replace a_3 by a b. Hence s does not connect two cycles of s_1. By hypothesis the transitive constituents of $\{s_1, s\}$ are in part icosaedral. If there are one or more tetraedral constituents $\{s_1, s\}$ is a direct product and of class less than 15. *Hence if* s

has just two new letters in one of its cycles, it has two new letters in every cycle, and $\{s, s_1\}$ *is a simple isomorphism between five alternating groups of order* 60. *If any two substitutions of degree* 15 *are so related that one connects cycles of the other they are commutative, and have at most nine letters in common.*

The last of the groups D is

$$\{s_1,\ s_2=a_1a_2a_4\,.\,b_1b_2b_4\,.\,c_1c_2c_4\,.\,d_1d_2d_4\,.\,e_1e_2e_4\}.$$

A substitution s_3 exists that connects two of the constituents of D, and it may be taken uniquely as

$$s_3=a_1b_1c_1\,.\,a_2b_2c_2\,.\,a_3b_3c_3\,.\,a_4b_4c_4\,.\,x_1x_2x_3\,.$$

Next, s_4 may be chosen with at most one letter new to $\{s_1, s_2, s_3\}$ in any cycle. Then there are just two distinct forms that s_4 may assume:

$$s_4=a_1d_1b_1\,.\,a_2d_2b_2\,.\,a_3d_3b_3\,.\,a_4d_4b_4\,.\,x_1y_1x_2,$$

and

$$s_4=a_1x_1a_2\,.\,b_1x_2b_2\,.\,c_1x_3c_2\,.\,d_1y_1d_2\,.\,e_1y_2e_2\,.$$

If we proceed with the second form of s_4, we are led at once to a substitution $s_5=a_1d—\ldots$, so that we are at liberty to use for s_4 the first only, and quite similarly we have for the next step

$$s_5=a_1e_1b_1\,.\,a_2e_2b_2\,.\,a_3e_3b_3\,.\,a_4e_4b_4\,.\,x_1z_1x_2,$$

and then finally

$$s_6=a_1x_1a_2\,.\,b_1x_2b_2\,.\,c_1x_3c_2\,.\,d_1y_1d_2\,.\,e_1z_1e_2$$

closes the series of generators of a transitive group H of degree 25 and order 360: the direct product of the two icosaedral groups

$$\begin{gathered}\{a_1a_2a_3\,.\,b_1b_2b_3\,.\,c_1c_2c_3\,.\,d_1d_2d_3\,.\,e_1e_2e_3,\quad a_1a_2a_4\,.\,b_1b_2b_4\,.\,c_1c_2c_4\,.\,d_1d_2d_4\,.\,e_1e_2e_4,\\ a_1a_2x_1\,.\,b_1b_2x_2\,.\,c_1c_2x_3\,.\,d_1d_2y_1\,.\,e_1e_2z_1\}\end{gathered}$$

and

$$\begin{gathered}\{a_1b_1c_1\,.\,a_2b_2c_2\,.\,a_3b_3c_3\,.\,a_4b_4c_4\,.\,x_1x_2x_3,\quad a_1b_1d_1\,.\,a_2b_2d_2\,.\,a_3b_3d_3\,.\,a_4b_4d_4\,.\,x_1x_2y_1,\\ a_1b_1e_1\,.\,a_2b_2e_2\,.\,a_3b_3e_3\,.\,a_4b_4e_4\,.\,x_1x_2z_1\}.\end{gathered}$$

The primitive group G can contain no other substitution of degree 15, so that H is an invariant subgroup, and *the degree of no primitive group of class* 15 *exceeds* 25. Now instead of the special case before us let us turn to the imprimitive group H' generated by

$$\begin{gathered}s_2=a_1a_2\,.\,b_1b_2\,.\,c_1c_2\ldots\,j_1j_2\,.\,k_1k_2,\\ s_3=a_1a_3\,.\,b_1b_3\,.\,c_1c_3\ldots\,j_1j_3\,.\,k_1k_3,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ s_k=a_1a_k\,.\,b_1b_k\,.\,c_1c_k\ldots\,j_1j_k\,.\,k_1k_k,\\ s'_2=a_1b_1\,.\,a_2b_2\,.\,a_3b_3\ldots\,a_jb_j\,.\,a_kb_k,\\ s'_3=a_1c_1\,.\,a_2c_2\,.\,a_3c_3\ldots\,a_jc_j\,.\,a_kc_k,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ s'_k=a_1k_1\,.\,a_2k_2\,.\,a_3k_3\ldots\,a_jk_j\,.\,a_kk_k,\end{gathered}$$

in which H (after obvious changes in the lettering) is a characteristic subgroup. H' is of order $(k!)^2$. A primitive group of class $2k$ is generated by H' and the following substitution:

$$\begin{array}{r}
t = (a_1)\,(a_2b_1)\,(a_3c_1)\,(a_4d_1)\ldots\ldots(a_jj_1)\,(a_kk_1),\\
(b_2)\,(b_3c_2)\,(b_4d_2)\ldots\ldots(b_jj_2)\,(b_kk_2),\\
(c_3)\,(c_4d_3)\ldots\ldots(c_jj_3)\,(c_kk_3),\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
(j_j)\,(j_kk_j),\\
(k_k).
\end{array}$$

This group is of order $2(k!)^2$ and is simply isomorphic to the intransitive group

$$(aa'\cdot bb'\cdot\ldots\ldots\cdot kk')\,(abc\ldots\ldots k)\text{ all }(a'b'c'\ldots\ldots k')\text{ all}.$$

It is the largest primitive group of degree k^2 in which H' is invariant.* The transitive subgroups of this group in which

$$(abc\ldots\ldots k)\text{ pos }(a'b'c'\ldots\ldots k')\text{ pos}$$

is invariant are

$$\begin{array}{l}
(aa'\cdot bb'\cdot cc'\cdot\ldots\ldots\cdot kk')\,(abc\ldots\ldots k)\text{ pos }(a'b'c'\ldots\ldots k')\text{ pos},\\
(aa'\cdot bb'\cdot cc'\cdot\ldots\ldots\cdot kk')\,\{(abc\ldots\ldots k)\text{ all }(a'b'c'\ldots\ldots k')\text{ all}\}\text{ pos},\\
(ab'ba'\cdot cc'\cdot dd'\cdot\ldots\ldots\cdot kk')\,\{(abc\ldots\ldots k)\text{ all }(a'b'c'\ldots\ldots k')\text{ all}\}\text{ pos},
\end{array}$$

of the orders $(k!)^2/2$, $(k!)^2$, $(k!)^2$, respectively.† Maximal subgroups of these three groups are

$$\begin{array}{l}
(aa'\cdot bb'\cdot cc'\cdot\ldots\ldots\cdot kk')\,(abc\ldots\ldots j)\text{ pos }(a'b'c'\ldots\ldots j')\text{ pos},\\
(aa'\cdot bb'\cdot cc'\cdot\ldots\ldots\cdot kk')\,\{(abc\ldots\ldots j)\text{ all }(a'b'c'\ldots\ldots j')\text{ all}\}\text{pos},\\
(ab'ba'\cdot cc'\cdot dd'\cdot\ldots\ldots\cdot kk')\,\{(abc\ldots\ldots j)\text{ all }(a'b'c'\ldots\ldots j')\text{ all}\}\text{ pos},
\end{array}$$

respectively, with respect to which each can be represented as a primitive group of degree k^2, when k is greater than 3. The first, of order $(k!)^2/2$, is generated by H and t; the second, of order $(k!)^2$, is generated by H, t and

$$s_2s_2' = (a_1b_2)\,(a_2b_1)\,(c_1c_2)\ldots\ldots(k_1k_2)\,(a_3b_3)\ldots\ldots(a_kb_k).$$

It contains the preceding group of order $(k!)^2/2$. The third group, also of order $(k!)^2$, is generated by H, s_2s_2' and

$$\begin{array}{r}
s_2t = (a_1b_1b_2a_2)\,(c_1b_3c_2a_3)\,(d_1b_4d_2a_4)\ldots\ldots(k_1b_kk_2a_k),\\
(c_3)\,(c_4d_3)\ldots\ldots\ldots\ldots\ldots(c_jj_3)\,(c_kk_3),\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
(j_j)\,(j_kk_j),\\
(k_k).
\end{array}$$

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* AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVIII (1906), p. 236.

† Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXI (1899), p. 299, Theorems III and IV.

It does not contain the first of the three as a subgroup. These groups are simply transitive and are of class $3k$, when k is greater than 3. This last statement can readily be proved in connection with the important proposition that the group $\{H', t\}$ is the largest group on the same k^2 letters in which H is invariant. Both statements are known to be true when k is 4.* Let G' be the largest group of degree k^2 in which H is invariant. It is a simply transitive primitive group because systems of imprimitivity of H with one letter in common can be chosen in only two ways. Its subgroup leaving fixed the letter k_k is intransitive, and has the same two sets of intransitivity as $\{s_2, \ldots, s_j, s'_2, \ldots, s'_j, t\}$, the subgroup of $\{H', t\}$ that leaves k_k fixed. The larger constituent of the latter subgroup is the primitive group of degree $(k-1)^2$ belonging to the family $\{H', t\}$. Since $\{H', t\}$ when k is 4 is not contained in a larger group in which H is invariant, the group G' coincides with $\{H', t\}$ for all values of k greater than 3. The smaller constituent in the subgroup G'_1 that leaves k_k fixed is the imprimitive group

$$(a_k k_1 \cdot b_k k_2 \cdot \ldots \cdot j_k k_j)\,(a_k b_k \ldots j_k) \text{ all } (k_1 k_2 \ldots k_j) \text{ all}$$

of degree $2(k-1)$ and order $2[(k-1)\,!]^2$; all the substitutions in its tail are of degree $2(k-1)$. Those substitutions of G'_1 that are in the division of G'_1 that involves only positive substitutions of

$$(a_k b_k \ldots j_k) \text{ all } (k_1 k_2 \ldots k_j) \text{ all}$$

are of degree $3k$ at least, if they are of degree $3(k-1)$ or more in the letters of the other constituent of G'_1. The substitutions of G'_1 which are in the tail with respect to the direct product of the two symmetric groups of degree $k-1$ are of degree $5(k-1)$ at least, if they are of degree $3(k-1)$ or more in the letters of the constituent of degree $(k-1)^2$. Then by a complete induction from the groups of degree 16, the three primitive groups above are of class $3k$. What intransitive subgroups of $\{H', t\}$ contain H, when this group is written as a transitive group of degree $2k$? Only

$$\{(abc \ldots k) \text{ all } (a'b'c' \ldots k') \text{ all}\} \text{pos},$$
$$(abc \ldots k) \text{ all } (a'b'c' \ldots k') \text{ pos},$$

and

$$(abc \ldots k) \text{ pos } (a'b'c' \ldots k') \text{ all},$$

include

$$(abc \ldots k) \text{ pos } (a'b'c' \ldots k') \text{ pos},$$

and they are imprimitive when represented on k^2 letters.

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* Miller, "On the Primitive Substitution Groups of Degree 16," AMERICAN JOURNAL OF MATHEMATICS, Vol. XX (1898), p. 234. It may be noted that G_{20} does not contain G_{17} as a self-conjugate subgroup.

In my paper on the groups of class 12 it is stated that a certain doubly transitive group of degree 28, as well as its maximal subgroup of degree 27, seem to be new. As a matter of fact the generators there given correspond respectively to the substitutions $[1,0,0,0,0,0]$, $[0,1,0,0,0,0]$, $[1,1,1,1,1,1]$, $[0,0,0,0,1,0]$, $[0,1,0,0,0,1]$, $[0,0,1,0,1,1]$, $[0,0,1,1,1,1]$, as is readily seen if we put a_1 for the letter $(0,0,1,1,0,0)$. Although Jordan gives this group and its sixty-three substitutions of order 2 and degree 12 in the *Traité* of 1870, page 229, he omits it and its subgroup of order 51840 from the list of primitive groups of the first thirteen classes published in 1872.* My existence proofs on pages 258 and 259 of the paper on the "Groups of Class 12" are faulty. The present identifications are designed to replace them. The substitutions $s_1, s_2, \ldots, s_5', s_6', s_7'$, of that paper, page 259, correspond to ab, ac, ad, de, df, dg, dh, respectively, and if we associate with ad the letter a_1 the symmetric group of degree 8 permutes the twenty-eight binary products ab, ac, ad, $\ldots$, according to the substitutions of our simply transitive primitive group of degree 28 and class 12.

To make reference easier a table of the seven primitive groups of class 15 is given below. The second group is doubly and the third triply transitive; all the others are simply transitive.

Degree	Order	Generating Substitutions
16	80	$G_1 = \{ab \,.\, cd \,.\, ef \,.\, gh \,.\, ij \,.\, kl \,.\, mn \,.\, op,\ boejc \,.\, dpknl \,.\, fhgim\}$
16	240	$G_2 = \{G_1,\ bmn \,.\, cik \,.\, deh \,.\, flo \,.\, gpj\}$
17	4080	$G_3 = \{G_2,\ aq \,.\, fk \,.\, dg \,.\, ej \,.\, il \,.\, mn \,.\, co \,.\, hp\}$
21	2520	$G_4 = \{agb \,.\, chl \,.\, dim \,.\, ejn \,.\, fko,\ ahi \,.\, blm \,.\, cpd \,.\, eqs \,.\, frt,\ ajk \,.\, bno \,.\, cqr \,.\, dst \,.\, euf\}$
25	7200	$G_5 = \{H,\ a_2b_1 \,.\, a_3c_1 \,.\, a_4d_1 \,.\, a_5e_1 \,.\, b_3c_2 \,.\, b_4d_2 \,.\, b_5e_2 \,.\, c_4d_3 \,.\, c_5e_3 \,.\, d_5e_4\}$
25	14400	$G_6 = \{G_5,\ a_1b_2 \,.\, a_2b_1 \,.\, c_1c_2 \,.\, d_1d_2 \,.\, e_1e_2 \,.\, a_3b_3 \,.\, a_4b_4 \,.\, a_5b_6\}$
25	14400	$G_7 = \{H,\ a_1b_2 \,.\, a_2b_1 \,.\, c_1c_2 \,.\, d_1d_2 \,.\, e_1e_2 \,.\, a_3b_3 \,.\, a_4b_4 \,.\, a_5b_5,\ a_1b_1b_2a_2 \,.\, c_1b_3c_2a_3 \,.\, d_1b_4d_2a_4 \,.\, e_1b_5e_2a_5 \,.\, c_4d_3 \,.\, c_5e_3 \,.\, d_5e_4\}$

In the last three groups

$$H = \{a_1a_2a_3 \,.\, b_1b_2b_3 \,.\, c_1c_2c_3 \,.\, d_1d_2d_3 \,.\, e_1e_2e_3,\quad a_1a_4a_5 \,.\, b_1b_4b_5 \,.\, c_1c_4c_5 \,.\, d_1d_4d_5 \,.\, e_1e_4e_5,$$
$$a_1b_1c_1 \,.\, a_2b_2c_2 \,.\, a_3b_3c_3 \,.\, a_4b_4c_4 \,.\, a_5b_5c_5,\quad a_1d_1e_1 \,.\, a_2d_2e_2 \,.\, a_3d_3e_3 \,.\, a_4d_4e_4 \,.\, a_5d_5e_5\}.$$

STANFORD UNIVERSITY, *September*, 1916.

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* *Comptes Rendus, loc. cit.*

Multiple Integrals Over Infinite Fields, and the Fourier Multiple Integral.[*]

BY BURTON H. CAMP.

INTRODUCTION.

1. Multiple integrals over infinite fields have been defined in recent years in different ways by Pierpont,[†] Vallée-Poussin,[‡] and Hardy.[§] The integrals of Pierpont and of Vallée-Poussin exist only when they exist absolutely, but this is not the case with Hardy's integral. It is shown here that, for functions whose integrals over finite fields exist, these three infinite integrals are equivalent when any one of them exists absolutely. This paper is divided into three parts. In the first part (§§ 2–5) are developed relations between Hardy's integral and the iterated integrals, certain new theorems on inversion of the order of integration, and some fundamental properties of multiple infinite integrals, most of which are known for the case of integrals over finite fields but not for the infinite case. Among these last is the second theorem of the mean. Heretofore, this has not been proved, apparently, in the infinite case even for functions of one variable. The second part (§§ 6–8) deals with the behavior of integrals with respect to parameters. I am mainly concerned here with the infinite integrals, but it is necessary incidentally to extend to the case of integrals over finite fields in space of several dimensions theorems which are known only for integrals over one-dimensional fields. To save space, and because novel methods would not be introduced, the proofs of many of the statements in this part are not given. They have to do, however, with fundamental matters, including what are, in one dimension, well-known tests for termwise integration and differentiation; in the infinite case they include the so-called fundamental theorem of the integral calculus. A résumé of the third part (§§ 9–11), which contains certain applications to Fourier's double and quadruple integrals, is given at the beginning of § 9.

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[*] Read in part before the American Mathematical Society, February 27, 1915, April 29, 1916.

[†] *Transactions of the American Mathematical Society,* Vol. VII (1906), p. 169.

[‡] *Cours d'Analyse,* 2d ed., Vol. II, pp. 69, 70, 108, 109.

[§] *Messenger of Mathematics,* Vol. XXXII (1902–3), pp. 92 ff.

FUNDAMENTAL THEORY, §§ 2–5.

2. **Elementary Theorems.** As we shall not be concerned usually with integration over finite fields, we make the assumption always that the functions used are *absolutely* L-integrable* in each *limited* measurable set of their domain of definition. Let R denote the infinite rectangle† where $x, y \geqq 0$, and $R_{\alpha,\beta}$ the finite rectangle where $0 \leqq x \leqq \alpha$, $0 \leqq y \leqq \beta$. Let $f(x, y)$ be defined in R. We may then define the following integrals, when the limits in question exist:

$$J_1(f) = \lim_{\alpha,\beta=\infty} \int_{R_{\alpha,\beta}} f dx dy,$$

$$J_2(f) = \lim_{\alpha=\infty} \lim_{\beta=\infty} \int_{R_{\alpha,\beta}} f dx dy,$$

$$J_3(f) = \lim_{\beta=\infty} \lim_{\alpha=\infty} \int_{R_{\alpha,\beta}} f dx dy,$$

$$J_4(f) = \lim_{\alpha=\infty} \int_0^{\alpha} dx \lim_{\beta=\infty} \int_0^{\beta} f dy = \int_0^{\infty} dx \int_0^{\infty} f dy,$$

$$J_5(f) = \lim_{\beta=\infty} \int_0^{\beta} dy \lim_{\alpha=\infty} \int_0^{\alpha} f dx = \int_0^{\infty} dy \int_0^{\infty} f dx.$$

The integral $J_1(f)$ is Hardy's definition referred to in the introduction, J_4 and J_5 are the usual iterated integrals, J_2 and J_3 are introduced chiefly for convenience in developing the theory of J_1, J_4 and J_5. When it is desired to integrate a function over a larger portion or over all of the infinite two-dimensional field, we let R_1 denote the rectangle $(x, y \geqq 0)$, R_2 the rectangle $(x < 0 \leqq y)$, R_3 the rectangle $(x, y < 0)$, and R_4 the rectangle $(y < 0 \leqq x)$, and let $R = R_1 + R_2 + R_3 + R_4$, and then define the integral over R as the sum of the integrals over R_1, R_2, R_3, and R_4. These definitions could be generalized slightly by using the concept of the Harnack-Lebesgue integral, but that is not worth while for my present purpose. By $\lim_{\alpha,\beta=\infty}$ is meant the double limit‡ as α, β diverge to plus infinity passing through all positive values. Most of the theorems, though stated only for two dimensions, may obviously be extended to n dimensions.

The two following theorems are almost immediate consequences of our definition, and the succeeding lemma is known.

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* *I. e.*, integrable in the sense of Lebesgue.

† The case of n variables is not essentially different from the case of two, but the treatment would be cumbersome. In § 6, however, where finite fields only are employed, and there is no loss of simplicity, n variables are used.

‡ *E. g.*, Pierpont, *Theory of Functions of Real Variables*, Vol. I (1905), p. 195.

THEOREM 1.[*] *A necessary and sufficient condition that $J_1(f)$ shall exist is that, for every pair of constants $h, k>0$,*

$$\lim_{\alpha,\beta=\infty}\int_r f dx dy=0,$$

where $r=R_{\alpha+h,\,\beta+k}-R_{\alpha,\,\beta}$.

THEOREM 2. *If $J_i(f)$ and $J_i(g)$ exist, and c is a constant, $J_i(f)+J_i(g)=J_i(f+g)$, and $J_i(cf)=cJ_i(f)$.*

LEMMA. *If for all values of $\Delta\alpha, \Delta\beta\geqq 0$, $\phi(\alpha+\Delta\alpha, \beta+\Delta\beta)\geqq\phi(\alpha,\beta)$,* and *$\phi(\alpha,\beta)$ is less than a fixed number M, then $lim_{\alpha,\beta}\,\phi(\alpha,\beta)$, $lim_\alpha\, lim_\beta\,\phi(\alpha,\beta)$, $lim_\beta\, lim_\alpha\,\phi(\alpha,\beta)$ all exist and are equal and not greater than M.*

THEOREM 3. *If, in general,*[†] *$0\leqq f(x,y)\leqq g(x,y)$, and if $J_i(g)$ exists, so does $J_i(f)$, and $J_i(f)\leqq J_i(g)$.*

If $i=1$, 2, or 3, this is an immediate consequence of the lemma. If $i=4$, by hypothesis there exists, except perhaps for a null set, a positive L-integrable function,

$$L(x)=\int_0^\infty g dy,$$

whose integral from 0 to α is a monotonic increasing function of α and has the limit $J_4(g)$. Except perhaps for a null set of x's,

$$\int_0^\beta f dy=\leqq\int_0^\beta g dy,$$

and the former is a monotonic increasing function of β whose limit $l(x)$ is less than or at most equal to $L(x)$. The integral of $l(x)$ from 0 to α is a monotonic increasing function of α and hence

$$\lim_\alpha\int_0^\alpha l(x)\,dx=J_4(f)\leqq\lim_\alpha\int_0^\alpha L(x)\,dx=J_4(g).$$

COROLLARY.[‡] *The function $J_i(f)$ exists if $J_i(g)$ does, when $f(x,y)=g(x,y)$ in a part of R and zero elsewhere, provided that the field is so divided that f is L-integrable over every limited, measurable set in each part.*

THEOREM 4. *A necessary and sufficient condition that $J_i(f)$ shall exist is that there shall exist two other functions $g(x,y)$, $G(x,y)$, for both of*

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[*] The proof of this depends on the following theorem, well known for functions of *one* variable: A necessary and sufficient condition that the double limit of $\phi(\alpha,\beta)$ shall exist, as α,β become infinite, is that, for every pair of constants $h, k>0$, $\lim_{\alpha,\,\beta=\infty}[\phi(\alpha+h, \beta+k)-\phi(\alpha,\beta)]=0$.

[†] *I. e.*, except perhaps at a set of points of measure zero. Such a set will be called a null set.

[‡] This is not necessarily true if g has both signs. Indeed, this property, together with the properties in Theorem 2, would be sufficient to show that $|f|$ would be integrable, no matter what the definition of integration for f.

which J_i *exists, such that in general* $g(x, y) < f(x, y) < G(x, y)$. *Then also* $J_i(g) \leqq J_i(f) \leqq J_i(g)$.

Since $0 < f - g < G - g$, $J_i(f-g) \leqq J_i(G-g)$. Then, since $J_i(g)$ exists, we have $J_i(f) \leqq J_i(G)$.

COROLLARY 1. *The same is true, with an appropriate change in the last statement, if* $g \leqq f \leqq G$.

COROLLARY 2. *The function* $J_i(f)$ *exists if it exists absolutely, or if* f *is the product of two positive functions which are less than unity and for each of which* J_i *exists, or if it is the p-th power* $(p > 1)$ *of such a function.*

COROLLARY 3. *Under the usual conditions the mean-value theorem,* $mJ_i(g) \leqq J_i(fg) \leqq MJ_i(g)$, *is valid.*

COROLLARY 4. *If* $J_i(f^2)$ *and* $J_i(g^2)$ *exist, then* $J_i(fg)$ *exists absolutely, and the inequality of Schwarz,* $J_i^2(fg) \leqq J_i(f^2) \cdot J_i(g^2)$, *is valid.*[*]

CASE I: $(i=1)$. Let $\psi = f$ and $\phi = 0$, $\gamma = g$ and $\delta = 0$ where $fg \geqq 0$, and let $\phi = f$ and $\psi = 0$, $\delta = g$ and $\gamma = 0$ where $fg < 0$. The same equalities hold when both sides are squared, and by the corollary of Theorem 3 the following exist: $J_1(\psi^2)$, $J_1(\phi^2)$, $J_1(\gamma^2)$, $J_1(\delta^2)$. By the inequality for finite fields,

$$\left(\int_{R_{\alpha,\beta}} \psi\gamma\, dx\, dy\right)^2 \leqq \left(\int_{R_{\alpha,\beta}} \psi^2 dx\, dy\right)\left(\int_{R_{\alpha,\beta}} \gamma^2 dx\, dy\right) \leqq J_1(\psi^2) \cdot J_1(\gamma^2).$$

Since $\psi\gamma \geqq 0$, the function of α, β on the left satisfies the conditions of the lemma to Theorem 3. Therefore $J_1^2(\psi\gamma)$ exists, and so does $J_1(\psi\gamma)$. Similarly, $J_1(\phi\delta)$ exists, and since by the early equalities $|fg| = \psi\gamma - \phi\delta$, so does $J_1(|fg|)$. Now, proceeding to the limit in the relation,

$$\left(\int_{R_{\alpha,\beta}} fg\, dx\, dy\right)^2 \leqq \left(\int_{R_{\alpha,\beta}} f^2 dx\, dy\right)\left(\int_{R_{\alpha,\beta}} g^2 dx\, dy\right),$$

we may establish the proposition.

CASE II: $(i > 1)$. This may be referred to Case I by the use of the corollary to Theorem 5 and the first part of Corollary 2, Theorem 4.

3. **The Integrals of Pierpont and Vallée-Poussin.**[†] In considering these definitions we shall assume as in § 2 that $f(x, y)$ is absolutely L-integrable in every limited measurable part of the field R. This is not done by Pierpont, but it is better for our present purpose to dispense with the extra element of generality which he obtains by omitting this condition. Let R be defined as before, and let $D_1 < D_2 < \ldots.$ be any infinite sequence of limited, measurable

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[*] It does not follow, as in the finite case, that f is integrable if f^2 is.

[†] Cf. § 1.

point sets of such a nature that each D is wholly in R and that every point of R is in some D.

Vallée-Poussin's Definition, $V(f)$. Let $f \geqq 0$. If the following limit exists and is independent of the choice of D's,

$$V(f)=\int_R f dx dy = \lim_{i=\infty} \int_{D_i} f dx dy. \tag{1}$$

Let f be unrestricted as to sign. Set $f=f_1-f_2$, where $f_1, f_2 \geqq 0$. When the integrals on the right exist,

$$V(f)=\int_R f dx dy = \int_R f_1 dx dy - \int_R f_2 dx dy. \tag{2}$$

Pierpont's Definition, $P(f)$. Let f be unrestricted as to sign. Let the following be an infinite sequence of limited measurable sets, $E_1, E_2, \ldots\ldots$; and let E_i contain every point whose distance from the origin is less than i; it may contain other points. When the following limit exists and is independent of the choice of E's,

$$P(f)=\int_R f dx dy = \lim_{i=\infty} \int_{E_i} f dx dy. \tag{3}$$

It will be shown that this definition is equivalent to the definition:

$$P(f)=\lim_{i=\infty} \int_{D_i} f dx dy. \tag{4}$$

It is obvious that if (3) exists (4) exists and gives the same result. To prove the converse we let (4) exist. Suppose that, for a certain E set, the limit in (3) should not exist, *i. e.*, suppose that the sequence

$$\int_{E_1} f dx dy, \quad \int_{E_2} f dx dy, \ldots\ldots \tag{5}$$

should diverge. In this sequence there would exist a divergent sub-sequence

$$\int_{E'} f dx dy, \quad \int_{E''} f dx dy, \ldots\ldots \tag{6}$$

for which $E' < E'' < \ldots\ldots$ For example, we might take $E'=E_1$, and then, since E_1 is limited, there would exist an E_i which would include all points as near to the origin as any of those of E_1; let $E''=E_i$, or some E farther on in the sequence, if necessary, in order to make (6) diverge. This sub-sequence would be a D sequence.

LEMMA. *If $f(x, y) \geqq 0$ and the following limit exists for one set of D's, it does for every set:*

$$\lim_{i=\infty} \int_{D_i} f dx dy.$$

THEOREM 5. *(a) If $f \geqq 0$, $V(f)=P(f)=J_1(f)$, provided one of these exists. (b) If $V(f)$ or $P(f)$ exists, it exists absolutely, and then $V(f)=P(f)=J_1(f)$. (c) $J_1(f)$ may exist when $V(f)$ and $P(f)$ do not.*

(*a*) Since $R_{a,\beta}$ is a special case of E_i, if P exists, so does J_1, and $J_1=P$. By the lemma, if J_1 exists, so does V, and $V=J_1$. By definition $V=P$.

(*b*) It is known and obvious that if $V(f)$ exists so does $V(|f|)$, and

$$V(|f|)=\lim_{i=\infty}\int_{D_i} f_1 dxdy+\lim_{i=\infty}\int_{D_i} f_2 dxdy,\quad V(f)=\lim_{i=\infty}\int_{D_i} f_1 dxdy-\lim_{i=\infty}\int_{D_i} f_2 dxdy,$$

where $|f|=f_1+f_2$, $f=f_1-f_2$. By (*a*), then, $J_1(f_1)=V(f_1)$, $J_1(f_2)=V(f_2)$. Therefore, $J_1(f)=J_1(f_1)-J_1(f_2)=V(f)$. A similar result may be obtained for $P(f)$, since it may be shown that this integral also obeys Theorem 2. Thus (*b*) is proved if the hypothesis be that $V(f)$ exists. It only remains to show that if $P(f)$ exists so does $V(f)$. The following equation is true if two of the limits exist:

$$P(f)=\lim_{i=\infty}\int_{D_i} f dxdy=\lim_{i=\infty}\int_{D_i} f_1 dxdy-\lim_{i=\infty}\int_{D_i} f_2 dxdy.$$

If it is true, $V(f)$ exists. If it is not true, but $P(f)$ exists; since $f_1, f_2 \geqq 0$, both of the last two integrals must diverge to plus infinity for some set of D's, and therefore by the lemma for all sets. On this hypothesis we will show that there exists a set of D's for which the first limit does not exist. Let R_1 be a circle of unit radius about the origin. Let

$$k_1=\int_{R_1} f dxdy.$$

Since the integral of f_1 over R_1 is not infinite, and since by definition $f_1=f$ where $f \geqq 0$, there must exist outside of R_1 a limited set A_1, in which $f \geqq 0$, such that

$$\int_{A_1} f_1 dxdy>1-k_1.$$

Hence,

$$\int_{R_1+A_1} f dxdy=\int_{R_1} f dxdy+\int_{A_1} f dxdy>k_1+1-k_1=1.$$

Let $D_1=R_1+A_1$. Let R_2 be a circle about the origin including all of D_1. Let

$$k_2=\int_{R_2} f dxdy.$$

The integral of f_1 over R_2 is not infinite, and therefore, as before, there exists a set A_2 outside of R_2 where $f \geqq 0$ such that

$$\int_{R_2+A_2} f dxdy=\int_{R_2} f dxdy+\int_{A_2} f dxdy=\int_{R_2} f dxdy+\int_{A_2} f_1 dxdy>2.$$

Let $D_2=R_2+A_2$; etc.

(*c*) Hardy[*] has given some interesting examples of this. One which will concern us later is $J_1(\cos xy) = \pi/2$.

COROLLARY.[†] *If* $J_i(|f|)$ *exists, so does* $J_j(|f|)$, *and* $J_i(|f|) = J_j(|f|)$, *and* $J_i(f) = J_j(f)$.

Vallée-Poussin has shown that if $V(|f|)$ exists, $J_4(|f|) = J_5(|f|) = V(|f|)$, and that $J_4(f) = J_5(f) = V(f)$. By the theorem, then, the corollary is proved for the cases $i, j = 1, 4, 5$. The remaining cases may easily be proved by the use of the lemma to Theorem 3.

4. **More Existence and Inversion Theorems.** THEOREM 6. *Let* $J_i(f)$ *exist. A necessary and sufficient condition that* $J_j(f)$ *shall exist is that there shall exist a function* $g(x, y)$ *which is in general less than* $f(x, y)$ *for which both* J*'s exist.*

Necessity. Let $q(x, y) > 0$, and let $J_j(q)$ exist. We may take $g = f - q$, for, since $J_j(f)$ and $J_j(q)$ exist, so does $J_j(f - q)$.

Sufficiency. Since $J_i(f - g)$ exists and $f - g \geqq 0$, $J_j(f - g)$ exists by the preceding corollary. Since now $J_j(g)$ exists, so does $J_j(f - g + g)$.

COROLLARY 1. *Let* $J_i(f)$ *exist. A necessary and sufficient condition that* $J_j(f) = J_i(f)$ *is that there shall exist a function* $g(x, y)$, *in general less than* $f(x, y)$, *for which this is true.*

$J_i(f - g) = J_j(f - g) = J_j(f) - J_j(g)$. Add the equality, $J_i(g) = J_j(g)$.

COROLLARY 2. *Let one of the iterated integrals* $J_4(f)$, $J_5(f)$ *exist. A necessary and sufficient condition that inversion of the order of iteration be allowable is that there shall exist a function* $g(x, y)$, *in general less than* $f(x, y)$, *for which the inversion is allowable.*

COROLLARY 3.[‡] *A necessary and sufficient condition that both iterated integrals of* f *shall exist and be equal is that this shall be true of two functions* $g(x, y)$, $G(x, y)$ *for which in general* $g < f < G$.

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[*] *Loc. cit.*, § 1, pp. 159 ff.

[†] Vallée-Poussin, *loc. cit.*, Vol. II, p. 121. Bromwich (*Proceedings of the London Mathematical Society*, Vol. I (1903), p. 181, third footnote) indicates that this result follows from the lemma to Theorem 3, but really only the last part, in which it is said that $J_1 = J_2 = J_3$, follows from that lemma. Indeed, if it did follow also that $J_1 = J_4 = J_5$, as he states, his result would antedate its discovery for the finite case by four years (see also Hobson, same *Proceedings*, Vol. VIII (1910), p. 22).

I am omitting from this paper the subject of transformation of variables in these integrals. It is true, however, that, if any J_i exists absolutely, it may be transformed into an equivalent double, or iterated, integral over the unit square. Another proof of the corollary may be devised by the use of transformations of this sort.

[‡] Cf. W. H. Young, same *Proceedings*, Vol. IX (1910–11), p. 323. Young's theorem gives only sufficient criteria that inversion of the order of iteration be permissible, and involves the additional hypothesis that this inversion be permissible for integrals of g and G over one-way infinite fields.

COROLLARY 4.* (*a*) *Inversion of the order of iteration of f over the one-way infinite field* $R_\alpha(0\leqq x\leqq \alpha, 0\leqq y)$ *is allowable if one of the iterated integrals of* $|f|$ *exists.* (*b*) *If one of the iterated integrals of f over* R_α *exists, a necessary and sufficient condition that inversion be allowable is that there shall exist a function* $g(x, y)$, *in general less than* $f(x, y)$, *for which it is allowable.* (*c*) *A necessary and sufficient condition that both iterated integrals over* R_α *shall exist and be equal is that there shall exist two functions* $g(x, y)$, $G(x, y)$ *for which this is so and for which in general* $g<f<G$.

COROLLARY 5. *The theorem and previous corollaries are true if we write, instead of* $g<f(f<G)$, $g\leqq f(f\leqq G)$, *or if, in the theorem and first two corollaries, we substitute* $G>f$ *for* $g<f$.

THEOREM 7. (*a*) *If* $J_1(f)$ *exists, and* $J_2(f)$ *exists over* R_α *for sufficiently large values of* α *(say* $\alpha>C$*) then* $J_2(f)=J_1(f)$; *similar statements may be made for* J_3 *and the one-way infinite field* $R_\beta(0\leqq y\leqq \beta, 0\leqq x)$. (*b*) *Let inversion*† *of the order of iteration over* $R_\alpha(\alpha>C)$ *be allowable; then* $J_2(f)=J_4(f)$ *if either exists; and similar statements may be made for* J_3 *and* J_5 *and* R_β.

COROLLARY. *Sufficient conditions that all the J's shall exist and be equal are that inversion be allowable over both* R_α *and* $R_\beta(\alpha, \beta>C)$, *and that* J_1 *and one of each of the pairs* (J_2, J_4), (J_3, J_5) *shall exist. Then also all the J's are equal.*

The proofs of the theorems and corollaries of this section are simple.

5. **Second Theorem of the Mean.**‡ THEOREM 8. *Let* $f(x, y)$ *be limited in R and L-integrable over every finite measurable set, and let* $g(x, y)$ *be absolutely L-integrable in R. Then, if* $l\leqq f(x, y)\leqq L$, *there exists a k such that*

$$J=\int_R f(x, y)\,g(x, y)\,dxdy=l\int_{R-A_k} g\,dxdy+L\int_{A_k} g\,dxdy;$$

where, if f has the property that the set of points E_k, *where* $f=k$, *is always null,* A_k *is defined as the set where* $f>k$; *and otherwise* A_k *is the set where*

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* Parts (*a*) and (*c*) are known. See also note to Theorem 7 (*b*). By the existence of the integral of f over R_α we mean the existence over R of the integral of the function which equals f in R_α and zero elsewhere.

† For criteria for this condition we have, not only Theorem 6, Corollary 4, but also the usual condition that, assuming $|f|$ L-integrable over finite fields, termwise integration with respect to x of the β sequence—viz: the integral of f with respect to y from 0 to β—be allowable in $(0, \alpha)$. This assumes implicitly the existence of one iterated integral.

‡ Proved for the case where R is finite by Lebesgue, *Annales de l'École Normale*, (3) Vol. XXVII (1910), p. 444; proved in a slightly more general form by the author, *Mathematische Annalen*, Vol. LXXV (1914), p. 285. There will be used in this section much of the notation and some of the results of the latter paper.

$f>k$, plus perhaps certain points where $f=k$. In the second case the set A_k may be defined by the use of an associated function $q(x,y)$.

CASE 1: $(l=0,\ \hat{E}_k=0)$.[*] Consider a monotonic sequence of circles whose union[†] is $R: R_1<R_2<\ldots\ldots$ Let A_k be the set of points where $f>k$. Let A'_{k_i} refer to the parts of A_{k_i} in R_i. By the finite case there exists for each R_i a k_i such that

$$\int_{R_i} fg = L\int_{A'_{k_i}} g. \tag{1}$$

By Theorem 4, fg is absolutely L-integrable in R, and by Theorem 5

$$J=\int_R fg=\lim_{i=\infty}\int_{R_i} fg = L\lim_{i=\infty}\int_{A'_{k_i}} g; \tag{2}$$

so the last limit exists. It only remains to show that there is a definite k and A_k for which

$$\lim_{i=\infty}\int_{A'_{k_i}} g=\int_{A_k} g. \tag{3}$$

The left-hand side may be considered as the limit of any infinite sub-sequence selected from the sequence

$$\int_{A'_{k_1}} g,\ \int_{A'_{k_2}} g,\ \ldots\ldots \tag{4}$$

Suppose the sub-sequence to be the terms corresponding to $i=a_1<a_2<\ldots.$, where these numbers are so chosen that, if k be taken as the lower limit of the sequence $k_1, k_2, \ldots.$, it is also the limit of either (5) or (6),

$$k_{a_1}\geqq k_{a_2}\geqq\ldots., \tag{5}$$

$$k_{a_1}\leqq k_{a_2}\leqq\ldots\ldots \tag{6}$$

To be assured that this is possible we first note that whenever the points $k_1, k_2, \ldots.$ constitute an infinite point set, there exists a lower limit, for the set is limited $(0\leqq k_i\leqq L)$. Whenever the points do not constitute an infinite point set, certain of these numbers are repeated an infinite number of times; let k be the least of these numbers. It is immaterial to the proof whether (5) or (6) holds. Assume (5); and to avoid the additional subscripts let it be assumed that (4) *is* the sequence corresponding to (5), *i. e.*, that k is the limit of k_i, and that

$$k_1\geqq k_2\geqq\ldots\ldots \tag{7}$$

Then

$$A'_{k_1}\leqq A'_{k_2}\leqq\ldots\ldots \tag{8}$$

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[*] By $\hat{E}$ is meant the measure of E, a concept which may be defined in the infinite case as the integral over E of unity.

[†] By "union" of (R_i) is meant the totality of points each of which belongs to some R_i.

The limit of this sequence may be an infinite field. For $\varepsilon > 0$ prescribed, let R_a be so large that

$$\int_{R-R_a} |g| < \varepsilon. \tag{9}$$

Let

$$A''_{k_1} \leqq A''_{k_2} \leqq \ldots . \tag{10}$$

refer to the portions of A'_{k_1}, etc., in R_a. If $i < \alpha$, A''_{k_i} may not refer to all of the points of R_a where $f > k_i$, for A'_{k_i} may not include them all; but if $i \geqq \alpha$, A''_{k_i} does include all those points, and only those. A''_k is the set in R_a where $f > k$; and so, by the reasoning of my *Annalen* paper,* A''_k is the union of A''_{k_i} and $\lim_{i=\infty} \hat{A}''_{k_i} = \hat{A}''_k$. Thus, on account of the absolute continuity of the integral of g in R_a, there exists an $i_\varepsilon > \alpha$ and so large that if $i > i_\varepsilon$ $(\hat{A}''_k - \hat{A}''_{k_i})$ is sufficiently small to make

$$\int_{A''_k - A''_{k_i}} |g| < \varepsilon. \tag{11}$$

Now the set $(A_k - A'_{k_i})$ is made up of two parts, $(A''_k - A''_{k_i})$ and B, where B is some set outside of R_a. Therefore, by (9) and (11),

$$\left| \int_{A_k - A'_{k_i}} g \right| < \int_{A''_k - A''_{k_i}} |g| + \int_B |g| < 2\varepsilon,$$

which proves (3), and hence the proposition.

CASE 2: ($l = 0$, $\hat{E}_k \geqq 0$). The only new difficulty here is with inequality (11). It is obvious that $(\hat{A}''_k - \hat{A}''_{k_i})$ may not become small at pleasure because of the non-null sets that may be included. Therefore a method must be found of dividing up these non-null E_k sets. The measures of some of them may be infinite. We first note that there are only an enumerable number of values of k, say $k_1, k_2, \ldots,$ where $\hat{E}_k > 0$. This follows from the fact that, if A_k be, as before, the set where $f > k$, $\hat{A}_k$ is a monotonic decreasing function of k, and k_i is a point of discontinuity of $\hat{A}_k$. It is known that such points are at most enumerably infinite for monotonic functions. We now define, corresponding to the parameter $\lambda (0 \leqq \lambda \leqq L+1)$, a family of point sets Ω_λ, and a limited function $q(x, y)$ which is associated in a certain way with $f(x, y)$, and which is constant at most in null sets, and has the property that the set of points where $q > \lambda$ is Ω_λ. Let c_i be any positive number such that

$$c_1 + c_2 + \ldots = 1. \tag{12}$$

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[*] *Loc. cit.*, p. 275, Lemma 2.

Let $q(x, y)$ be defined as the limit of the monotonic sequence,

$$f(x, y) \leqq q_1(x, y) \leqq q_2(x, y) \leqq \ldots . \leqq f(x, y) + 1, \tag{13}$$

where $q_1, q_2, \ldots .$ are defined as follows: at points where $f < k_1$, let q_1 equal f; at points where $f = k_1$, let q_1 equal $c_1 e^{-x^2-y^2} + f$; and at points where $f > k_1$, let q_1 equal $f + c_1$. Here the function $c_1 e^{-x^2-y^2} + k_1$ has the properties that it is constant only in null sets, that it lies between k_1 and $k_1 + c_1$, and that the set of points where it is greater than any number between these limits is a circle. Other definitions of q_1 might obviously be chosen, resulting in other associated functions q. Next, let q_2 be equal to q_1 where $f < k_2$, equal to $c_2 e^{-x^2-y^2} + q_1$ where $f = k_2$, and equal to $q_1 + c_2$ where $f > k_2$, etc. Let Ω_λ be the set of points where $q(x, y) > \lambda \, (0 \leqq \lambda \leqq L+1)$. The proof that q is constant at most in null sets is omitted because the fact seems obvious.

It will now be shown that each Ω_λ is divisible into two parts; one is part or all of the set where f equals a certain corresponding k_λ, and the other is the whole of the set where $f > k_\lambda$. Consider the set A_k where $f > k$, and suppose the k_i's (7) which are not greater than k to be denoted by α_i's and their corresponding c_i's by s_i's, thus:

$$\alpha_1, \alpha_2, \ldots . \leqq k; \; s_1, s_2, \ldots . .$$

In A_k,

$$q \geqq f + s_1 + s_2 + \ldots . > k + s_1 + s_2 + \ldots . . \tag{14}$$

For example, if $k_1 = \alpha_1$, $k_2 > k$, $k_3 = \alpha_2, \ldots .$; in A_k, $q_1 = f + s_1$, $q_2 \geqq q_1 = f + s_1$, $q_3 = q_2 + s_2 \geqq f + s_1 + s_2$, etc., and finally $q_1 \leqq q_2 \leqq \ldots . \leqq q$. In $R - A_k$,

$$q \leqq f + s_1 + s_2 + \ldots . \leqq k + s_1 + s_2 + \ldots ., \tag{15}$$

for here, for every j, $q_j \leqq f + s_1 + s_2 + \ldots .$, and finally $q = \lim_j q_j$. Thus, if $\lambda = k + s_1 + s_2 + \ldots .$, $q > \lambda$ in A_k and $q \leqq \lambda$ in $R - A_k$, and so in this case $\Omega_\lambda = A_k$. If now it could be shown that to every λ there corresponds a k such that $\lambda = k + s_1 + s_2 + \ldots .$, the part of our proposition considered in this paragraph would be established, but this is not always true. It can, however, be proved except for an enumerable number of intervals where

$$\lambda_{k_i} - c_i < \lambda \leqq \lambda_{k_i}. \tag{16}$$

For $s_1 + s_2 + \ldots .$ is a monotonic increasing function of k, and therefore, considered as a function of k, $\lambda = k + s_1 + s_2 + \ldots .$ is always increasing, and is defined for each k in $(0, L)$. Moreover, it is always continuous on the right, for, for any $\bar{k}$ which is not a k_i, there exists a δ so that the interval $(k, k+\delta)$ does not contain any of the points $k_1, k_2, \ldots ., k_n$ (n finite). Therefore,

$$|\lambda(\bar{k} + \delta) - \lambda(\bar{k})| < |\delta| + \sum_n^\infty c_i < \varepsilon,$$

if n and δ are chosen properly. This is true for $\delta>0$ even if $\bar{k}$ is a k_i, but in this case every interval $(\bar{k}, \bar{k}+\delta)$, $\delta<0$, would include k_i, and hence at such points λ would have a discontinuity on the left equal to c_i. The correspondence desired is proved. We proceed to consider values of λ of (16). First, all points of A_{k_i} are in Ω_λ, for, if $f>k_i$, $q>\lambda_{k_i}=k_i+s_1+s_2+\ldots\ldots (c_i=s_1)$, that is $q>\lambda$. Secondly, Ω_λ contains no other points, except some where $f=k_i$; for, if P is a point where $f(P)<k_i$, let $f(P)=k_i-\gamma$. To this value of k there is a corresponding $\bar{\lambda}=k_i-\gamma+\sigma_1+\sigma_2+\ldots\ldots$, where $\sigma_1, \sigma_2, \ldots\ldots$ are the s_i's for $k_i-\gamma$; their sum is less than $s_1+s_2+\ldots\ldots$ by at least $c_i=s_1$. By (15)

$$q(P)\leqq k_i-\gamma+\sigma_1+\sigma_2+\ldots\ldots\leqq k_i-\gamma+s_1+s_2+\ldots\ldots-s_1<\lambda_{k_i}-c_i<\lambda,$$

and therefore P is not in Ω_λ. Thus, in every case there is a k_λ corresponding to λ for which the conditions are satisfied.

It may now be shown that

$$J=L\int_{\Omega_\lambda} g.$$

As in Case 1 (1),

$$\int_{R_i} fg=L\int_{\Omega'_{\lambda_i}} g,$$

where Ω'_{λ_i} is the part of Ω_{λ_i} in R_i. For q is for R_i precisely an associated function in the sense of my paper for the case when R is limited. The rest of the reasoning of Case 1 now holds, *mutatis mutandis.*

CASE 3: $(l\neq 0)$. We may obviously consider Case 1 as a special case of Case 2, where $q=f$. By Case 2,

$$\int_R [f(x,y)-l]\,g(x,y)=(L-l)\int_{\Omega_\lambda} g=L\int_{\Omega_\lambda} g-l\int_R g+l\int_{R-\Omega_\lambda} g.$$

Add

$$\int_R lg=l\int_R g.$$

COROLLARY. *If $f(x)$ is monotonic decreasing in (a,∞) and $l\leqq f(x)\leqq L$, and if $g(x)$ is absolutely L-integrable in the same interval, there exists a $\lambda\,(a\leqq\lambda\leqq\infty)$ such that*

$$\int_a^\infty fg=l\int_\lambda^\infty g+L\int_a^\lambda g.$$

It is to be noted also that the theorem gives even for one dimension a much more general statement than this corollary.

INTEGRALS CONTAINING PARAMETERS, §§ 6–8.

6. **Integrals over Finite Fields.**[*] As a part of my discussion of integrals containing parameters I intend to extend to integrals over infinite fields certain theorems of W. H. Young[†] on integrals of functions of one variable. It is necessary first to make the extension to n dimensions for finite fields, and in this section I shall cite certain of these results. They may be deduced more simply from Vitali's test[‡] of equi-absolute continuity, than by extending to n dimensions Young's idea of upper (lower) semi-continuity, but I will not give the proofs. In passing, however, I call attention to one theorem (11) which is not related to Young's, but which is a generalization of a very useful theorem on integrals not containing parameters.

THEOREM 9. *Let A be a limited measurable point set in space of n dimensions. Let $lim_{n,m} f_{n,m}(x, y, \ldots.) = f(x, y, \ldots.)$ in A, in general. The necessary and sufficient condition that*

$$\lim_{n,m=\infty} \int_A f_{nm}\,dA = \int_A f\,dA$$

is that there shall exist two functions $g_{n,m}(x, y, \ldots.)$, $G_{n,m}(x, y, \ldots.)$ whose limits are $g(x, y, \ldots.)$ and $G(x, y, \ldots.)$, respectively, for which this is true, and such that, in general, $g_{n,m} < f_{n,m} < G_{n,m}$.

COROLLARY 1. *An analogous theorem is true for iterated integrals.*

COROLLARY 2. *If $f_{n,m}(x, y, \ldots.) \geq 0$, and its limit is $f(x, y, \ldots.)$, and if also the limit of its integral over A is the integral of f over A, then a similar statement is true for the limit of its integral over any measurable part of A.*

THEOREM 10. *If the limit of a monotonic sequence of integrable functions $0 \leq f_1(x, y, \ldots.) \leq f_2(x, y, \ldots.) \leq \ldots.$, is integrable in A, the sequence is integrable termwise.*

THEOREM 11.[§] *If in A $f(x, y, \ldots.)$ is the limit of a function $f_{n,m}(x, y, \ldots.)$ which is nowhere negative, and the limit of whose integral is zero, then in general $f = 0$.*

The proof is immediate if it can be shown that the integral of f over A exists, for in that case the integral of $f_{n,m}$ is obviously equi-absolutely continuous and approaches the integral of f as a limit. Thus the integral of f is zero, and the theorem of which this is a generalization applies. Suppose the integral of f over A did not exist. We could then let A_k be the set where $f>k$, and set ϕ_k equal to zero in A_k and to f in $(A-A_k)$. Then the integral of ϕ_k over A would diverge to plus infinity as k became infinite. This would be impossible, for suppose that for each k $\phi_{n,m;k}$ equals 0 in A_k and $f_{n,m}$ elsewhere. Then for each k $\phi_{n,m;k}$ would satisfy the conditions of the theorem and ϕ_k would be integrable. Therefore the integral of ϕ_k over A would equal zero for each k.

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[*] It is not assumed in this section, as it is elsewhere, that the functions in question have integrals over finite measurable fields. The case of n parameters is not essentially different from that of two.

[†] *Loc. cit.*, pp. 315 ff.

[‡] *Rendiconti del Circolo di Palermo*, Vol. XXIII (1907) pp. 137 ff. Vitali enunciated his test only for one dimension, but his method is valid for n dimensions.

[§] This is a generalization of a well-known theorem. Cf. Pierpont, *The Theory of Functions of Real Variables*, Vol. II (1911), p. 385, § 402 (2).

7. **Continuity.** LEMMA 1. *If* $0\leqq f(x,y)\leqq g(x,y)$, *and* $J_i(g)$ *exists, then*

$$0\leqq J_i(f)-\int_{R_{\alpha,\beta}} f\,dxdy\leqq J_i(g)-\int_{R_{\alpha,\beta}} g\,dxdy.$$

LEMMA 2. *If for each* (n, m) *the function* $f_{n,m}(x, y)$ *satisfies the conditions imposed on* f *in Lemma* 1, *and if* $f(x,y)=lim_{n,m}f_{n,m}(x,y)$ *in general, then* $J_i(f_{n,m})$ *exists and its limit is* $J_i(f)$.

Obviously $0\leqq f\leqq g$, and so by Theorem 3 $J_i(f_{n,m})$ and $J_i(f)$ exist. For an arbitrary $\varepsilon>0$, let α, β be chosen so large that

$$J_i(g)-\int_{R_{\alpha,\beta}} g<\varepsilon.$$

By Lemma 1, then,

$$0\leqq J_i(f)\quad-\int_{R_{\alpha,\beta}} f\leqq J_i(g)\quad-\int_{R_{\alpha,\beta}} g<\varepsilon, \tag{1}$$

$$0\leqq J_i(f_{n,m})-\int_{R_{\alpha,\beta}} f_{n,m}\leqq J_i(g)-\int_{R_{\alpha,\beta}} g<\varepsilon. \tag{2}$$

By Theorem 9 there exists for these fixed values of α, β an n_ε so large that, if n, $m>n_\varepsilon$,

$$\left|\int_{R_{\alpha,\beta}} f-\int_{R_{\alpha,\beta}} f_{n,m}\right|<\varepsilon. \tag{3}$$

From (1), (2), and (3),

$$|J_i(f)-J_i(f_{n,m})|<3\varepsilon, \qquad (n,\ m>n_\varepsilon).$$

LEMMA 3. *Except perhaps for a null set in* R, *let* $g_{n,m}(x,y)\geqq 0$, $lim_{n,m}g_{n,m}(x,y)=g(x,y)$, *and* $lim_{n,m}J_i(g_{n,m})=J_i(g)$. *Then, for all values of* α, β *uniformly,*

$$\lim_{n,m}\int_{R_{\alpha,\beta}} g_{n,m}\,dxdy=\int_{R_{\alpha,\beta}} g\,dxdy.$$

Let $\phi_{n,m}=g_{n,m}$ and $\psi_{n,m}=g$ where $g_{n,m}\geqq g$, and let $\phi_{n,m}=g$ and $\psi_{n,m}=g_{n,m}$ where $g_{n,m}<g$. Then $0\leqq\psi_{n,m}<\phi_{n,m}$,

$$0\leqq\phi_{n,m}-\psi_{n,m}=|g_{n,m}-g|, \tag{1}$$

$\lim_{n,m}\phi_{n,m}=\lim_{n,m}\psi_{n,m}=g$, and $0\leqq\psi_{n,m}\leqq g$. By Lemma 2, then,

$$\lim_{n,m}J_i(\psi_{n,m})=J_i(g). \tag{2}$$

By definition, $\phi_{n,m}+\psi_{n,m}=g_{n,m}+g$, and hence $\lim_{n,m}J_i(\phi_{n,m}+\psi_{n,m})=2J_i(g)$. This, with (1) and (2), proves that $\lim_{n,m}J_i(|g_{n,m}-g|)=0$. Now, uniformly in α, β,

$$\left|\int_{R_{\alpha,\beta}}(g_{n,m}-g)\right|\leqq\int_{R_{\alpha,\beta}}|g_{n,m}-g|\leqq J_1(|g_{n,m}-g|).$$

LEMMA 4. *Except perhaps for a null set in R, let $0\leqq f_{n,m}(x,y)\leqq g_{n,m}(x,y)$, let $\lim_{n,m}f_{n,m}(x,y)=f(x,y)$, and $\lim_{n,m}g_{n,m}(x,y)=g(x,y)$, and $\lim_{n,m}J_i(g_{n,m})=J_i(g)$. Then $J_i(f_{n,m})$ exists and its limit is $J_i(f)$.*

Since obviously $0\leqq f\leqq g$, we have, by Lemma 1, for all values of n, m; α, β,

$$0\leqq J_i(f)\quad-\int_{R_{\alpha,\beta}}f\quad\leqq J_i(g)\quad-\int_{R_{\alpha,\beta}}g, \tag{1}$$

$$0\leqq J_i(f_{n,m})-\int_{R_{\alpha,\beta}}f_{n,m}\leqq J_i(g_{n,m})-\int_{R_{\alpha,\beta}}g_{n,m}. \tag{2}$$

For each $\varepsilon>0$ there exists a number n_ε so that, by hypothesis and by Lemma 3, for all values of n, $m>n_\varepsilon$ and all values of α, β uniformly,

$$|J_i(g_{n,m})-J_i(g)|,\ \left|\int_{R_{\alpha,\beta}}g-\int_{R_{\alpha,\beta}}g_{n,m}\right|<\varepsilon, \tag{3}$$

and so

$$\left|\left(J_i(g_{n,m})-\int_{R_{\alpha,\beta}}g_{n,m}\right)-\left(J_i(g)-\int_{R_{\alpha,\beta}}g\right)\right|<2\varepsilon. \tag{4}$$

But, if α, $\beta>$ some number α_ε,

$$\left|J_i(g)-\int_{R_{\alpha,\beta}}g\right|<\varepsilon, \tag{5}$$

and, therefore, by (3) and (5),

$$\left|J_i(g_{n,m})-\int_{R_{\alpha,\beta}}g_{n,m}\right|<3\varepsilon,\qquad(n,\ m>n_\varepsilon;\alpha,\ \beta>\alpha_\varepsilon).$$

This, with (1) and (2), shows that

$$\left|J_i(f)-J_i(f_{n,m})+\int_{R_{\alpha,\beta}}f_{n,m}-\int_{R_{\alpha,\beta}}f\right|<4\varepsilon(n,\ m>n_\varepsilon;\alpha,\ \beta>\alpha_\varepsilon). \tag{6}$$

Now let α, β be held fast at some values greater than α_ϵ. There exists a number n'_ϵ so that

$$\left| \int_{R_{\alpha,\beta}} f_{n,m} - \int_{R_{\alpha,\beta}} f \right| < \epsilon, \qquad (n, m > n'_\epsilon), \tag{7}$$

for, by Lemma 3, and since $0 \leqq f_{n,m} \leqq g_{n,m}$, Theorem 9 applies. Thus (6) and (7) prove the lemma.

THEOREM 12. *Let* $lim_{n,m} f_{n,m}(x, y) = f(x, y)$ *in general. In order that the following integrals shall exist and that*

$$lim_{n,m} J_i(f_{n,m}) = J_i(f),$$

it is necessary and sufficient that there shall exist two convergent sequences of functions, $lim_{n,m} g_{n,m}(x, y) = g(x, y)$, $lim_{n,m} G_{n,m}(x, y) = G(x, y)$, *for which this is true, such that in general* $g_{n,m}(x, y) < f_{n,m}(x, y) < G_{n,m}(x, y)$.

To prove the necessary part we may take $g_{n,m} = f_{n,m} - 1/(mnx^2y^2)$, $G_{n,m} = f_{n,m} + 1/(mnx^2y^2)$. To prove the sufficient part we note that $0 \leqq f_{n,m} - g_{n,m} \leqq G_{n,m} - g_{n,m}$, and that $lim_{n,m} J_i(G_{n,m} - g_{n,m}) = J_i(G - g)$, and apply Lemma 4.

COROLLARY 1. *Let* $lim_{n,m} f_{n,m}(x, y) = f(x, y)$ *in general. Any one of the following is sufficient to make* $lim_{n,m} J_i(f_{n,m}) = J_i(f)$:

(*a*) $g(x, y) \leqq f_{n,m}(x, y) \leqq G(x, y)$, *and* $J_i(g)$, $J_i(G)$ *shall exist;*

(*b*) $f(x, y)$ *shall be the limit of a monotonic sequence,*

$$f_1(x, y) \leqq f_2(x, y) \leqq \ldots,$$

and $J_i(f)$ *and* $J_i(f_1)$ *shall exist;*

(*c*) $lim_{n,m} J_i(|f_{n,m}|) = J_i(|f|)$;

(*d*) $|f_{n,m}(x, y)| \leqq G(x, y)$ *and* $J_i(G)$ *shall exist;*

(*e*) $f_{n,m}(x, y) = h_{n,m}(x, y) g_{n,m}(x, y)$, *where* $0 \leqq h_{n,m}, g_{n,m} \leqq 1$, *and one of the convergent sequences* $h_{n,m}$ *and* $g_{n,m}$ *shall be integrable termwise in R;*

(*f*) $f_{n,m} = (g_{n,m})^p$, $0 \leqq g_{n,m} \leqq 1 < p$, *and the sequence* $g_{n,m}$ *shall be integrable termwise in R.*

COROLLARY 2.* *The double series of which the general term is* $u_{n,m}(x, y)$ *is absolutely convergent and absolutely integrable termwise in R if there exists a never negative function* $v(x, y)$ *which is integrable in R and a convergent double series of positive terms of the type* $a_{n,m}$ *such that* $|u_{n,m}(x, y)| \leqq a_{n,m} v(x, y)$.

For, if $s_{n,m}$ refer to the sum of the absolute values of the nm terms of the u series, and $\sigma_{n,m}$ to the sum of the corresponding terms of the a series, $0 \leqq s_{n,m} \leqq \sigma_{n,m} v(x, y) \leqq \lim_{n,m} \sigma_{n,m} v(x, y)$, and (*d*) of Corollary 1 applies.

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* Cf. W. H. Young, *loc. cit.*, p. 324. The same statement holds good if *R* is a finite field.

8. **Differentiation and Integration.** The next theorem may be derived from what precedes in much the same manner as may be proved the theorem* of which it is a generalization.

THEOREM 13. *Let* $g_n(x, y) \leqq f_n(x, y) \leqq G_n(x, y)$, *and* $f_n - g_n$, $G_n - f_n$ *be monotonic increasing functions of* n *at each point* (x, y), *and let the derivative of* $J_i(g_n) = J_i(dg_n/dn)$, *and the derivative of* $J_i(G_n) = J_i(dG_n/dn)$. *Then the derivative with respect to* n *of* $J_i(f_n) = J_i(df_n/dn)$ *wherever*† df_n/dn *exists. The same is true if the right (left) hand derivative be used throughout instead of the derivative.*

The following lemma may be proved by the use of the theorem of Young analogous to Theorem 9, and the theorem of Fubini analogous to the corollary of Theorem 5.

LEMMA. *Let the interval* $(a \leqq n \leqq b)$ *be finite. Sufficient conditions that the following exist and be equal,*

$$\int_a^b J_1[g(x, y; n)]\,dn, \quad J_1\left[\int_a^b g(x, y; n)\,dn\right],$$

are that (1°) *the integral of* g *with respect to* x *and* y *over* $R_{\alpha,\beta}$, *for all values of* $\alpha, \beta > \alpha_0$, *be absolutely less than an integrable function of* n, (2°) $J_1(g)$ *exist for each* n, *and* g *be absolutely integrable with respect to* x, y, *and* n *in the parallelopiped*, $0 \leqq x \leqq \alpha$, $0 \leqq y \leqq \beta$, $a \leqq n \leqq b$.

THEOREM 14. *Sufficient conditions that the equation,*

$$\int_a^n J_1\left(\frac{df}{dn}\right)dn = J_1(f_n) - J_1(f_a),$$

be valid in $(a \leqq n \leqq b)$ *are that* $J_1(f)$ *exist at one point of the interval* (a, b), *and that* df/dn *exist and satisfy the conditions imposed on* g *in the lemma.*

By the lemma

$$\int_a^n J_1\left(\frac{df}{dn}\right)dn = J_1 \int_a^n \frac{df}{dn}\,dn.$$

By a known theorem due to Vallée-Poussin,‡ this equals $J_1(f_n) - J_1(f_a)$.

COROLLARY.§ *Under the same conditions the derivative, with respect to* n, *of* $J_1(f_n)$ *equals* J_1 *of the derivative of* f_n, (*a*) *in general in* (a, b) *including all*

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[*] See preceding note, same reference, p. 321.

[†] More generally, for each value of n for which, except perhaps for a null set in the xy-plane, df/dn exists.

[‡] *Loc. cit.*, Vol. I, p. 263.

[§] See reference of previous note, pp. 267, 268.

points where the derivative of $J_1(f_n)$ exists, and (b) at all points of (a, b) provided $J_1(df/dn)$ is continuous in (a, b).

THE FOURIER INTEGRALS, §§ 9–11.

9. **Introduction.** The iterated integral

$$J_4 = \frac{1}{\pi}\int_0^\infty du \int_{-\infty}^\infty f(t)\cos u(t-x)\,dt$$

is sometimes called the Fourier double integral. We shall reserve this name for

$$J_1 = D = \lim_{\alpha,\beta=\infty}\frac{1}{\pi}\int_0^\alpha du\int_0^\beta f(t)\cos u(t-x)\,dt + \lim_{\alpha,\beta=\infty}\frac{1}{\pi}\int_0^\alpha du\int_{-\beta}^0 f(t)\cos u(t-x)\,dt.$$

Here J_1 and J_4 have the same relative significance as in the preceding sections. It is evident that J_5 does not exist. A good deal has been written[*] recently on the problem of finding the most general conditions that one can impose on $f(t)$ and make J_4 equal to $f(x)$. In the following section, it is shown that J_1 equals $f(x)$ under circumstances more general than those derived for J_4. Indeed, if $f(x)=1$, J_4 does not exist, and if a value be given it by the usual summation method, that value is zero, but $J_1=1$. In Section 11 the corresponding integral which arises from three-dimensional problems is considered, and some of the questions connected with it. For simplicity x, y, and z are restricted to non-negative values. This enables one to dispense with the second term of D. No generality is lost thereby.

10. **The Fourier Double Integral.** LEMMA. *(a) For all $\alpha, \beta \geqq 0$ uniformly,*

$$\left|\int_0^\alpha \frac{\sin\beta t}{t}\,dt\right| \leqq \frac{\pi}{2}.$$

(b) Let $0 \leqq f(t+x) \leqq M$, and let f be a monotonic decreasing function of $t\,(t \geqq 0)$ for each x. Then, uniformly with respect to $\alpha \geqq \alpha_0 > 0$, and boundedly[†] *with respect to x in the first case, and uniformly in the second:*

$$\lim_{\beta=\infty}\int_0^\alpha f(t+x)\frac{\sin\beta t}{t}\,dt = \frac{\pi}{2}f(+0+x), \quad \lim_{\beta=\infty}\int_{\alpha_0}^\alpha f(t+x)\frac{\sin\beta t}{t}\,dt = 0.$$

(c) If also uniformly with respect to x $\lim_{t=0} f(t+x) = f(+0+x)$, the convergence in the first case under (b) is uniform.

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[*] A. Pringsheim, *Mathematische Annalen,* Vol. LXVIII (1910), pp. 367–408. W. H. Young, *Transactions of the Cambridge Philosophical Society,* Vol. XXI (1910), pp. 428–451; *Proceedings of the Royal Society of Edinburgh,* Vol. XXXI (1911), pp. 559–586.

[†] $F_n(x, y, \ldots)$ approaches its limit "boundedly" with respect to $x, y, \ldots$ if there exists a fixed number greater than $|F_n|$ uniformly for all values of $x, y, \ldots$.

The lemma is known except perhaps for the boundedness and uniformity of the convergence. By the more familiar part, however, we have

$$\frac{\pi}{2} f(+0+x) - \lim_{\beta=\infty} \int_0^a f(t+x) \frac{\sin \beta t}{t}\, dt$$
$$= \lim_{\beta=\infty} \int_0^a [f(+0+x) - f(t+x)] \frac{\sin \beta t}{t}\, dt = \lim_{\beta=\infty} S.$$

By (a) and the second theorem of the mean,

$$|S| \leqq M \max_a \left| \int_0^a \frac{\sin \beta t}{t} \right| \leqq \frac{\pi M}{2}.$$

Thus the convergence is bounded. In the cases where it is uniform the demonstration is the same except that M is to be chosen arbitrarily.

THEOREM 15. *Let* $0 \leqq f(x) \leqq M$, *and let* $f(x)$ *be a monotonic decreasing function of* $x (x \geqq 0)$. *Then* $D = 1/2 [f(x-0) + f(x+0)]$, *and the proper integral of which* D *is the limit approaches* D *boundedly with respect to* x *if* $x \geqq a > 0$.

CASE 1: $(x=0)$. By the lemma

$$\pi D = \lim_{\alpha,\beta=\infty} \int_0^\alpha du \int_0^\beta f(t) \cos tu\, dt = \lim_{\alpha,\beta=\infty} \int_0^\beta f(t) \frac{\sin \alpha t}{t}\, dt = \frac{\pi}{2} f(+0).$$

CASE 2: $(0 < a \leqq x)$. In the formal expression for D substitute $t - x = s$. Then

$$\pi D = \lim_{\alpha,\beta=\infty} \int_0^\alpha du \int_{-x}^{\beta-x} f(s+x) \cos su\, ds = \lim_{\alpha,\beta=\infty} \int_{-x}^{\beta-x} f(s+x) \frac{\sin \alpha s}{s}\, ds$$
$$= \lim_{\alpha,\beta=\infty} \int_{-x}^{0} f(s+x) \frac{\sin \alpha s}{s}\, ds + \lim_{\alpha,\beta=\infty} \int_0^{\beta-x} f(s+x) \frac{\sin \alpha s}{s}\, ds.$$

Substituting $t = -s$ in the first integral, and separating the last integral into two parts by the point x,

$$\pi D = \lim_{\alpha,\beta=\infty} \int_0^x [f(-t+x) + f(t+x)] \frac{\sin \alpha t}{t}\, dt + \lim_{\alpha,\beta=\infty} \int_x^{\beta-x} f(s+x) \frac{\sin \alpha s}{s}\, ds,$$

which equals $\frac{1}{2}[f(x-0) + f(x+0)]$ by (b) of the lemma, and the convergence is bounded.

COROLLARY 1. *If also* $f(x)$ *is continuous in* $(0 < a \leqq x \leqq b)$, *the convergence is uniform there.*

Apply (c) of the lemma.

COROLLARY 2. *The theorem and corollary are true if* $f(x) = f_1(x) - f_2(x)$, *where* $f_1(x)$ *and* $f_2(x)$ *have the properties there ascribed to* $f(x)$.

Examples: $e^{-x}, 1 - e^{-x}, f(x)$ equal to a polynomial in $(0, 1)$ and to $1/x$ elsewhere.

THEOREM 16. *Let* (1°) $f(x)=g(x)p(x)$, *where* $0\leqq f, g, p\leqq M, f(x)$ *has limited variation in every finite*[*] *interval, and* $g(x)$ *is monotonic decreasing* $(x\geqq x_0)$, *and* $p(x)$ *is periodic* $(x\geqq x_0)$. *Let* (2°), *except perhaps a null set, for* $x\geqq x_0$,

$$p(x)=\sum_{i=0}^{\infty}(a_i \sin kix+b_i \cos kix),$$

where the a_i*'s and* b_i*'s are the usual Fourier coefficients, and* (3°) $\Sigma(|a_i|+|b_i|)=H$. Then for each $x\geqq 0$, $D=1/2[f(x-0)+f(x+0)]$.

Let $f(t)=\phi(t)+\psi(t)$, where $\phi=f$ and $\psi=0$ at points where $t\leqq 2x$, and $\phi=0$ and $\psi=f$ where $t>2x$, x being the point at which the development is to be examined. Referring to the preceding corollary we see that it holds when ϕ replaces f. Referring to the proof of the previous theorem we see that, by formal processes,

$$\pi D-\pi/2[f(x-0)+f(x+0)]=\lim_{\alpha,\beta=\infty}\int_0^x[\psi(-t+x)+\psi(t+x)]\frac{\sin \alpha t}{t}dt$$
$$+\lim_{\alpha,\beta=\infty}\int_x^{\beta-x}\psi(s+x)\frac{\sin \alpha s}{s}ds=\mathrm{I}+\mathrm{II}.$$

By definition of ψ, I$=0$. To prove that II$=0$ we have essentially to establish for the f of our present theorem only the second part of (b) of the lemma to Theorem 15, with the omission of the references to boundedness and uniformity with respect to x. An apparent exception occurs when $x=0$, but it may be avoided by separating the interval (α, β) into two parts by means of a fixed point. We proceed, therefore, to show that, $\alpha_0>0$, $\varepsilon>0$, x being prescribed, there exists a β_ε so that the following is true uniformly in α and β:

$$\left|\int_{\alpha_0}^{\alpha}g(t+x)p(t+x)\frac{\sin \beta t}{t}dt\right|<\varepsilon, \qquad (\beta>\beta_\varepsilon).$$

The function $g(t+x)/t$ is monotonic decreasing if $t>x_0$. Let α_1 be so large that $\alpha_1>x_0$ and also that $g(t+x)/\alpha_1<\varepsilon$. Since[†] gp/t is absolutely L-integrable in (α_0, α_1),

$$\lim_{\beta=\infty}\int_{\alpha_0}^{\alpha_1}\frac{gp}{t}\sin \beta t dt=0.$$

Now

$$\left|\int_{\alpha_1}^{\alpha}\frac{g(t+x)}{t}p(t+x)\sin \beta t dt\right|<\varepsilon \max_{\lambda}\left|\int_{\alpha_1}^{\lambda}p(t+x)\sin \beta t dt\right|$$
$$=\varepsilon \max_{\lambda}\left|\int_{\alpha_1}^{\lambda}\sin \beta t\left[\sum_0^{\infty}a_i \sin ki(t+x)+b_i \cos ki(t+x)\right]dt\right|.$$

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[*] These conditions could be generalized a little by requiring that certain of them should hold necessarily only in the neighborhood of x.

[†] Lebesgue, *Annales de la Faculté de Toulouse*, Ser. 3, Vol. I (1909), p. 52.

It may be seen in several ways that this series is integrable termwise; for example, by 3° its remainder is uniformly limited. Thus the above expression equals

$$\varepsilon \max_{\lambda} \left| \sum_{0}^{\infty} \left[a_i \int_{a_1}^{\lambda} \sin ki(t+x) \sin \beta t dt \right.\right.$$
$$\left.\left. + b_i \int_{a_1}^{\lambda} \cos ki(t+x) \sin \beta t dt \right] \right| \leqq 8\varepsilon \Sigma(|a_i| + |b_i|) = 8\varepsilon H.$$

Corollary. *The theorem is true if $f(x) = f_1(x) - f_2(x)$, where f_1 and f_2 separately enjoy the properties there ascribed to f.*

Example: $f = x$ in $(0, \pi)$, $= 2\pi - x$ in $(\pi, 2\pi)$, etc.

Theorem 17. *If $f(x)$ has limited variation in a neighborhood $(x-c, x+c)$, $c > 0$, of x, and if $f(x)/x$ is absolutely L-integrable in the infinite interval $(x \geqq c)$, then, for each $x \geqq 0$, $D = 1/2[f(x-0) + f(x+0)]$.*

As before

$$D = \lim_{\alpha, \beta=\infty} \frac{1}{\pi} \int_{-x}^{\beta - x} f(t+x) \frac{\sin \alpha t}{t} dt = \lim_{\alpha, \beta=\infty} \frac{1}{\pi} \left[\int_{-x}^{-\frac{c}{2}} \frac{f \sin \alpha t}{t} dt \right.$$
$$\left. + \int_{-\frac{c}{2}}^{\frac{c}{2}} \frac{f \sin \alpha t}{t} dt + \int_{\frac{c}{2}}^{\beta - x} \frac{f \sin \alpha t}{t} dt \right] = \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

$$\mathrm{II} = \lim_{\alpha, \beta=\infty} \frac{1}{\pi} \int_{0}^{\frac{c}{2}} \tfrac{1}{2}[f(-t+x) + f(t+x)] \frac{\sin \alpha t}{t} dt = \tfrac{1}{2}[f(x-0) + f(x+0)],$$

since the part of the integrand in brackets has limited variation in $(0, c/2)$. Since $f(t+x)/t$ is absolutely L-integrable in $(-x, -c/2)$, by Lebesgue's[*] theorem $\mathrm{I} = 0$; and, similarly, a part of III,

$$\lim_{\alpha, \beta=\infty} \frac{1}{\pi} \int_{\frac{c}{2}}^{\beta_\varepsilon} f(t+x) \frac{\sin \alpha t}{t} dt = 0,$$

where β_ε is so chosen that

$$\int_{\beta_\varepsilon}^{\infty} \left| \frac{f(t+x)}{t} \right| dt < \varepsilon.$$

Thus $|\mathrm{III}| < \varepsilon$.

Example: $f(x) = p(x)e^{-x}$, where p is any limited function which has limited variation in the neighborhood of x and is L-integrable in every finite interval.

It could now be shown that, if in this form of the Fourier integral the integrand were multiplied by e^{-uy} (or by $e^{-c^2u^2t}$), the usual thermostatic (or thermodynamic) equations would be satisfied, in general, and that the functions defined thereby would enjoy the other properties desired for the solutions of the corresponding problems in heat and electricity. But this would not be useful, for, although some texts imply that the Fourier "double" (in my sense, iterated) integral is needed in these physical problems, it never is, and the appropriate propositions can be proved for the simple integrals that are needed in a much more general fashion than I can prove them even for the double integrals considered here.[*] The theorems of this section serve as lemmas to the next section, where their practical value becomes apparent. However, as notably in the case of Fourier's series,[†] so to a less degree in the case of Fourier's integral, the theory of the representation of an arbitrary function has enjoyed a development and interest that have nothing to do with its value as a physical tool.

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[*] *Loc. cit.*

11. **The Fourier Quadruple Integral.** If one seeks the value of the potential function V in an infinite rectangular parallelopiped $(x, y, z \geqq 0)$, given that the surface $z=0$ is kept at a potential $V_0 = f(x, y)$, and that the surface $x=0$ is insulated, one is led to LaPlace's equation, $\nabla^2(V)=0$, and to the formula:[‡]

$$V_z = \lim_{\alpha,\beta,\gamma,\delta=\infty} \int_0^\alpha dt \int_0^\beta du \int_0^\gamma ds \int_0^\delta e^{-z\sqrt{s^2+w^2}} f(t, u) \cos s(t-x) \cos w(u-y)\, dw.$$

THEOREM 18. *Let* $f(x, y) = h(x)k(y)$, *where* h *and* k *satisfy the conditions imposed on* f *in one of the theorems* 15, 16, *or* 17, *or are the differences of two such functions. Then* $V_0 = 1/4\,[f(x-0, y-0) + f(x-0, y+0) + f(x+0, y-0) + f(x+0, y+0)]$.

For $V_0 = \lim\limits_{\alpha,\beta,\gamma,\delta} \left[\int_0^\alpha dt \int_0^\gamma h(t) \cos s(t-x)\, ds\right] \left[\int_0^\beta du \int_0^\delta k(y) \cos w(u-y)\, dw\right]$, which equals $1/4\,[h(x-0) + h(x+0)]\,[k(y-0) + k(y+0)]$.

COROLLARY. *The convergence is uniform with respect to* x (*or* y) *if* h (*or* k) *also satisfies the condition of continuity imposed in the corollary to Theorem* 15.

THEOREM 19. *If* $f(x, y)$ *is absolutely L-integrable in the infinite rectangle* $(x, y \geqq 0)$, $V_z (z>0)$ *satisfies LaPlace's equation.*

It is sufficient to show that the signs of partial differentiation may be passed over the limit sign and over the signs of integration in the expression for V_z. For this purpose we refer to corollary (b) of Theorem 14. Using the notation there, but replacing that f by the present integrand, which we call ϕ, and thinking of n as representing in turn x, y, and z, we see that we need to prove: (1) that $J_1(d\phi/dn)$ is continuous near n, n being the point at which the derivative of J_1 is being examined; (2) that $J_1(\phi)$ exists at n; (3) that $d\phi/dn$ exists, and is absolutely L-integrable over finite three-dimensional fields; and (4) that there exists a function, $\Phi(n)$, whose integral with respect to n exists over a finite interval enclosing the particular point n in question, such that, for all values of $\alpha, \beta, \gamma, \delta \geqq 0$ uniformly,

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[*] Cf. W. H. Young, *Proceedings of the Royal Society of Edinburgh*, Vol. XXXI (1911), p. 587; *Encyklopädie Mathematischen Wissenschaften*, II, A 12, p. 1089.

[†] Cf. VanVleck, "The Influence of Fourier's Series upon the Development of Mathematics," *Science*, N. S., Vol. XXXIX (1914), Number 995, pp. 113–124.

[‡] In the usual formula the iterated integral is used.

$$\left| \int_{R_{\alpha,\beta,\gamma,\delta}} \frac{d\phi}{dn}\, dt\, du\, ds\, dw \right| < \Phi(n).$$

These results are needed in order to show that the first derivative of J_1 may be replaced by the first derivative of its integrand. The process must be repeated for the second derivative. We shall then have conditions (5), (6), (7), and (8) exactly like (1), (2), (3), and (4), respectively, except that $d\phi/dn$ replaces ϕ, and in (4) $\Phi(n)$ is replaced by some other function having the same properties. It is obvious that (3) and (7) are true, and that (6) is included in (1). We first prove (4). For all values of x, y, z uniformly,

$$|\phi| \leqq |f(t,u)|\, e^{-z\sqrt{s^2+w^2}} \leqq |f|\, e^{-z\left(\frac{s+w}{2}\right)} = |f|\, e^{-\frac{zs}{2}}\, e^{-\frac{zw}{2}},$$

$$\left| \int_{R_{\alpha,\beta,\gamma,\delta}} \phi \right| \leqq M \left(\int_0^\infty e^{-\frac{zs}{2}}\, ds \right)^2 = \frac{4M}{z^2},$$

where

$$M = \lim_{\alpha,\beta} \int_0^\alpha dt \int_0^\beta |f(t,u)|\, du.$$

Thus (2) $J_1(\phi)$ exists absolutely. Now, first let $n=z$.

$$\left.\begin{aligned} &\frac{\partial\phi}{\partial z} = -\sqrt{s^2+w^2\phi}, \quad \frac{\partial^2\phi}{\partial z^2} = (s^2+w^2)\phi, \quad \left|\frac{\partial\phi}{\partial z}\right| \leqq |f|\,(s+w)\, e^{-\frac{zs}{2}}\, e^{-\frac{zw}{2}}, \\ &\qquad \left| \int_{R_{\alpha,\beta,\gamma.\delta}} \frac{\partial\phi}{\partial z} \right| \leqq 2M \left(\int_0^\infty s e^{-\frac{zs}{2}}\, ds \int_0^\infty e^{-\frac{zw}{2}}\, dw \right) = \frac{16M}{z^3}. \end{aligned}\right\} \quad (9)$$

Thus (4) is satisfied ($z>0$). We can similarly prove (8), for

$$\left| \int_{R_{\alpha,\beta,\gamma,\delta}} \frac{\partial^2\phi}{\partial z^2} \right| \leqq 2M \int_0^\infty s^2 e^{-\frac{zs}{2}}\, ds \int_0^\infty e^{-\frac{zw}{2}}\, dw = \frac{64M}{z^4}.$$

Now, by referring to Corollary 1 (d°) of Theorem 12, it will be seen that in (9) we have established (1) and (5); *e. g.*, if $z \geqq c > 0$,

$$|f|\,(s+w)\, e^{-\frac{z}{2}(s+w)} \leqq |f|\,(s+w)\, e^{-\frac{c}{2}(s+w)},$$

which is an integrable function in R. Secondly, let $n = x \geqq 0$.

$$\frac{\partial \phi}{\partial x} = s e^{-z\sqrt{s^2+w^2}} f(t, u) \sin s(t-x) \cos w(u-y),$$

$$\frac{\partial^2 \phi}{\partial x^2} = -s^2 e^{-z\sqrt{s^2+w^2}} \cos s(t-x) \cos w(u-y).$$

$$\left| \int_{R_{\alpha,\beta,\gamma,\delta}} \frac{\partial \phi}{\partial x} \right| < M \int_0^\infty s e^{-\frac{zs}{2}} ds \int_0^\infty e^{-\frac{zw}{2}} dw = \frac{8M}{z^3}.$$

$$\left| \int_{R_{\alpha,\beta,\gamma,\delta}} \frac{\partial^2 \phi}{\partial x^2} \right| < M \int_0^\infty s^2 e^{-\frac{zs}{2}} ds \int_0^\infty e^{-\frac{zw}{2}} dw = \frac{32M}{z^4}.$$

Hence the conditions in question are satisfied. The case, $n = y \geqq 0$, is identical with this one.

WESLEYAN UNIVERSITY, MIDDLETOWN, CONN., *October*, 1916.

The Potential of a Lens, and Allied Physical Problems.

BY G. GREENHILL.

As a sequel to the article in this Journal, Vol. XXXIII, p. 311, on the "Potential of a Spherical Segment (a Lens)", some further applications may be cited, by which the solution can be given of allied problems in Electricity, Attraction, and Hydrodynamics.

1. Starting with U, the potential function (P. F.) of a spherical bowl of radius c cm, and uniform superficial density, σ, in g/cm^2 super, it was shown in the former article (cited as A. J. M.) that G denoting the gravitation constant, 666×10^{-10} in C. G. S. units,

$$\frac{U}{G\sigma}=c\Omega+r'\Omega', \tag{1}$$

(A. J. M., § 8), and this is Maxwell's P of § 670, "Electricity and Magnetism" (E. and M.); and then, as in A. J. M., § 13 (18),

$$\frac{1}{c}\frac{d}{dr}(Pr)=\frac{d}{dr}(r\Omega+c\Omega')=\Omega+r\frac{d\Omega}{dr}+c\frac{d\Omega'}{dr}=\Omega. \tag{2}$$

The P. F. U in (1) is composed of two terms, of which Ω in the first is the apparent area at P of the bowl or its base AB (fig. 1), and Ω is a P. F. satisfying Laplace's equation; and as U is a P. F. at P, it follows that the second term $r'\Omega'$ is also a P. F. at P, while Ω' is a P. F. at P', the inverse point of P in the spherical surface, Ω' representing the apparent area of the bowl or its base AB at P'.

Interpreted physically, Ω will represent the magnetic potential at P of a plate bounded by the circle AB and magnetized normally, or the equivalent electro-magnetic potential of a current round the rim; or it will represent the illumination on a page at P parallel to AB, due to skylight coming through the circle AB, as inside a chimney shaft or down a well, where the illumination of a sky of uniform brightness would be reduced by the fraction

$$\frac{\Omega}{2\pi}=1-\frac{h}{\sqrt{(h^2+a^2)}},$$

at a depth h in the middle of a shaft or well of radius a.

2. The radial component F of the attraction along PC is given by

$$\frac{F}{G\sigma}=-\frac{d}{dr}\,(c\Omega+r'\Omega')=-c\,\frac{d\Omega}{dr}-r'\,\frac{d\Omega'}{dr}+\frac{c^2}{r^2}\,\Omega'=\frac{c^2}{r^2}\,\Omega'. \tag{1}$$

Thus with P on the axis, and outside the convex side of the bowl at G, when P' is inside at O,

$$\Omega'=2\pi,\quad \frac{F}{G\sigma}=2\pi\frac{CA^2}{CG^2}=2\pi\cos^2\gamma. \tag{2}$$

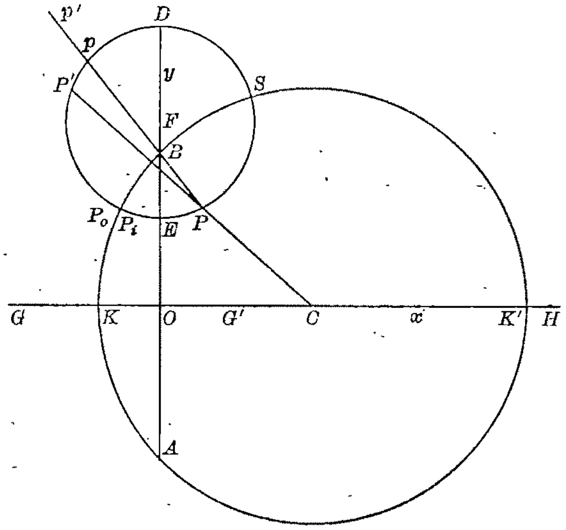

FIG. 1.

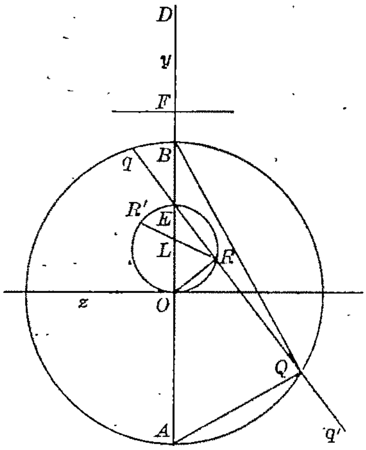

FIG. 2.

With P inside at O, and P' outside at G, Ω' is negative,

$$\Omega'=-4\pi+2\pi\frac{OG}{OA}=-4\pi+2\pi\sin\gamma,$$

$$\frac{F}{G\sigma}=2\pi(-2+\sin\gamma)\,\frac{CA^2}{CO^2}=2\pi\frac{-2+\sin\gamma}{\cos^2\gamma}. \tag{3}$$

With P at K', $c=r=x'$, $\Omega=\Omega'$;

$$\frac{F}{G\sigma}=\Omega=2\pi\frac{OK'}{AK'}=2\pi\cos\tfrac{1}{2}\gamma; \tag{4}$$

P outside, close to K, and P' inside, $r=c=r'$,

$$\frac{F}{G\sigma}=\Omega'=2\pi\frac{KO}{KA}=2\pi\sin\tfrac{1}{2}\gamma; \tag{5}$$

P inside, and P' outside, close to K,

$$\frac{F}{G\sigma}=\Omega'=-4\pi+2\pi\sin\tfrac{1}{2}\gamma. \tag{6}$$

A particle at K, inside or outside, will stick to the surface of the bowl, in stable equilibrium; for if slightly displaced on a small smooth spot at K, it will beat time with a pendulum of length (A. J. M., § 7, p. 386),

$$\frac{g}{G\sigma\pi}\,\frac{KA^3}{OA^2}, \tag{7}$$

this length is $\frac{4}{3}\,\frac{\Delta}{\sigma}\,\frac{KA^3}{OA^2}$ of the earth's radius with a mean density Δ; or the beat is equal to $\sqrt{\left(\frac{1}{3}\,\frac{\Delta}{\sigma}\,\frac{KA^3}{OA^2}\right)}$ of the period of the grazing satellite.

3. In the previous demonstration it was assumed that the bowl was the segment of a sphere made by a plane; but as the result is independent of the size of the segment, it holds true when the segment is made small; and then by summation the result in (1) § 1 is seen to be unaltered in form when the bowl is bounded by any other curve.

This is evident by elementary geometry in fig. 3; the element dS of the spherical surface at E has the potential $\frac{dS}{EP}$ at P, and

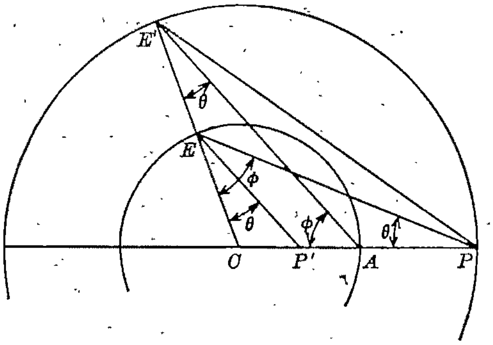

FIG. 3.

$$EP=CE\cos\phi+CP\cos\theta, \tag{1}$$

so that

$$\frac{dU}{G\sigma}=\frac{dS}{EP}=\frac{dS}{EP^2}(CE\cos\phi+CP\cos\theta)$$

$$=CE\,\frac{dS\cos\phi}{EP^2}+CP'\,\frac{dS\cos\theta}{EP'^2}\left(\text{because }\frac{CP}{CP'}=\frac{EP^2}{EP'^2}\right)$$

$$=CE\cdot d\Omega+CP'\cdot d\Omega', \tag{2}$$

reckoning Ω, Ω' positive when their aspect from P is the concave side of the bowl; and then by summation,

$$\frac{U}{G\sigma}=c\Omega+r'\Omega', \tag{3}$$

whatever the shape of the boundary rim of the bowl.

And for F, the radial force along PC of attraction at P,

$$\frac{dF}{G\sigma}=\frac{dS\cos\theta}{EP^2}=\frac{CP'}{CP}\frac{dS\cos\theta}{EP'^2}=\frac{r'}{r}\,d\Omega'=\frac{c^2}{r^2}\,d\Omega', \tag{4}$$

$$\frac{F}{G\sigma}=\frac{c^2}{r^2}\,\Omega'. \tag{5}$$

These two or three lines of geometry can thus replace some pages of analysis in Maxwell's E. and M.

For a complete sphere, and

$$P \text{ inside,}\quad \Omega=4\pi,\ \Omega'=0,\quad \frac{U}{G\sigma}=4\pi c,\quad F=0, \tag{6}$$

$$P \text{ outside,}\quad \Omega=0,\quad \Omega'=4\pi,\quad \frac{U}{G\sigma}=4\pi r'=\frac{4\pi c^2}{r},\quad \frac{F}{G\sigma}=\frac{4\pi c^2}{r^2}, \tag{7}$$

the well-known results for a spherical shell, and thence for a solid sphere given first by Newton in the "Principia," and of pioneering interest in justifying his theory of gravitation.

Because evidence has been found recently, by Prof. J. C. Adams, that Newton laid aside his calculations for nineteen years, till 1684, not only on account of his erroneous estimate of the size of the earth, at sixty land miles to the degree of latitude instead of sixty-nine; but also because Newton wanted to prove that the attraction of a spherical body like the earth on an external body, like an apple at the surface, was the same as if the earth was condensed into a particle at the centre, an assumption good enough for its attraction on a distant body like the moon, but requiring justification for an apple on the surface. As soon as this was clear, he set to work at once on the "Principia."

4. The theorem in § 1 is general, that if any P. F. at P is given as a function of r, $\mu=\cos\phi$, ψ, by $V=V(r,\mu,\psi)$, so that $V'=V\left(r'=\frac{c^2}{r},\mu,\psi\right)$ is a P. F. at the inverse point P', then

$$r'V(r',\mu,\psi)=\frac{c^2}{r}\,V\left(\frac{c^2}{r},\mu,\psi\right) \text{ is a P. F. at } P. \tag{1}$$

(W. D. Niven, L. M. S., VIII, 1876, p. 66; W. Burnside, L. M. S., XXV, 1893, p. 99).

The theorem is proved at once by Laplace's operator

$$r^2\frac{d^2}{dr^2}+2r\frac{d}{dr}+\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu}+\frac{1}{1-\mu^2}\frac{d^2}{d\psi^2}, \tag{2}$$

which changes, for $r'=\frac{c^2}{r}$, into

$$r'^2\frac{d^2}{dr'^2}+\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu}+\frac{1}{1-\mu^2}\frac{d^2}{d\psi^2}, \tag{3}$$

and this operator, acting as an annihilator on $r'V'$, reduces to Laplace's equation

$$r'^2\frac{d^2V'}{dr'^2}+2r'\frac{dV'}{dr'}+\frac{d}{d\mu}(1-\mu^2)\frac{dV'}{d\mu}+\frac{1}{1-\mu^2}\frac{d^2V'}{d\psi^2}=0. \tag{4}$$

Or in the algebraical form employed by Niven, if V in $f(V, x, y, z)=0$ satisfies Laplace's operator $\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0$, so also does the function

$$f\left(\frac{r}{c}V,\quad \frac{c^2}{r^2}x,\quad \frac{c^2}{r^2}y,\quad \frac{c^2}{r^2}z\right)=0.$$

A combination of V and $\frac{r'}{c}V'$, or $\frac{c}{r}V'$ will then give a P. F., where $V=V'$ on the sphere $r=c=r'$.

5. For a distribution symmetrical about the axis Ox, the P. F. V satisfies Laplace's equation in the form

$$\frac{d}{dx}\left(y\frac{dV}{dx}\right)+\frac{d}{dy}\left(y\frac{dV}{dy}\right)=0; \tag{1}$$

so that a function N can be assigned, the Stokes or stream function (S. F.), such that

$$y\frac{dV}{dx}=\frac{dN}{dy},\quad y\frac{dV}{dy}=-\frac{dN}{dx}, \tag{2}$$

and the meridian curves of the surface of constants V and N are orthogonal.

The factor 2π of N in A. J. M., § 11, introduced by Maxwell has been omitted here, and the sign changed; so that here the slope or gradient of V is changed into the gradient of N by a rotation through a right angle against the clock or sun, *widdershins*. Also

$$\frac{dV}{dx}=\frac{1}{y}\frac{dN}{dy},\quad \frac{dV}{dy}=-\frac{1}{y}\frac{dN}{dx}, \tag{3}$$

$$\frac{d^2V}{dxdy}=\frac{d}{dy}\left(\frac{1}{y}\frac{dN}{dy}\right)=-\frac{d}{dx}\left(\frac{1}{y}\frac{dN}{dx}\right), \tag{4}$$

$$\frac{d}{dx}\left(\frac{1}{y}\frac{dN}{dx}\right)+\frac{d}{dy}\left(\frac{1}{y}\frac{dN}{dy}\right)=0. \tag{5}$$

Better then use the ordinary x, y coordinates instead of z, ϖ, or Maxwell's b, A; because at a large distance from the axis Ox, these equations (1) and (5) degenerate into

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}=0, \quad \frac{d^2N}{dx^2}+\frac{d^2N}{dy^2}=0, \tag{6}$$

as for plane orthogonal conjugate functions, V and N.

In polar coordinates in fig. 1, with

$$x = c\cos\gamma - r\cos\phi, \quad y=r\sin\phi, \tag{7}$$

$$\frac{d}{dr}=-\cos\phi\frac{d}{dr}+\sin\phi\frac{d}{dy}, \quad \frac{d}{rd\phi}=\sin\phi\frac{d}{dx}+\cos\phi\frac{d}{dy}, \tag{8}$$

$$\begin{aligned}\frac{dN}{dr}&=-\cos\phi\frac{dN}{dx}+\sin\phi\frac{dN}{dy}=y\cos\phi\frac{dV}{dy}+y\sin\phi\frac{dV}{dx}\\&=y\frac{dV}{rd\phi}=\sin\phi\frac{dV}{d\phi},\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{dN}{rd\phi}&=\sin\phi\frac{dN}{dx}+\cos\phi\frac{dN}{dy}=-y\sin\phi\frac{dV}{dy}+y\cos\phi\frac{dV}{dr}\\&=-y\frac{dV}{dr}=-\sin\phi\frac{rdV}{dr},\end{aligned} \tag{10}$$

and the V gradient is changed into the N gradient by a rotation through a right angle with the clock and sun, *deasil*.

But here x and $r\cos\phi$ are antagonistic, so that V, *widdershins* with N with respect to x and y, becomes *deasil* in the polar coordinates r, ϕ.

With dn the outward normal element of an equipotential V, and ds the element of the meridian curve of V, normal element of N,

$$\frac{dN}{ds}=y\frac{dV}{dn}=4\pi\sigma y, \quad N_1-N_2=\int 4\pi\sigma y ds, \tag{11}$$

and this is twice the charge or induction through the zone S_1-S_2 on the equipotential V; σ denoting the electrical density of induction.

6. A relation similar to (1) § 4, holds for a Stokes stream function (S. F.) at P; denoting it by N, or $N(r, \mu=\cos\phi)$ and by $N'=N\left(r'=\frac{c^2}{r}, \mu\right)$ at P', then

$$\frac{r}{c}N'=\frac{c}{r'}N' \tag{1}$$

is a S. F. at P; and Laplace's operator in (2) § 4, is replaced for the S. F. by

$$r^2\frac{d^2}{dr^2}+(1-\mu^2)\frac{d^2}{d\mu^2}, \tag{2}$$

changing with $r=\dfrac{c^2}{r'}$ into

$$r'^2\frac{d^2}{dr'^2}+2r'\frac{d}{dr'}+(1-\mu^2)\frac{d^2}{d\mu^2}, \tag{3}$$

so that, operating as an annihilator on $\dfrac{cN'}{r'}$,

$$r'^2\frac{d^2N'}{dr'^2}+(1-\mu^2)\frac{d^2N'}{d\mu^2}=0. \tag{4}$$

Thus, if

$$y^2M(r,\mu)=r^2\sin^2\phi M(r,\cos\phi), \tag{5}$$

is a S. F. at P, then $y'^2M(r',\mu)$ is a S. F. at P'; and

$$\begin{aligned}\frac{r}{c}\,y'^2M\left(\frac{c^2}{r},\mu\right)&=\frac{r}{c}\,r'^2\sin^2\phi M\left(\frac{c^2}{r},\mu\right)\\&=\frac{c^3}{r}\sin^2\phi M\left(\frac{c^2}{r},\mu\right)=\frac{c^3}{r^3}\,y^2M\left(\frac{c^2}{r},\mu\right)\end{aligned} \tag{6}$$

is also a S. F. at P.

Axial differentiation $\dfrac{d}{dx}$ will give a new P. F. and S. F., such as $\dfrac{dV}{dx}$ and $\dfrac{dN}{dx}$; and since, in Maxwell's notation, with $A=y\cos\psi$,

$$\frac{dV}{dy}=\frac{dV}{dA}\cos\psi=\frac{1}{A}\frac{dN}{db}\cos\psi; \tag{7}$$

this transverse differentiation will give a new P. F., tesseral of the first order, suitable for use in a uniform field perpendicular to the axis Ox.

7. For a plane circular plate AB, not dished as a bowl, $c=\infty$, $\Omega'=0$, and the result in (1) § 1, changes, in Maxwell's coordinates A, b in E. and M. § 701, to the expression in A. J. M., § 3, p. 378.

$$\frac{W}{G\sigma}=aP-AQ-b\Omega, \tag{1}$$

$$\frac{1}{G\sigma}\left(\frac{dW}{da},\ \frac{dW}{dA},\ \frac{dW}{db}\right)=P,\ -Q,\ -\Omega; \tag{2}$$

illustrating the application of the complete Elliptic Integral, First, Second and Third, in P, Q, Ω; as P here represents the potential of the rim of the plate, Q and Ω the magnetic potential for uniform magnetization, in the plate and normal to it.

For any other boundary of a plane plate the normal component of the attraction, or the magnetic potential of uniform normal magnetization, is still given at a point P by the conical angle Ω of the plate as seen from P.

For if $d\alpha$ denotes a small element of area round a point Q on the plate, the normal component of the attraction of the element is

$$\frac{G\sigma d\alpha}{PQ^2} \cos \text{(angle between } PQ \text{ and normal)} = G\sigma d\Omega \qquad (3)$$

and so in $G\sigma\Omega$ for the whole area.

Thus, for an infinite plate, $\Omega=2\pi$, and the field of the attraction is uniform and $2\pi G\sigma$; changing to $-2\pi G\sigma$ in crossing the plate, a total change of $4\pi G\sigma$, in accordance with a general theorem.

The S. F. L of the plate, with the sign changed to that in A. J. M., § 12, p. 391, and omitting 2π, is then given by

$$\frac{L}{G\sigma} = \tfrac{1}{2}abP + \tfrac{1}{2}bAQ + \tfrac{1}{2}(a^2-A^2)\Omega, \qquad (4)$$

$$\frac{dL}{dA} = A\frac{dU}{db} = -G\sigma A\Omega, \quad \frac{dL}{db} = -A\frac{dU}{dA} = G\sigma AQ, \qquad (5)$$

as in § 3; and the S. F. of P, the rim P. F., is $bP + a\Omega$.

8. Here, as in A. J. M., § 20, p. 405, for the flat circular plate, with $PQ=r$, $\theta=2\omega$, $r^2=r_1^2\cos^2\omega+r_2^2\sin^2\omega$,

$$Q=\int_0^{2\pi}\frac{-a\cos\theta d\theta}{r}=\frac{8a}{r_1+r_2}\,\frac{K-E(\kappa)}{\kappa}, \quad P-Q=\int\frac{a(1+\cos\theta)\,d\theta}{r}$$

$$=\frac{8a}{r_1}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\omega d\omega}{\Delta\omega}=\frac{8a}{r_1}\,\frac{E(\gamma)-\gamma'^2G}{\gamma^2}=\frac{8a}{r_1+r_2}\,\frac{E(\kappa)-(1-\kappa)K}{\kappa}, \qquad (1)$$

$$P=\int_0^{2\pi}\frac{ad\theta}{r}=\int_0^{\frac{1}{2}\pi}\frac{4ad\omega}{\sqrt{(r_1^2\cos^2\omega+r_2^2\sin^2\omega)}}=\int_{r_2}^{r_1}\frac{4adr}{\sqrt{(r_1^2-r^2\cdot r^2-r_2^2)}}$$

$$=\frac{4aG}{r_1}=\frac{8aK}{r_1+r_2}\to\frac{2\pi a}{R}, \qquad (2)$$

where G, K is the complete quarter period to comodulus $\gamma'=\frac{r_2}{r_1}$, or modulus $\kappa=\frac{r_1-r_2}{r_1+r_2}$, and R is Gauss's arithmetic-geometric mean (A. G. M) of r_1 and r_2.

Inserting some further intermediate values of the series of quadric transformations, such as

$$\lambda=\frac{1-\varkappa'}{1+\varkappa'}=\left(\frac{\sqrt{r_1}-\sqrt{r_2}}{\sqrt{r_1}+\sqrt{r_2}}\right)^2,\quad K=(1+\lambda)L=\frac{r_1+r_2}{\frac{1}{2}(\sqrt{r_1}+\sqrt{r_2})^2}L;$$

$$\mu=\frac{1-\lambda'}{1+\lambda'}=\left(\frac{\sqrt{\frac{r_1+r_2}{2}}-\sqrt[4]{r_1r_2}}{\sqrt{\frac{r_1+r_2}{2}}+\sqrt[4]{r_1r_2}}\right)^2,\quad L=(1+\mu)M=\frac{(\sqrt{r_1}+\sqrt{r_2})^2}{\frac{1}{2}\left(\sqrt{\frac{r_1+r_2}{2}}+\sqrt[4]{r_1r_2}\right)^2}M;$$

$$\nu=\frac{1-\mu'}{1+\mu'}=\left(\frac{\frac{\sqrt{r_1}+\sqrt{r_2}}{2}-\sqrt[8]{r_1r_2}\sqrt[4]{\frac{r_1+r_2}{2}}}{\frac{\sqrt{r_1}+\sqrt{r_2}}{2}+\sqrt[8]{r_1r_2}\sqrt[4]{\frac{r_1+r_2}{2}}}\right)^2,$$

$$M=(1+\nu)N=\frac{\left(\sqrt{\frac{r_1+r_2}{2}}+\sqrt[4]{r_1r_2}\right)^2}{\frac{1}{2}\left(\frac{\sqrt{r_1}+\sqrt{r_2}}{2}+\sqrt[8]{r_1r_2}\sqrt[4]{\frac{r_1+r_2}{2}}\right)^2}N; \tag{3}$$

the ring potential of a mass m round the circle AB is given by

$$\frac{m}{r_1}\frac{G}{\frac{1}{2}\pi}=\frac{m}{\frac{1}{2}(r_1+r_2)}\frac{K}{\frac{1}{2}\pi}=\frac{m}{\frac{1}{4}(\sqrt{r_1}+\sqrt{r_2})^2}\frac{L}{\frac{1}{2}\pi}=\frac{m}{\frac{1}{4}\left(\sqrt{\frac{r_1+r_2}{2}}+\sqrt[4]{r_1r_2}\right)^2}\frac{M}{\frac{1}{2}\pi}$$

$$=\frac{m}{\frac{1}{4}\left(\frac{\sqrt{r_1}+\sqrt{r_2}}{2}+\sqrt[8]{r_1r_2}\sqrt[4]{\frac{r_1+r_2}{2}}\right)^2}\frac{N}{\frac{1}{2}\pi}\rightarrow\frac{m}{R} \tag{4}$$

For a point close to the rim, r_2 is small, and r_1, $2A$ may be replaced by $2a$; P is then large, and Q too, but $P-Q$ is finite; and as in A. J. M., § 22, writing it

$$P=\int_0^{\frac{1}{2}\pi}\frac{4a\,d\omega}{r}=\int\frac{4a\sin\omega\,d\omega}{r}+\int\frac{4a(1-\sin\omega)\,d\omega}{r}, \tag{5}$$

the first integral

$$\int_\omega^{\frac{1}{2}\pi}\frac{4a\sin\omega\,d\omega}{r}=\int_{r_2}^{r}\frac{4a\sin\omega\,dr}{\sqrt{(r_1^2-r^2\cdot r^2-r_2^2)}}$$

$$=\frac{4a}{\sqrt{(r_1^2-r_2^2)}}\int\frac{dr}{\sqrt{(r^2-r_2^2)}}=\frac{4a}{\sqrt{(r_1^2-r_2^2)}}\operatorname{ch}^{-1}\frac{r}{r_2}, \tag{6}$$

$$\int_0^{\frac{1}{2}\pi}\frac{4a\sin\omega\,d\omega}{r}=2\sqrt{\frac{a}{A}}\operatorname{ch}^{-1}\frac{r_1}{r_2}=2\sqrt{\frac{a}{A}}\log\frac{r_1+\sqrt{(r_1^2-r_2^2)}}{r_2}\rightarrow 2\log\frac{2r_1}{r_2}$$

and the second integral, with $r>r_1\cos\omega$, $\Delta\omega>\cos\omega$,

$$\int_0^{\frac{1}{2}\pi}\frac{4a(1-\sin\omega)\,d\omega}{r}<\frac{4a}{r_1}\int\frac{1-\sin\omega}{\cos\omega}d\omega \text{ or } 2\int\frac{\cos\omega\,d\omega}{1+\sin\omega}=2\log 2, \tag{8}$$

so that we can take

$$P=2\log\frac{4r_1}{r_2}+\text{small terms, making}$$

$$G\rightarrow\log\frac{4r_1}{r_2}=\log\frac{4}{\gamma'}\text{, practically, when }\gamma'\text{ is small; and then}$$

$$G=2\log 2\frac{1+\kappa}{\kappa'}=2\log\frac{4}{\kappa'}=2K;\ \text{ or }\gamma'=4e^{-G}=4e^{-\frac{1}{2}\pi\frac{G}{G'}},\ \kappa'=4e^{-\frac{1}{2}\pi\frac{K}{K'}},$$

as they may be written, with $G'=K'=\frac{1}{2}\pi$; (9)

$$P-Q=\frac{8a}{r_1}\int\frac{\cos^2\omega d\omega}{\Delta\omega}<\frac{8a}{r_1}\int\cos\omega d\omega,\text{ or } 4; \tag{10}$$

$$P+Q=2P-4+\text{small terms}=4\log\frac{4r_1}{er_2}+\text{small terms}; \tag{11}$$

and this is large as r_2 is small.

Thus, for example, the capacity of a ring AB, of small circular cross section πc^2 may be taken, with $r_2=c$, $r_1=2a$,

$$\frac{2\pi a}{P}=\frac{\pi a}{\log\frac{8a}{c}}=\frac{\text{circumference of }AB}{\log\dfrac{64\text{ area of circle }AB}{\text{area of cross section}}} \tag{12}$$

For instance, with $a=10c$, the capacity is $\dfrac{\pi a}{\log_e 80}=\dfrac{\pi a}{4.382}=0.7168a$.

So also for the potential of a circular plate, in exact functions tabulated numerically, instead of in an approximation by series, as in Thomson and Tait, § 546.

9. When the bowl in fig. 1 is insulated and electrified, the electrical potential can be written, in analogy with the potential of the bowl itself in (1), § 1,

(W. Thomson, Liouville, Oct. 8, 1845, "Electrical Papers," XVIII, p. 178; J. C. Maxwell, "Scientific Papers," II, p. 303; Ferrers, Q. J. M., XVIII, 1881, p. 97; Gallop, Q. J. M., XXI, 1886, p. 229.)

$$V=\omega+\frac{c}{r}\omega'=\omega+\frac{r'}{c}\omega',\quad cV=c\omega+r'\omega', \tag{1}$$

where, in fig. 1, ω and ω' are plane angles, given by

$$\sin\omega=\frac{2a}{r_1+r_2},\quad \sin\omega'=\frac{2a}{r_1'+r_2'}=\frac{2r\sin\gamma}{r_1+r_2}, \tag{2}$$

$$r_1=PA,\quad r_2=PB,\quad r_1'=P'A,\quad r_2'=P'B,\quad OP=r,\quad OP'=r'=\frac{c^2}{r}, \tag{3}$$

$$AB=2a,\quad ACB=2\gamma,\quad a=c\sin\gamma,\quad OC=c\cos\gamma,\quad OA=c, \tag{4}$$

$$\frac{r_1'}{r}=\frac{AP'}{AP}=\frac{DP'}{DP}=\frac{r'-c}{c-r}=\frac{r'}{c}=\frac{c}{r}=\frac{r_2'}{r_2}=\frac{r_1'+r_2'}{r_1+r_2}=\frac{\sin\omega}{\sin\omega'}, \tag{5}$$

the ratio of the line elements or relative magnification at P and P'; and $\sin\omega$, $\sin\omega'$ is the excentricity of the ellipse with foci at A, B, passing through P and P'.

The statement in (1) is verified, because ω and $\frac{c}{r}\omega'$ are P. F.'s at P; and over the bowl AKB, $r=r'=c$, $r_1'=r_1$, $r_2'=r_2$, $\sin\omega'=\sin\omega$, $\omega'=\pi-\omega$, so that $V=\pi$; while $\omega=\omega'$, $V=2\omega$ over the remaining part $AK'B$ of the spherical surface. At infinity, $r_1=r_2=r=\infty$, $r'=0$,

$$\omega=\sin\omega=\frac{a}{r}=\frac{c\sin\gamma}{r},\ \sin\omega'=\sin\gamma,\ V(\infty)=\frac{c\sin\gamma}{r}+\frac{c\gamma}{r}=\frac{E}{r}, \tag{6}$$

so that the charge $E=c(\gamma+\sin\gamma)$.

The term ω in V is the potential of the electrification of the flat circular disc AB, insulated and at potential $\frac{1}{2}\pi$; but the term $\frac{c}{r}\omega'$ is obtained by inversion of the disc AB with respect to C, as the potential of a bowl AHB on the base AB, part of a spherical surface passing through C, when this bowl is earthed and influenced by a point charge $-\frac{1}{2}\pi c$ at C; because $\omega'=\frac{1}{2}\pi$ over AHB.

The sum of the two terms is then the electric potential of the insulated bowl on the base AB, centre at C.

10. The difference of the two terms

$$V'=-\omega+\frac{c}{r}\omega' \tag{1}$$

is also a P. F., zero over the spherical bowl $AK'B$, where $r=c$, $\omega'-\omega=0$; but over AKB,

$$\omega'+\omega=\pi,\quad V=\pi-2\omega=2\omega'-\pi. \tag{2}$$

At infinity, $r_1=r_2=r=\infty$; $r'=0$;

$$\omega=\sin\omega=\frac{a}{\gamma}=\frac{c\sin\gamma}{r},\quad \sin\omega'=\sin\gamma,\quad V'=\frac{c}{r}(\gamma-\sin\gamma), \tag{3}$$

so that the charge is $c(\gamma-\sin\gamma)$.

At the centre C, where $r=0$, $\omega=\gamma$,

$$\frac{c}{r}\omega'=\sin\gamma,\quad V=-\gamma+\sin\gamma. \tag{4}$$

Mr. J. R. Wilton gives (*Messenger of Mathematics*, p. 96, August, 1914),

$$\phi=-\omega+\frac{c}{r}(\pi-\omega') \tag{5}$$

as the P. F. of the bowl AKB, uninsulated, in presence of a point charge πc at the centre C.

11. The S. F. A of the P. F. ω is then found to be given by

$$A=\tfrac{1}{2}\surd[AB^2-(PA-PB)^2]=\surd(PA\cdot PB)\sin\tfrac{1}{2}APB, \qquad (1)$$

so that A is the semi-conjugate axis of the confocal hyperbola through P, and the meridian curves of constant ω and A are confocal ellipses and hyperbolas.

More generally, any oblate spheroid of which the disc AB is the focal circle, if insulated and electrified with a charge E, will have a P. F. V and S. F. N given at an external point P in the meridian plane APB by

$$V=\frac{2E}{AB}\sin^{-1}\frac{AB}{PA+PB},\quad N=E\sqrt{\left[1-\left(\frac{PA-PB}{AB}\right)^2\right]} \qquad (2)$$

as this verifies at infinity, where $PA=PB=PO=$ infinity, and

$$V=\frac{2E}{AB}\sin^{-1}\frac{AB}{2PO}=\frac{2E}{AB}\cdot\frac{AB}{2PO}=\frac{E}{PO}. \qquad (3)$$

The electrical density σ at a point Q on the spheroid will be given by

$$\sigma=\frac{E}{2\pi}\cdot\frac{1}{QA+QB}\cdot\frac{1}{\surd(QA\cdot QB)}=\frac{\text{electrical force}}{4\pi}, \qquad (4)$$

$$\text{electrical force}=4\pi\sigma=\frac{E}{\tfrac{1}{2}(QA+QB)\surd(QA\cdot QB)}, \qquad (5)$$

electrical charge on the zone $QK=$ half the difference of the S. F. at K and $Q=\frac{1}{2}E\left\{1-\sqrt{\left[1-\left(\frac{QA-QB}{AB}\right)^2\right]}\right\}$. (6)

12. Putting

$$\sqrt{\left[\left(\frac{r_1+r_2}{2}\right)^2-a^2\right]}=B,\text{ with}\sqrt{\left[a^2-\left(\frac{r_1-r_2}{2}\right)^2\right]}=A, \qquad (1)$$

as before in (1) § 11, where

$$r_1^2=x^2+(y+a)^2,\quad r_2^2=x^2+(y-a)^2,\quad r_1^2-r_2^2=4ay, \qquad (2)$$

$$\frac{dr_1}{dx}=\frac{x}{r_1},\quad \frac{dr_2}{dx}=\frac{x}{r_2},\quad \frac{dr_1}{dy}=\frac{y+a}{r_1},\quad \frac{dr_2}{dy}=\frac{y-a}{r_2}, \qquad (3)$$

$$B=\surd(\tfrac{1}{2}\cdot r_1r_2+x^2+y^2-a^2),\quad A=\surd(\tfrac{1}{2}\cdot r_1r_2-x^2-y^2+a^2), \qquad (4)$$

$$B^2+A^2=r_1r_2,\quad B^2-A^2=x^2+y^2-a^2,\quad AB=ax, \qquad (5)$$

$$A,\ B=\tfrac{1}{2}\surd(r_1r_2+2ax)\mp\tfrac{1}{2}\surd(r_1r_2-2ax), \qquad (6)$$

$$\omega=\sin^{-1}\frac{2a}{r_1+r_2}=\cos^{-1}\frac{2B}{r_1+r_2}, \qquad (7)$$

$$\frac{d\omega}{dx}=\frac{-\dfrac{2a}{r_1+r_2}\left(\dfrac{dr_1}{dx}+\dfrac{dr_2}{dx}\right)}{2B}=-\frac{ax}{Br_1r_2}=-\frac{A}{r_1r_2} \qquad (8)$$

$$\frac{dA}{dy}=\frac{-\frac{1}{4}(r_1-r_2)\left(\frac{dr_1}{dy}-\frac{dr_2}{dy}\right)}{A}=\frac{-(r_1-r_2)\left(\frac{y+a}{r_1}-\frac{y-a}{r_2}\right)}{4A}$$

$$=\frac{y(r_1-r_2)^2-a(r_1^2-r_2^2)}{4Ar_1r_2}=\frac{y(a^2-A^2)-a^2y}{Ar_1r_2}=-\frac{yA}{r_1r_2}=y\frac{d\omega}{dx}, \quad (9)$$

$$\frac{d\omega}{dy}=\frac{\frac{-2a}{r_1+r_2}\left(\frac{y+a}{r_1}+\frac{y-a}{r_2}\right)}{2B}=\frac{-ay(r_1+r_2)+a^2(r_1-r_2)}{Br_1r_2(r_1+r_2)}$$

$$=\frac{-ay(r_1^2-r_2^2)+a^2(r_1-r_2)^2}{Br_1r_2(r_1^2-r_2^2)}=\frac{-a^2y^2+a^2(a^2-A^2)}{Br_1r_2ay}$$

$$=\frac{ax^2-aB^2}{Br_1r_2y}=\frac{xAB-aB^2}{Br_1r_2y}=\frac{xA-aB}{r_1r_2y}, \quad (10)$$

$$\frac{dA}{dx}=\frac{-\frac{1}{4}(r_1-r_2)\left(\frac{x}{r_1}-\frac{x}{r_2}\right)}{A}=\frac{x(r_1-r_2)^2}{4Ar_1r_2}$$

$$=\frac{x(-A^2+a^2)}{Ar_1r_2}=\frac{-xA^2+aAB}{Ar_1r_2}=\frac{-xA+aB}{r_1r_2}=-y\frac{d\omega}{dy} \quad (11)$$

which proves, as defined in (2), § 5, that A is the S. F. of the P. F. ω, as stated above in § 11.

Similarly, we prove that B is the S. F. of the P. F.

$$\omega_1=\text{ch}^{-1}\frac{2a}{r_1-r_2}=\text{sh}^{-1}\frac{2A}{r_1-r_2}, \quad (12)$$

and ω_1 can be the P. F. of the electrification of the infinite plate with the circular hole AB cut out, or of any confocal hyperboloid of revolution, the electric charge being infinite.

Then B is the semi-minor axis of the confocal ellipse through P, and

$$B=a\cot\omega=x\coth\omega_1=\surd(r_1r_2)\cos\tfrac{1}{2}APB, \quad (13)$$

$$A=x\tan\omega=a\ \ \text{th}\,\omega_1=\surd(r_1r_2)\sin\tfrac{1}{2}APB; \quad (14)$$

these relations help to settle a doubtful sign.

13. The S. F. at P of the P. F. ω being

$$A=\surd(r_1r_2)\sin\tfrac{1}{2}APB=\sqrt{\left[a^2-\left(\frac{r_1-r_2}{2}\right)^2\right]}, \quad (1)$$

as proved by the preceding differentiations, the S. F. at P' of the P. F. ω' is

$$A'=\surd(r_1'r_2')\sin\tfrac{1}{2}AP'B=\sqrt{\left[a^2-\left(\frac{r_1'-r_2'}{2}\right)^2\right]}$$

$$=\sqrt{\left[c^2\sin^2\gamma-\frac{c^2}{r^2}\left(\frac{r_1-r_2}{2}\right)^2\right]}=\frac{c}{r}\sqrt{\left[r^2\sin^2\gamma-\left(\frac{r_1-r_2}{2}\right)^2\right]}, \quad (2)$$

and then by (1), § 6,

$$C=\sqrt{(r_1r_2)}\sin\tfrac{1}{2}AP'B=\sqrt{\left[r^2\sin^2\gamma-\left(\frac{r_1-r_2}{2}\right)^2\right]} \qquad (3)$$

is a S. F. at P.

But we must not suppose that it has the P. F. $\dfrac{c}{r}\omega'$; the S. F. of this P. F. $\dfrac{c}{r}\omega'$ must be determined to have the S.F. by addition of the whole P.F. $\omega+\dfrac{c}{r}\omega'$.

By analogy with the S. F. of the material of the bowl, given in A. J. M., § 12, where, with the sign changed, AQ is the S. F. of the P. F. Ω, satisfying the relations of A. J. M., § 13,

$$\frac{dAQ}{dA}=-A\frac{d\Omega}{db},\quad \frac{dAQ}{db}=A\frac{d\Omega}{dA}, \qquad (4)$$

and so there the P. F. $\dfrac{c}{r}\Omega'$ has the S. F. $aP-c\Omega\cos V-c\Omega'\cos\phi$; the analogy shows that the S. F. of the P. F. $\dfrac{c}{r}\omega'$ will contain a term $c\omega'\cos\phi$, and Mr. Wilton has found (*Messenger of Mathematics*, p. 70, August, 1914), that the complete expression is

$$c\omega-c\omega'\cos\phi,\ \text{the S. F. of the P. F. } \frac{c}{r}\omega'. \qquad (5)$$

Thus the P. F.

$$V=\omega+\frac{c}{r}\omega' \qquad (6)$$

has the S. F.

$$N=A+c\omega-c\omega'\cos\phi. \qquad (7)$$

This is verified by the differentiation

$$\begin{aligned}\frac{dN}{dx}+y\frac{dV}{dy}&=\frac{dA}{dx}+y\frac{d\omega}{dy}+c\frac{d\omega}{dx}-c\frac{d\omega'}{dx}\cos\phi+c\omega'\sin\phi\frac{\sin\phi}{r}\\&\qquad+r\sin\phi\left(\frac{c}{r}\frac{d\omega'}{dy}-\frac{c}{r^2}\omega'\sin\phi\right)\\&=c\frac{d\omega}{dx}+c\left(-\frac{d\omega'}{dx}\cos\phi+\frac{d\omega'}{dy}\sin\phi\right)=c\left(\frac{d\omega}{dx}+\frac{d\omega'}{dr}\right)=0, \qquad (8)\end{aligned}$$

$$\begin{aligned}\frac{dN}{dy}-y\frac{dV}{dx}&=\frac{dA}{dy}-y\frac{d\omega}{dx}+c\frac{d\omega}{dy}-c\frac{d\omega'}{dy}\cos\phi+c\omega'\sin\phi\frac{\cos\phi}{r}\\&\qquad-r\sin\phi\left(\frac{c}{r}\frac{d\omega'}{dx}+\frac{c}{r^2}\omega'\cos\phi\right)\\&=c\frac{d\omega}{dy}-c\left(\frac{d\omega'}{dx}\sin\phi+\frac{d\omega'}{dy}\cos\phi\right)=c\left(\frac{d\omega}{dy}-\frac{d\omega'}{rd\phi}\right)=0. \qquad (9)\end{aligned}$$

These relations in (8) (9) are verified by putting

$$B' = \sqrt{\left[\left(\frac{r_1' + r_2'}{2}\right)^2 - a^2\right]} = \frac{c}{r} D, \text{ with } A' = \sqrt{\left[a^2 - \left(\frac{r_1' - r_2'}{2}\right)^2\right]} = \frac{c}{r} C, \quad (10)$$

as before; and then, differentiating,

$$\frac{d\omega'}{rd\phi} = \frac{r'}{r}\frac{d\omega'}{r'd\phi} = \frac{c^2}{r^2}\left(\frac{d\omega'}{dx'}\sin\phi + \frac{d\omega'}{dy'}\cos\phi\right)$$

$$= \frac{c^2}{r^2}\left(\frac{-A'\sin\phi}{r_1'r_2'} + \frac{x'A' - aB'}{r_1'r_2'y'}\cos\phi\right) = -\frac{A'\sin\phi}{r_1r_2} + \frac{x'A' - aB'}{r_1r_2\frac{c^2}{r^2}y}\cos\phi$$

$$= -\frac{\frac{c}{r}C\sin\phi}{r_1r_2} + \frac{\left(c\cos\gamma - \frac{c^2}{r}\cos\phi\right)\frac{c}{r}C\cos\phi - c\sin\gamma\frac{c}{r}D\cos\phi}{r_1r_2\frac{c^2}{r^2}y}$$

$$= \left[-\frac{c}{r}Cr\sin^2\phi + \left(c\cos\gamma - \frac{c^2}{r}\cos\phi\right)\frac{r}{c}C\cos\phi - c\sin\gamma\frac{r}{c}D\cos\phi\right] \div r_1r_2y$$

$$= [(-c + r\cos\phi\cos\gamma)C - Dr\sin\gamma\cos\phi] \div r_1r_2y$$

$$= [x(-C\cos\gamma + D\sin\gamma) - a(C\sin\gamma + D\cos\gamma)] \div r_1r_2y$$

$$= \frac{xA - aB}{r_1r_2y} = \frac{d\omega}{dy}; \quad (11)$$

because *AE*, *BE* bisect the angles *PAP′*, *PBP′* if the spherical surface cuts *CPP′* in *E*; and (*APC*, *AEC*, *AP′C*) (*BPC*, *BEC*, *BP′C*) are angles in arithmetical progression, and so is their difference, so that

$$\tfrac{1}{2}APB + \tfrac{1}{2}AP'B = AEB = \gamma, \quad (12)$$

$$A = \sqrt{(r_1r_2)}\sin\tfrac{1}{2}APB = \sqrt{(r_1r_2)}\sin(\gamma - \tfrac{1}{2}AP'B) = D\sin\gamma - C\cos\gamma, \quad (13)$$

$$B = \sqrt{(r_1r_2)}\cos\tfrac{1}{2}APB = \sqrt{(r_1r_2)}\cos(\gamma - \tfrac{1}{2}APB) = D\cos\gamma + C\sin\gamma, \quad (14)$$

as if *A*, *B* and *C*, *D* were orthogonal components of the same vector.

Differentiating again

$$\frac{d\omega'}{dr} = -\frac{r'}{r}\frac{d\omega'}{dr'} = -\frac{c^2}{r^2}\left(-\frac{d\omega'}{dx'}\cos\phi + \frac{d\omega'}{dy'}\sin\phi\right)$$

$$= -\frac{c^2}{r^2}\left(\frac{A'\cos\phi}{r_1'r_2'} + \frac{x'A' - aB'}{r_1'r_2'y'}\sin\phi\right)$$

$$= -\frac{\frac{c}{r}C\cos\phi}{r_1r_2} - \frac{\left(c\cos\gamma - \frac{c^2}{r}\cos\phi\right)\frac{c}{r}C\sin\phi - c\sin\gamma\frac{c}{r}D\sin\phi}{r_1r_2\frac{c^2}{r}\sin\phi}$$

$$= \left[-\frac{c}{r}C\cos\phi - \left(\cos\gamma - \frac{c}{r}\cos\phi\right)C + D\sin\gamma\right] \div r_1r_2$$

$$= \frac{-C\cos\gamma + D\sin\gamma}{r_1r_2} = \frac{A}{r_1r_2} = -\frac{d\omega}{dx}. \quad (15)$$

Mr. Wilton has considered also the S. F.

$$a\frac{\cos\omega+\cos\omega'}{\sin\omega}=B+D, \tag{16}$$

which vanishes over the bowl, and has the P. F.

$$\omega\sin\gamma+\omega_1(1+\cos\gamma), \tag{17}$$

and investigated the physical interpretation. So also

$$a\frac{\operatorname{sh}\omega_1+\operatorname{sh}\omega_1'}{\operatorname{ch}\omega_1}=A+C, \tag{18}$$

is a S. F. with a P. F.

$$\omega(1-\cos\gamma)+\omega_1\sin\gamma. \tag{19}$$

Other combinations can be made and interpreted (J. R. Wilton, *Messenger of Mathematics*, p. 75, August, 1914).

14. Over the bowl AKP_iB,

$$r=c,\quad r_1=2c\sin\tfrac{1}{2}(\gamma+\phi),\quad r_2=2c\sin\tfrac{1}{2}(\gamma-\phi),\quad r_1+r_2=4c\sin\tfrac{1}{2}\gamma\cos\tfrac{1}{2}\phi, \tag{1}$$

$$\sin\omega=\frac{2a}{r_1+r_2}=\frac{2c\sin\gamma}{4c\sin\frac{1}{2}\gamma\cos\frac{1}{2}\phi}=\frac{\cos\frac{1}{2}\gamma}{\cos\frac{1}{2}\phi}=\frac{BK'}{PK'}. \tag{2}$$

But over $BK'A$, the rest of the spherical surface,

$$r_1=2c\sin\tfrac{1}{2}(\phi+\gamma),\quad r_2=2c\sin\tfrac{1}{2}(\phi-\gamma),\quad r_1+r_2=4c\cos\tfrac{1}{2}\gamma\sin\tfrac{1}{2}\phi \tag{3}$$

$$\sin\omega=\frac{\sin\frac{1}{2}\gamma}{\sin\frac{1}{2}\phi}=\frac{AK}{QK}. \tag{4}$$

The P. F. $V=\omega+\dfrac{c}{r}\omega'$ must be adjusted as a single valued function, although ω and ω' are multiple-valued.

Starting from a point S on the spherical surface $AK'B$, where

$$\omega=\omega'=\sin^{-1}\frac{AK}{SK}, \tag{5}$$

taken as the acute angle, and $V=2\omega$, let the point P travel from S to P_i on the interior of the bowl, and for simplicity along the orthogonal circle, with limiting points A and B.

Then P' will travel from S along this circle in the opposite direction, and will reach a point P_0 on the outside of the bowl, coincident with P_i. In this path P' does not cross the base AB of the bowl, and ω' never reaches $\frac{1}{2}\pi$, and $\cos\omega'$ remains positive.

But P in crossing the base AB makes $\omega=\frac{1}{2}\pi$; and beyond AB and up to P_i we must take $\omega>\frac{1}{2}\pi$, and $\cos\omega$ negative.

Thus if ω_0, ω_i denotes the value of ω according as it is reached from S by a path to the outside at P_0, or inside at P_i of the bowl,

$$\sin\omega_0 = \sin\omega_i = \frac{BK'}{PK'}, \quad \cos\omega_0 = -\cos\omega_i, \quad \omega_0 + \omega_i = \pi, \tag{6}$$

making π the potential on the bowl; and we can put

$$\omega_0 = \tfrac{1}{2}(\pi - \theta), \quad \omega_i = \tfrac{1}{2}(\pi + \theta), \quad \cos\tfrac{1}{2}\theta = \frac{BK'}{PK'}, \tag{7}$$

where $\frac{1}{2}\theta$ is a positive acute angle. At S, the S. F. is

$$N(S) = \tfrac{1}{2}\sqrt{[AB^2 - (SA - SB)^2]} + c(1 - \cos\phi)\sin^{-1}\frac{AK}{SK}; \tag{8}$$

and travelling from S to P_i,

$$\begin{aligned} N(P_i) = A + c\omega_i - c\omega_0\cos\phi = \tfrac{1}{2}\sqrt{[AB^2 - (PA - PB)^2]} \\ + \tfrac{1}{2}(\pi + \theta)c - \tfrac{1}{2}(\pi - \theta)c\cos\phi. \end{aligned} \tag{9}$$

At B, $\theta = 0$, $A = 0$, $\phi = \gamma$, $N(B) = \frac{1}{2}\pi c(1 - \cos\gamma)$, (10)

$$N(P_i) - N(B) = A + \tfrac{1}{2}(\pi + \theta)c - \tfrac{1}{2}(\pi - \theta)c\cos\phi - \tfrac{1}{2}\pi c(1 - \cos\gamma). \tag{11}$$

At K, $\theta = \gamma$, $\phi = 0$, $A = a = c\sin\gamma$,

$$N(K) = c\sin\gamma + \tfrac{1}{2}(\pi + \gamma)c - \tfrac{1}{2}(\pi - \gamma)c = c\sin\gamma + c\gamma, \tag{12}$$

$$N(K_i) - N(B) = c\sin\gamma + c\gamma + \tfrac{1}{2}\pi c(1 - \cos\gamma). \tag{13}$$

15. Denoting by σ_i the electrical density at P_i, on the inside of the bowl,

$$4\pi\sigma_i = +\frac{dV_i}{dr} = -\frac{1}{y}\frac{dN(P_i)}{cd\phi} \tag{1}$$

and the charge on the interior of the bowl swept out by the revolution of the arc P_iB is

$$\int 2\pi\sigma_i y c d\phi = \tfrac{1}{2}N(P_i) - \tfrac{1}{2}N(B), \tag{2}$$

and the total charge on the interior of the bowl is

$$\tfrac{1}{2}N(K_i) - \tfrac{1}{2}N(B) = \tfrac{1}{2}(c\sin\gamma + c\gamma) - \tfrac{1}{4}\pi c(1 - \cos\gamma). \tag{3}$$

At P_0 on the exterior of the bowl the sign of A must be changed in the S. F., and ω_0, ω_i interchanged, making

$$\begin{aligned} N(P_0) &= -A + c\omega_0 - c\omega_i\cos\phi \\ &= -\tfrac{1}{2}\sqrt{[AB^2 - (PA - PB)^2]} + \tfrac{1}{2}(\pi - \theta)c - \tfrac{1}{2}(\pi + \theta)c\cos\phi, \end{aligned} \tag{4}$$

$$N(K_0) = -c\sin\gamma - c\gamma. \tag{5}$$

The electrical density σ_0 on the outside of the bowl, at P_0, is then given by

$$4\pi\sigma_0 = -\frac{dV_0}{dr} = \frac{1}{y}\frac{dN(P_0)}{cd\phi}, \tag{6}$$

and the charge on the outside of the part of the bowl swept out by P_0B is

$$\int 2\pi\sigma_0 y c d\phi = \tfrac{1}{2}N(B) - \tfrac{1}{2}N(P_0), \tag{7}$$

and the total charge on the outside is

$$\tfrac{1}{2}N(B)-\tfrac{1}{2}N(K_0)=\tfrac{1}{2}(c\gamma+c\sin\gamma)+\tfrac{1}{2}\pi c(1-\cos\gamma). \tag{8}$$

The total charge on the bowl, outside and inside, is then

$$\tfrac{1}{2}N(K_0)-\tfrac{1}{2}N(K_i)=c\gamma+c\sin\gamma \tag{9}$$

at potential π, so that the capacity of the bowl is

$$\frac{c\gamma+c\sin\gamma}{\pi}=\frac{\text{arc } AKB+\text{chord } AOB}{2\pi}=\frac{\text{meridian girth}}{2\pi}, \tag{10}$$

this is the radius of a sphere of the same girth.

This verifies for the complete sphere, and a flat circular disc AB.

The difference of the charge on the part of the bowl swept out by KP, outside and inside, is

$$\begin{aligned}\int 2\pi(\sigma_0-\sigma_i)yc\,d\phi &= \tfrac{1}{2}N(P_0)-\tfrac{1}{2}N(K_0)+\tfrac{1}{2}N(K_i)-\tfrac{1}{2}N(P_i)\\ &=\tfrac{1}{2}(\pi-\theta)c-\tfrac{1}{2}(\pi+\theta)c\cos\phi-\tfrac{1}{2}(\pi-\theta)c\cos\phi+\tfrac{1}{2}(\pi+\theta)c\\ &=\pi c(1-\cos\phi)=\frac{\text{surface of the part}}{2c},\end{aligned} \tag{11}$$

so that the difference $\sigma_0-\sigma_i$ is constant $=\dfrac{1}{4c}$, $\quad 4\pi(\sigma_0-\sigma_i)=\dfrac{\pi}{c}$. (12)

16. At P_i on the bowl

$$\begin{aligned}A&=\tfrac{1}{2}\sqrt{(4c^2\sin^2\gamma-16c^2\cos^2\tfrac{1}{2}\gamma\sin^2\tfrac{1}{2}\phi)}\\ &=2\cos^2\tfrac{1}{2}\gamma\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)};\end{aligned} \tag{1}$$

$$\omega_i=\tfrac{1}{2}(\pi+\theta),\quad \omega_0=\tfrac{1}{2}(\pi-\theta), \tag{2}$$

$$\frac{d\omega_i}{d\phi}=-\frac{d\omega_0}{d\phi}=\tfrac{1}{2}\frac{d\theta}{d\phi}=\frac{\tfrac{1}{2}\cos\tfrac{1}{2}\gamma\tan\tfrac{1}{2}\phi}{\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}}, \tag{3}$$

$$\cos\tfrac{1}{2}\theta=\frac{\cos\tfrac{1}{2}\gamma}{\cos\tfrac{1}{2}\phi},\quad \sin\tfrac{1}{2}\theta=\frac{\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}}{\cos\tfrac{1}{2}\phi}, \tag{4}$$

$$\tan\omega_0=\cot\tfrac{1}{2}\theta=\frac{\cos\tfrac{1}{2}\gamma}{\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}}, \tag{5}$$

$$N(P_i)=2c\cos\tfrac{1}{2}\gamma\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}+c\omega_i-c\omega_0\cos\phi, \tag{6}$$

$$\begin{aligned}\frac{dN(P_i)}{c\,d\phi}&=-\frac{\tfrac{1}{2}\cos\tfrac{1}{2}\gamma\sin\phi}{\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}}-\tfrac{1}{2}(1+\cos\phi)\frac{d\theta}{d\phi}+\omega_0\sin\phi\\ &=-\frac{\cos\tfrac{1}{2}\gamma\sin\phi}{\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}}+\omega_0\sin\phi=-(\tan\omega_0-\omega_0)\sin\phi,\end{aligned} \tag{7}$$

$$4\pi\sigma_i=-\frac{1}{c\sin\phi}\frac{dN(P_i)}{c\,d\phi}=\frac{\tan\omega_0-\omega_0}{c}, \tag{8}$$

$$4\pi(\sigma_0-\sigma_i)=\frac{\pi}{c}=\frac{\omega_i+\omega_0}{c}, \tag{9}$$

$$4\pi\sigma_0=\frac{\omega_i+\tan\omega_0}{c}=\frac{\omega_i-\tan\omega_i}{c}. \tag{10}$$

If the potential is U, instead of π, these values of σ and the charge in (9), § 15, must be multiplied by $\frac{U}{\pi}$, making

$$4\pi\sigma_i = \frac{U}{\pi c}(\tan\omega_0 - \omega_0), \quad 4\pi\sigma_0 = \frac{U}{\pi c}(\omega_i - \tan\omega_i), \tag{11}$$

$$4\pi\sigma_i = \frac{U}{\pi c}\left[\frac{AK'}{\sqrt{(AK^2 - KP_0^2)}} - \tan^{-1}\frac{AK'}{\sqrt{(AK^2 - KP_0^2)}}\right], \tag{12}$$

$$\tan\omega_0 = -\tan\omega_i = \frac{AK'}{\sqrt{(AK^2 - KP_0^2)}}. \tag{13}$$

Draw the circle, centre K' and radius $K'P_0$, cutting KA in R, R'; then,

$$AR^2 = K'R^2 - K'A^2 = K'P_0^2 - K'A^2 = KA^2 - KP_0^2;$$

$$ARK' = \omega_0, \quad AK'R = \tfrac{1}{2}\theta, \quad RK'R' = \theta. \tag{14}$$

Anywhere on the axis G on the convex side of the bowl,

$$A = 0, \quad \sin\omega = \frac{OA}{AG} = \sin CGA, \quad \sin\omega' = \frac{CG\sin\gamma}{AG} = \sin CAG,$$

$$\omega = CGA, \quad \omega' = \pi - CAG, \quad \phi = 0, \tag{15}$$

$$N(G) = A + c\omega - c\omega' = a + c(CGA + CAG) a - c\gamma = c(\sin\gamma - \gamma), \tag{16}$$

$$N(G') = A + c\omega' - c\omega = a + c\gamma = a(\sin\gamma + \gamma). \tag{17}$$

And anywhere on the axis at H on the concave side of the bowl, beyond C,

$$A = 0, \quad \omega = CHA, \quad \omega' = CAH, \quad \phi = \pi, \tag{18}$$

$$N(H) = a + c\omega + c\omega' = a + c\gamma = c(\sin\gamma + \gamma). \tag{19}$$

Thus N is $a + c\gamma$ on the concave side along the axis HK, and changes to $a - c\gamma$ along the prolongation KG, in crossing the bowl to the convex side.

17. For the hydrodynamical application to the axial motion U of the circular base AB through an infinite liquid, the velocity function (V. F.) V and stream function (S. F.) N are given by

$$V = Ux\frac{\tan\omega - \omega}{\frac{1}{2}\pi}, \quad N = \tfrac{1}{2}Uy^2\frac{\omega - \frac{1}{2}\sin 2\omega}{\frac{1}{2}\pi} \tag{1}$$

at P; and at P',

$$V' = Ux'\frac{\tan\omega' - \omega'}{\frac{1}{2}\pi}, \quad N' = \tfrac{1}{2}Uy'^2\frac{\omega' - \frac{1}{2}\sin 2\omega'}{\frac{1}{2}\pi}; \tag{2}$$

and the combination

$$V + \frac{c}{r}V', \text{ with } N + \frac{r}{c}N', \tag{3}$$

will serve for Basset's expression of the V. F. and S. F. in the liquid motion due to the axial velocity U of the bowl, in agreement with that given in his "Hydrodynamics," I, pp. 153–156.

Fixed in the current, in an axial stream U past the circular base AB,

$$V=-Ux\left(1-\frac{\tan\omega-\omega}{\frac{1}{2}\pi}\right),\quad N=\tfrac{1}{2}Uy^2\left(1-\frac{\omega-\frac{1}{2}\sin 2\omega}{\frac{1}{2}\pi}\right), \tag{4}$$

with $\omega=\frac{1}{2}\pi$, $N=0$ over the disc AB; while the V. F.

$$\frac{2N\cos\psi}{y}=Uy\left(1-\frac{\omega-\frac{1}{2}\sin 2\omega}{\frac{1}{2}\pi}\right)\cos\psi \tag{5}$$

will give the potential of the electric field, uniform and of potential $Uy\cos\psi$ across the axis Ox, when the field is disturbed by the disc AB, earthed to zero potential; with similar extensions when the disc is replaced by the bowl.

If the bowl is to earth in an axial field Ux, the potential of the field is changed to

$$V=U\left(\frac{c\omega\cos\gamma}{\pi}+x\frac{\tan\omega-\omega}{\pi}\right)+U\frac{c}{r}\left(\frac{c\omega'\cos\gamma}{\pi}+x'\frac{\tan\omega'-\omega'}{\pi}\right)$$
$$-U(c\cos\gamma-x). \tag{6}$$

(Basset, I, p. 154; Gallop, Q. J. M., XXI, p. 256), with the S. F.

$$N=U\left(\frac{Ac\cos\gamma}{\pi}+\tfrac{1}{2}y^2\frac{\omega-\frac{1}{2}\sin 2\omega}{\pi}\right)$$
$$+U\frac{r}{c}\left(\frac{A'c\cos\gamma}{\pi}+\tfrac{1}{2}y'^2\frac{\omega'-\frac{1}{2}\sin 2\omega'}{\pi}\right)-\tfrac{1}{2}Uy^2. \tag{7}$$

18. Similar extensions can be made with applications of the conical angle Ω, Ω', instead of plane angles ω, ω'.

With the P. F. of the material of the bowl

$$V=\Omega+\frac{c}{r}\Omega'=\Omega+\frac{r'}{c}\Omega', \tag{1}$$

$V=\Omega+\Omega'=2\Omega$ over the spherical surface $AK'B$; but over the bowl AKB,

$$\Omega_i=2\pi+\Omega_1,\quad \Omega_0=-2\pi+\Omega_1,\quad \Omega+\Omega'=\Omega_i+\Omega_0=2\Omega_1, \tag{2}$$

so that V is not constant over the bowl AKB, or the rest of the sphere.

To obtain a constant potential over the surface, take

$$V=\Omega-\frac{c}{r}\Omega', \tag{3}$$

which is zero over $AK'B$, and 4π over AKB; and this V can serve for the electric potential when the part $AK'B$ of the sphere is to earth, and the other part AKB is changed to potential 4π, the insulation being made perfect along the line of separation of the two parts AKB, $AK'B$.

The associated S. F. can be written down, as the difference of those given in § 13,

$$N=-AQ-aP+c\Omega\cos\gamma+c\Omega'\cos\phi. \tag{4}$$

But along the insulating circle AB, $A=a$, $P+Q=\infty$; and the charge on each bowl is infinite, with an infinite electromotive force across the line of perfect insulation, so here is an electrical version of the old dynamical paradox of an irresistible force, push, or blow, applied to an immovable body.

The question was proposed for the special case of two hemispheres in the *Mathematical Tripos*, II, 1912, and the answer given in a series, but the series is divergent and the result infinite, as it should be.

19. With the potential of the bowl,

$$V=\Omega+\frac{c}{r}\Omega', \tag{1}$$

$$\frac{dV}{dr}=\frac{d\Omega}{dr}+\frac{c}{r}\frac{d\Omega'}{dr}-\frac{c}{r^2}\Omega'=-\frac{c}{r^2}\Omega', \tag{2}$$

$$\frac{dV}{rd\phi}=\frac{d\Omega}{rd\phi}+\frac{c}{r}\frac{d\Omega'}{rd\phi}=-\frac{Q}{r}, \tag{3}$$

$$\frac{dV}{dx}=-\frac{dV}{dr}\cos\phi+\frac{dV}{rd\phi}\sin\phi=\frac{c}{r^2}\Omega'\cos\phi-\frac{Q}{r}\sin\phi; \tag{4}$$

and as this is a P. F. at P, there is another P. F. at P',

$$\frac{c}{r'^2}\Omega\cos\phi-\frac{Q'}{r'}\sin\phi=\frac{r^2}{c^3}(\Omega\cos\phi-Q\sin\phi), \tag{5}$$

or another P. F. at P,

$$\frac{r}{c^2}(\Omega\cos\phi-Q\sin\phi). \tag{6}$$

The associated S. F. can be obtained from

$$\begin{aligned}\frac{dN}{dx}&=-r\sin\phi\frac{dV}{dy}=-r\sin\phi\left(\frac{dV}{dr}\sin\phi+\frac{dV}{rd\phi}\cos\phi\right)\\&=r\sin\phi\left(\frac{c}{r^2}\Omega'\sin\phi+\frac{Q}{r}\cos\phi\right)\\&=\frac{c}{r}\Omega'\sin^2\phi+Q\sin\phi\cos\phi=M_1,\end{aligned} \tag{7}$$

suppose, and there is another S. F. at P,

$$M_2=\frac{r}{c}\left(\frac{c}{r'}\Omega\sin^2\phi+Q'\sin\phi\cos\phi\right)=\frac{r^2}{c^2}(\Omega\sin^2\phi+Q\sin\phi\cos\phi). \tag{8}$$

Then
$$M_2-M_1=\left(\frac{r^2}{c^2}\Omega-\frac{c}{r}\Omega'\right)\sin^2\phi+\left(\frac{r^2}{c^2}-1\right)Q\sin\phi\cos\phi \tag{9}$$

is a S. F., zero over $AK'B$, where $r=c$, $\Omega=\Omega'$, and over the bowl AKB,

$$M_1-M_2=4\pi\sin^2\phi=4\pi\frac{y^2}{c^2} \tag{10}$$

Thus $M_1 - M_2$ is the S. F. of the liquid motion at the instant when the spherical lid *AKB* is dropped on the spherical bowl *AK'B*, or lifted again, the liquid velocity being great where the windage is small.

An interpretation can also be given to the motion where the V. F. is

$$\frac{M_1-M_2}{r\sin\phi}\cos\psi=\left[\left(\frac{r}{c^2}\,\Omega-\frac{c}{r^2}\,\Omega'\right)\sin\phi+\left(\frac{r}{c^2}-\frac{1}{r}\right)Q\cos\phi\right]\cos\psi, \tag{11}$$

the lid *AKB* sliding across the bowl *AK'B*.

20. Figures of revolution can also be made in fig. 1 about the axis Oy. For a uniform rod AB, or a confocal prolate spheroid with an electric charge E,

$$V=\frac{2E}{AB}\,\text{th}^{-1}\frac{AB}{PA+PB},\quad N=E(PA-PB), \tag{1}$$

obtained from the integral for the potential of a uniform rod

$$\begin{aligned}\int_{-a}^{a}\frac{dy'}{\sqrt{[x^2+(y'-y)^2]}}&=\text{sh}^{-1}\frac{y+a}{x}-\text{sh}^{-1}\frac{y-a}{x}=\text{ch}^{-1}\frac{r_1}{x}-\text{ch}^{-1}\frac{r_2}{x}\\&=\text{sh}^{-1}\frac{a(r_1+r_2)-y(r_1-r_2)}{x^2}=\text{ch}^{-1}\frac{r_1r_2-y^2+a^2}{x^2}\\&=2\text{th}^{-1}\sqrt{\frac{r_1r_2-x^2-y^2+a^2}{r_1r_2+x^2-y^2+a^2}}=2\text{th}^{-1}\frac{2a}{r_1+r_2}=2\text{th}^{-1}\frac{2y}{r_1-r_2}.\end{aligned} \tag{2}$$

Then

$$V=\text{sh}^{-1}\frac{y+a}{x}=\text{ch}^{-1}\frac{r_1}{x},\quad N=r_1, \tag{3}$$

gives the P. F. and S. F. for a positive semi-infinite rod Ay, and negative rod Ay'. With the break at O, and $a=0$,

$$V=\text{sh}^{-1}\frac{y}{x}=\text{sh}^{-1}\tan\theta=\text{ch}^{-1}\sec\theta=\log(\sec\theta+\tan\theta),\quad N=r, \tag{4}$$

so that the stream sheets are spherical and of uniform thickness, the flow issuing from a pole, and disappearing at the other pole, with velocity inversely as the distance from the axis.

In the conformal representation on a Mercator chart, there would be a uniform current, North and South. The S. F.

$$N=E(PA+PB)\text{ has the P. F. }\frac{2E}{AB}\,\text{th}^{-1}\frac{2y}{r_1+r_2}=\frac{2E}{AB}\,\text{th}^{-1}\frac{PA-PB}{AB}, \tag{5}$$

as required for confocal hyperboloids of revolution about the focal line AB, and excentricity $AB \div (PA-PB)$.

The typical element of a P. F. is $\dfrac{1}{r}$, as of a point source or sphere; and then the S. F. is $\cos\theta$ or $\sin\theta$ according as Ox or Oy is the axis.

But the simplest element of a S. F. is r, as in (4), for a line source Oy, and a line sink Oy'.

The single line source Oy would have the

$$\text{S. F. } N=r-y, \text{ and V. F. } V=\text{sh}^{-1}\frac{y}{x}+\log x=\log(r+y), \tag{6}$$

and surfaces of constant N and V are orthogonal confocal paraboloids, with focus at O.

21. For a magnetic molecule, or sphere magnetized uniformly or moving in infinite liquid in direction Ox,

$$V=-\frac{d}{dx}\frac{1}{r}=\frac{x}{r^3},\quad N=\frac{d}{dx}\cos\phi=-\sin\phi\frac{d\phi}{dx}=\frac{\sin^2\phi}{r}=\frac{y^2}{r^3}, \tag{1}$$

$$\frac{dN}{dx}=-y\frac{dV}{dy}=-\frac{3xy^2}{r^5},\quad \frac{dN}{dy}=y\frac{dV}{dx}=\frac{2y}{r^3}-\frac{3xy^2}{r^5}. \tag{2}$$

For a magnetized oblate spheroid, on a focal circle AB, magnetized axially, or for the liquid motion due to axial velocity U,

$$V=-\frac{UxA(\lambda)}{2B(0)},\qquad N=\frac{\frac{1}{2}Uy^2B(\lambda)}{B(0)}, \tag{3}$$

$$A(\lambda)=\tan\omega-\omega,\quad B(\lambda)=\tfrac{1}{2}\omega-\tfrac{1}{4}\sin 2\omega,\quad \sin\omega=\frac{AB}{PA+PB}, \tag{4}$$

and $A(0)$, $B(0)$ the value of $A(\lambda)$, $B(\lambda)$ over the surface of the spheroid.

For a prolate spheroid with a focal line AB, magnetized or moving axially,

$$V=-\frac{UyA(\lambda)}{2B(0)},\qquad N=\frac{\frac{1}{2}Uy^2B(\lambda)}{B(0)}, \tag{5}$$

$$A(\lambda)=\tfrac{1}{4}\text{sh}2\zeta-\tfrac{1}{2}\zeta,\quad B(\lambda)=\zeta-\text{th}\zeta,\quad \text{th}\zeta=\sin=\frac{AB}{PA+PB}=\frac{2a}{r_1+r_2}, \tag{6}$$

and then

$$B(\lambda)=\text{th}^{-1}\frac{2a}{r_1+r_2}-\frac{2a}{r_1+r_2},$$

$$A(\lambda)=\tfrac{1}{4}\left(\frac{2}{\frac{r_1+r_2}{2a}-\frac{2a}{r_1+r_2}}-\text{sh}^{-1}\frac{2}{\frac{r_1+r_2}{2a}-\frac{2a}{r_1+r_2}}\right) \tag{7}$$

22. The expression for the potential W of the solid homogeneous ellipsoid

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1 \tag{1}$$

can be derived from the components of the attraction, in the manner of §§ 8, 10,

A. J. M., p. 387, for the lens, by treating the potential as a homogeneous function of the second degree in x, y, z, and α, β, or γ, so that

$$2W=\alpha\frac{dW}{d\alpha}+x\frac{dW}{dx}+y\frac{dW}{dy}+z\frac{dW}{dz}. \tag{2}$$

It is not so difficult then (Thomson and Tait, § 494), to prove, as in Dirichlet's manner that the components of the attraction are given by

$$\frac{dW}{dx},\ \frac{dW}{dy},\ \frac{dW}{dz}=-\int_\lambda^\infty\frac{4\pi G\rho\alpha\beta\gamma(x,\,y,\,z)}{\psi+\alpha^2,\ \psi+\beta^2,\ \psi+\gamma^2}\,\frac{d\psi}{P(\psi)}, \tag{3}$$

$$P^2(\psi)=4\cdot\psi+\alpha^2\cdot\psi+\beta^2\cdot\psi+\gamma^2, \tag{4}$$

and the confocal ellipsoid through the attracted point $(x,\,y,\,z)$ is

$$\frac{x^2}{\lambda+\alpha^2}+\frac{y^2}{\lambda+\beta^2}+\frac{z^2}{\lambda+\gamma^2}=1; \tag{5}$$

while $\alpha\dfrac{dW}{d\alpha}$, due to the uniform swelling of the ellipsoid in (1), whereby $(\alpha,\,\beta,\,\gamma,\,p)$ increases slightly by $k(\alpha,\,\beta,\,\gamma,\,p)$, p denoting the perpendicular from the centre on a tangent plane.

This makes dW the P. F. of a film, or coat of matter like paint, of thickness kp and superficial density $\sigma=k\rho p$, equivalent of an electrical film of superficial electrical density $kG\rho p$, and charge

$E=kG\rho\int pdS=kG\rho$ (three times the volume of the ellipsoid, or $4\pi\alpha\beta\gamma$),

and then $\sigma=\dfrac{Ep}{4\pi G\alpha\beta\gamma}$. (6)

Assuming the expression for the potential of this electrical distribution in equilibrium as

$$\int_\lambda^\infty\frac{Ed\psi}{P(\psi)}, \tag{7}$$

this makes, with $d\alpha$ for $k\alpha$,

$$dW=\int\frac{4\pi kG\rho\alpha\beta\gamma d\psi}{P(\psi)},\quad \alpha\frac{dW}{d\alpha}=\int\frac{4\pi G\rho\alpha\beta\gamma d\psi}{P(\psi)} \tag{8}$$

Thence the complete expression of W as

$$W=\int_\lambda^\infty\left(1-\frac{x^2}{\psi+\alpha^2}-\frac{y^2}{\psi+\beta^2}-\frac{z^2}{\psi+\gamma^2}\right)\frac{2\pi G\rho\alpha\beta\gamma d\psi}{P(\psi)}, \tag{9}$$

in which each term has received a physical interpretation.

If (x, y, z) is inside the ellipsoid, the lower limit λ must be replaced by zero.

23. In Professor Andrew Gray's method, "On the Attraction of a Spherical and Ellipsoidal Shell" (*Proceedings Edinburgh Mathematical*

Society, 1914), he produces CE in figure 3 to meet the concentric sphere through P in E', and then AE' is parallel to $P'E$ and equal to EP,

$$\frac{dF}{G\sigma}=\frac{dS\cos\theta}{Ep^2}=\frac{CA^2}{CP^2}\frac{dS'\cos\theta}{AE'^2}=\frac{c^2}{r^2}d\Omega', \tag{1}$$

where $d\Omega'$ is the conical angle subtended at A by dS' at E'.

He extends the method to the ellipsoidal shell through E in figure 4, drawing the confocal ellipsoid through P, and taking A, P, and E, E' corresponding points on the ellipsoids, so that $AE'=EP$.

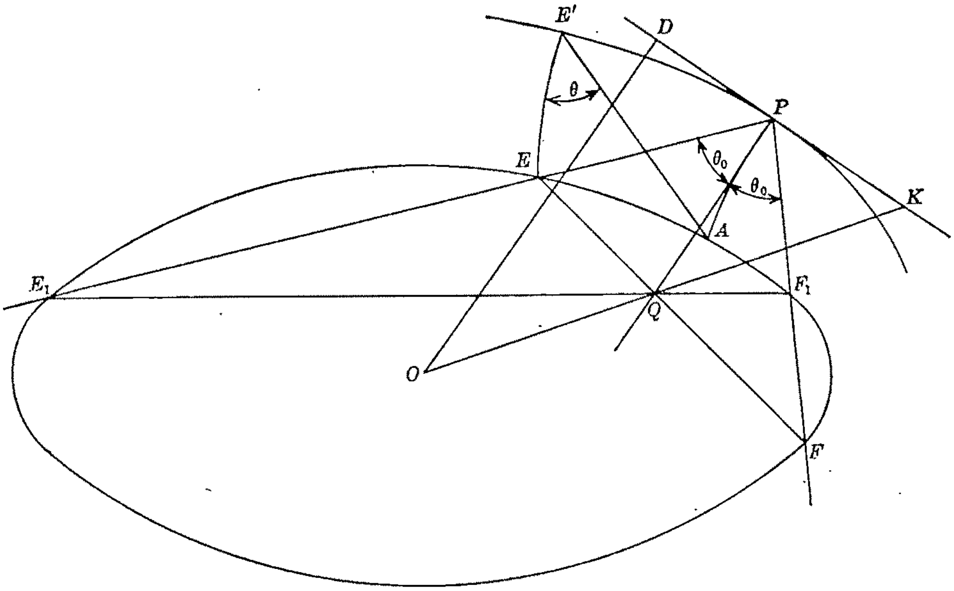

FIG. 4.

Draw the diameter parallel to AE, cutting the tangent plane at A, E in W_0, W; and denote the perpendicular from the centre O on the tangent plane at A, E by p_0, p; and denote the angle which AE makes with the normal at A, E by ϕ_0, ϕ; then $p_0\sec\phi_0=OW_0$, $p\sec\phi=OW$; and the plane through O and the line of intersection of the tangent planes at A, E bisects AE, so that

$$OW_0=OW,\quad p_0\sec\phi_0=p\sec\phi. \tag{2}$$

Similarly with ϖ_0, ϖ the perpendicular from the centre on the tangent planes at P, E', and θ_0, θ the angles which EP, AE' make with the normal at P, E', Professor Gray finds

$$\varpi_0\sec\theta_0=\varpi\sec\theta; \tag{3}$$

the equivalent of $OV_0=OV$, if OV_0, OV, parallel to EP, AE' cut the tangent planes at P, E' in V_0, V.

For if (x, y, z), (x', y', z') are the coordinates of P, E,

$$\frac{PE}{OV_0}=\frac{\text{perpendicular from } E \text{ on the tangent plane at } P}{\text{perpendicular from } O \text{ on the tangent plane at } P}$$

$$=1-\frac{xx'}{\lambda+\alpha^2}-\frac{yy'}{\lambda+\beta^2}-\frac{zz'}{\lambda+\gamma^2}, \tag{4}$$

and this ratio is unaltered in $\frac{AE'}{OV}$, because the coordinates of the corresponding points A, E' are

$$x''=x\sqrt{\frac{\alpha^2}{\lambda+\alpha^2}},\quad y''=y\sqrt{\frac{\beta^2}{\lambda+\beta^2}},\quad z''=z\sqrt{\frac{\gamma^2}{\lambda+\gamma^2}}, \tag{5}$$

$$x'''=x'\sqrt{\frac{\lambda+\alpha^2}{\alpha^2}},\quad y'''=y'\sqrt{\frac{\lambda+\beta^2}{\beta^2}},\quad z'''=z'\sqrt{\frac{\lambda+\gamma^2}{\gamma^2}}, \tag{6}$$

so that

$$x''x'''=xx',\quad y''y'''=yy',\quad z''z'''=zz', \tag{7}$$

$$\frac{AE'}{OV}=1-\frac{xx'}{\lambda+\alpha^2}-\frac{yy'}{\lambda+\beta^2}-\frac{zz'}{\lambda+\gamma^2}=\frac{PE}{OV_0} \tag{8}$$

But from the fundamental property of corresponding points

$$AE'^2=\left(x\sqrt{\frac{\alpha^2}{\lambda+\alpha^2}}-x'\sqrt{\frac{\lambda+\alpha^2}{\alpha^2}}\right)^2+\dots$$

$$=x^2\frac{\alpha^2}{\lambda+\alpha^2}-2xx'+x'^2\frac{\lambda+\alpha^2}{\alpha^2}+\dots$$

$$=x^2\left(1-\frac{\lambda}{\lambda+\alpha^2}\right)-2xx'+x'^2\left(\frac{\lambda}{\alpha^2}+1\right)+\dots$$

$$=x^2+y^2+z^2-\lambda-2xx'-2yy'-2zz'+\lambda+x'^2+y'^2+z'^2$$

$$=(x-x')^2+(y-y')^2+(z-z')^2=PE^2. \tag{9}$$

$$OV_0=OV,\quad \varpi_0\sec\theta_0=\varpi\sec\theta. \tag{10}$$

Draw a cone of small conical angle $d\Omega$ from the vertex at P to cut out area elements dS, dS_1 of the ellipsoid at E, E_1, the axis PEE_1 at an angle ϕ, ϕ_1 with the normal at E, E_1, so that

$$dS=PE^2\sec\phi d\Omega,\quad dS_1=PE_1^2\sec\phi_1 d\Omega. \tag{11}$$

Then with superficial density on the ellipsoid, proportioned to p the perpendicular on the tangent plane, the ratio of the attraction of these elements at E, E_1 on P is

$$\frac{\dfrac{pdS}{PE^2}}{\dfrac{p_1dS_1}{PE_1^2}}=\frac{p\sec\phi}{p_1\sec\phi_1}=1. \tag{12}$$

Thus, if P was inside the ellipsoid, the attraction of the elements would be equal and opposite, and the attraction of the shell is zero.

But with P external the normal PQ of the confocal through P is the axis of the enveloping cone; another elementary cone PF_1F can be drawn equally inclined to PQ, of the same small conical angle $d\Omega$, and contributing an equal attraction; also EF, E_1F_1 pass through Q, and make equal angles with PQ, where Q is the pole of the tangent plane at P with respect to the ellipsoid through E.

The surface of the ellipsoid can be exhausted by such pairs of cones of equal attraction, equally inclined to the normal PQ, so that the resultant attraction at P of the ellipsoidal shell is along the normal of the confocal through P, which confocal is thus a level surface.

Moreover, if the shell is divided by the plane of contact of the tangent cone from P, the two parts of the shell exert equal attraction on P.

24. With dS' the element of surface at E' on the ellipsoid PE', corresponding to dS at E on AE,

$$\frac{\varpi dS'}{p dS} = \text{ratio of corresponding conical elements of volume with vertex at } O$$

$$= \text{ratio of whole volume of ellipsoids} = \sqrt{\frac{\lambda+\alpha^2\cdot\lambda+\beta^2\cdot\lambda+\gamma^2}{\alpha^2\beta^2\gamma^2}} \qquad (1)$$

and then with superficial density $k\rho p$ at E, the normal component dF of the attraction at P of the element dS at E is given by

$$\frac{dF}{kG\rho} = \frac{p\cos\theta_0 dS}{PE^2} = \frac{\varpi\cos\theta_0 dS'}{PE^2}\sqrt{\frac{\alpha\beta\gamma}{(\lambda+\alpha^2\cdot\lambda+\beta^2\cdot\lambda+\gamma^2)}}$$

$$= \frac{\varpi_0\cos\theta dS'}{AE'^2}\frac{2\alpha\beta\gamma}{P(\lambda)} = \varpi_0 d\Omega'\frac{2\alpha\beta\gamma}{P(\lambda)}. \qquad (2)$$

Thus, for the whole ellipsoid, $\Omega'=4\pi$, as A is inside the ellipsoid PE', and

$$\frac{F}{kG\rho} = 4\pi\omega_0\frac{2\alpha\beta\gamma}{P(\lambda)} = 4\pi\varpi_0\frac{\text{volume of ellipsoid } AE}{\text{volume of ellipsoid } PE'}. \qquad (3)$$

The potential U of this shell is then the work required to carry P off to infinity, and so

$$U = \int F d\varpi_0, \text{ with } \varpi_0 d\varpi_0 = \tfrac{1}{2}d\lambda. \qquad (4)$$

The electric charge E, or mass of the shell, is given by

$$E = \int kG\rho p dS = 4\pi kG\rho\alpha\beta\gamma, \qquad (5)$$

$$U = \int_\lambda^\infty \frac{4\pi kG\rho\alpha\beta\gamma d\lambda}{P(\lambda)} = \int_\lambda^\infty \frac{E d\lambda}{P(\lambda)} \qquad (6)$$

and the level surfaces are the confocal ellipsoids of the system

$$\frac{x^2}{\lambda+\alpha^2}+\frac{y^2}{\lambda+\beta^2}+\frac{z^2}{\lambda+\gamma^2}=1. \tag{7}$$

And conversely, U in (6) is the potential of a mass E, if its external level surfaces are given by the system in (7).

This is Professor Gray's proof; but our desideratum is to refer Ω' to some internal point P', analogous in the sphere of figure 3 to the inverse point of P.

The pole of the tangent plane at P with respect to the ellipsoid AE will be at Q on the normal PQ, with coordinates

$$\frac{\alpha^2x}{\lambda+\alpha^2},\quad \frac{\beta^2y}{\lambda+\beta^2},\quad \frac{\gamma^2z}{\lambda+\gamma^2}; \tag{8}$$

and PQ will be the normal at Q of the ellipsoid through Q, homothetic with the ellipsoid through P.

If EQ makes an angle θ with the normal at E,

$$\cos\theta=\frac{px'}{a^2}\frac{x'-\dfrac{\alpha^2x}{\lambda+\alpha^2}}{EQ}+\dots, \tag{9}$$

$$EQ\cos\theta=p\left(1-\frac{xx'}{\lambda+\alpha^2}-\frac{yy'}{\lambda+\beta^2}-\frac{zz'}{\lambda+\gamma^2}\right), \tag{10}$$

$$EP\cos\theta_0=\varpi_0\left(1-\frac{xx'}{\lambda+\alpha^2}-\frac{yy'}{\lambda+\beta^2}-\frac{zz'}{\lambda+\gamma^2}\right), \tag{11}$$

$$\frac{EQ}{EP}=\frac{p\sec\theta}{\varpi_0\sec\theta_0}, \tag{12}$$

$$\frac{dF}{kG\rho}=\frac{p\cos\theta_0 dS}{EP^2}=\frac{EQ}{EP^3}\varpi_0\cos\theta dS=\frac{EQ^3}{EP^3}\varpi_0 d\Omega'. \tag{13}$$

But this is not a constant multiple of $d\Omega'$ on the ellipsoid, but only on the sphere, and so the analogy breaks down with P' at Q, and some other position must be found for P', say on the line of force PA.

25. In a reduction to a standard form of the elliptic integral, take

$$\alpha^2<\beta^2<\gamma^2,\ \text{and then}\ s_1>s_2>s_3, \tag{1}$$

on putting

$$\psi+\alpha^2,\ \psi+\beta^2,\ \psi+\gamma^2=m^2(s-s_1,\ s-s_2,\ s-s_3), \tag{2}$$

$$\frac{d\psi}{P(\psi)}=\frac{ds}{m\sqrt{S}},\quad \int_\lambda^\infty\frac{\sqrt{(\gamma^2-\alpha^2)}\,d\psi}{P(\psi)}=\int_\sigma^\infty\frac{\sqrt{(s_1-s_3)}\,ds}{\sqrt{S}}=eK, \tag{3}$$

$$\text{sn}^2eK=\frac{s_1-s_3}{\sigma-s_3}=\frac{\gamma^2-\alpha^2}{\lambda+\gamma^2},\quad \text{cn}^2eK=\frac{\lambda+\alpha^2}{\lambda+\gamma^2},\quad \text{dn}^2eK=\frac{\lambda+\beta^2}{\lambda+\gamma^2} \tag{4}$$

$$\kappa^2=\frac{\gamma^2-\beta^2}{\gamma^2-\alpha^2},\quad \kappa'^2=\frac{\beta^2-\alpha^2}{\gamma^2-\alpha^2}. \tag{5}$$

Then, putting $\lambda=0, E=1$ in (6), § 24, the reciprocal of the capacity of the ellipsoid (1), § 22, is

$$\int_0^\infty \frac{d\psi}{P(\psi)}=\frac{eK}{\sqrt{(\gamma^2-\alpha^2)}}=\frac{\operatorname{cn}^{-1}\dfrac{\alpha}{\gamma}}{\sqrt{(\gamma^2-\alpha^2)}}. \tag{6}$$

Thus, to determine λ for an ellipsoid of double capacity, we take the formulas for $\frac{1}{2}eK$,

$$\operatorname{sn}^2\tfrac{1}{2}eK=\frac{1-\operatorname{cn}eK}{1+\operatorname{dn}eK},\quad \frac{\gamma^2-\alpha^2}{\lambda+\gamma^2}=\frac{\gamma-\alpha}{\gamma+\beta}, \tag{7}$$

$$\lambda+\gamma^2=\gamma+\alpha\cdot\gamma+\beta,\quad \lambda+\beta^2=\beta+\gamma\cdot\beta+\alpha,\quad \lambda+\alpha^2=\alpha+\beta\cdot\alpha+\gamma, \tag{8}$$

(Hargreave's *Messenger of Mathematics*, August, 1912).

For an oblate ellipsoid, $\beta=\gamma=\alpha\sec\theta$, $\varkappa=0$,

$$\int_0^\infty \frac{d\psi}{P(\psi)}=\int\frac{d\psi}{2(\psi+\beta^2)\sqrt{(\psi+\alpha^2)}}=\frac{1}{\sqrt{(\gamma^2-\alpha^2)}}\cos^{-1}\frac{\alpha}{\gamma}$$
$$=\frac{1}{\sqrt{(\gamma^2-\alpha^2)}}\sin^{-1}\frac{\sqrt{(\gamma^2-\alpha^2)}}{\gamma}=\frac{\theta}{\gamma\sin\theta}=\frac{\theta}{\alpha\tan\theta}. \tag{9}$$

For a prolate ellipsoid, $\alpha=\beta$, $\varkappa=1$,

$$\int_0^\infty \frac{d\psi}{P(\psi)}=\int\frac{d\psi}{2(\psi+\alpha^2)\sqrt{(\psi+\gamma^2)}}=\frac{1}{\sqrt{(\gamma^2-\alpha^2)}}\operatorname{ch}^{-1}\frac{\gamma}{\alpha}$$
$$=\frac{1}{\sqrt{(\gamma^2-\alpha^2)}}\operatorname{sh}^{-1}\frac{\sqrt{(\gamma^2-\alpha^2)}}{\alpha}; \tag{10}$$

giving the capacity of the sphere by the radius, when $\alpha=\beta=\gamma$.

For an elliptic plate $\alpha=0$, $\operatorname{cn}^{-1}0=K$, $\varkappa'=\dfrac{\beta}{\gamma}$, making the capacity $\gamma\div K$; agreeing with the circular plate when $\beta=\gamma$, $\varkappa=0$, $K=\frac{1}{2}\pi$. Or, as in § 8, the capacity of the elliptic plate is the Gauss A. G. M. of β and γ, divided by $\frac{1}{2}\pi$.

To determine λ for an ellipsoid of n-fold capacity of the elliptic plate requires the elliptic functions of K/n.

Thus, as before in (8), for double capacity, $\lambda=\beta\gamma$ with $\alpha=0$.

For three-fold capacity, from the formula

$$\operatorname{sn}\tfrac{1}{3}K+\operatorname{cn}\tfrac{2}{3}K=1,\quad \operatorname{sn}\tfrac{1}{3}K+\frac{\varkappa'\operatorname{sn}\frac{1}{3}K}{\operatorname{dn}\frac{1}{3}K}=1, \tag{11}$$

$$\frac{\sqrt{(\lambda+\gamma^2)}}{\gamma}+\frac{\beta\sqrt{(\lambda+\gamma^2)}}{\gamma\sqrt{(\lambda+\beta^2)}}=1,\quad \frac{\gamma}{\sqrt{(\lambda+\gamma^2)}}-\frac{\beta}{\sqrt{(\lambda+\beta^2)}}=1, \tag{12}$$

leading on rationalisation to

$$\lambda^4-6\beta^2\gamma^2\lambda^2-4(\beta^2+\gamma^2)\beta^2\gamma^2\lambda-3\beta^4\gamma^4=0, \tag{13}$$

a Jacobian quartic for λ.

Or, for a condenser of capacity 50 per cent greater than the plate,

$$\frac{\operatorname{cn}\frac{2}{3}K}{\operatorname{dn}\frac{2}{3}K}+\operatorname{cn}\tfrac{2}{3}K=1,\ \sqrt{\frac{\lambda}{\lambda+\beta^2}}+\sqrt{\frac{\lambda}{\lambda+\gamma^2}}=1,\ \ \frac{1}{\sqrt{(\lambda+\beta^2)}}+\frac{1}{\sqrt{(\lambda+\gamma^2)}}=\frac{1}{\sqrt{\lambda}},\quad(14)$$

$$3\lambda^4+4(\beta^2+\gamma^2)\lambda^3+6\beta^2\gamma^2\lambda^2-\beta^4\gamma^4=0,\quad(15)$$

the same Jacobian quartic (13) as before, with the substitution $\left(\lambda, \frac{\beta\gamma}{\lambda}\right)$.

26. With (9), § 22, for the homogeneous ellipsoid of mass M,

$$\frac{W}{\frac{3}{2}GM}=D(\lambda)-x^2A(\lambda)-y^2B(\lambda)-z^2C(\lambda),\quad(1)$$

changing in the interior to

$$\frac{W}{\frac{3}{2}GM}=D(0)-x^2A(0)-y^2B(0)-z^2C(0),\quad(2)$$

$$D(\lambda)=\int_\lambda^\infty\frac{d\psi}{P(\psi)}=\frac{eK}{\sqrt{(\gamma^2-\alpha^2)}},\quad \operatorname{sn}^2eK=\frac{\gamma^2-\alpha^2}{\lambda+\gamma^2},\quad \varkappa^2=\frac{\gamma^2-\beta^2}{\gamma^2-\alpha^2},\quad(3)$$

$$A(\lambda),\ B(\lambda),\ C(\lambda)=\int_\lambda^\infty\frac{1}{\psi+\alpha^2,\ \psi+\beta^2,\ \psi+\gamma^2}\,\frac{d\psi}{P(\psi)}=-\frac{2dD(\lambda)}{d(\alpha^2,\ \beta^2,\ \gamma^2)},\quad(4)$$

$$A(\lambda)+B(\lambda)+C(\lambda)=\frac{2}{P(\lambda)},\quad(5)$$

and then with

$$\frac{1}{\psi+\alpha^2},\ \frac{1}{\psi+\beta^2},\ \frac{1}{\psi+\gamma^2}=\frac{1}{\gamma^2-\alpha^2}\left(\frac{\operatorname{sn}^2u}{\operatorname{cn}^2u},\ \frac{\operatorname{sn}^2u}{\operatorname{dn}^2u},\ \operatorname{sn}^2u\right)\quad(6)$$

$$A(\lambda),\ B(\lambda),\ C(\lambda)=\frac{1}{(\gamma^2-\alpha^2)^{\frac{3}{2}}}\int_0^{eK}\left(\frac{\operatorname{sn}^2u}{\operatorname{cn}^2u},\ \frac{\operatorname{sn}^2u}{\operatorname{dn}^2u},\ \operatorname{sn}^2u\right)du,\quad(7)$$

three Elliptic Integrals of the Second Kind,

$$(\gamma^2-\alpha^2)^{\frac{3}{2}}[A(\lambda),\ B(\lambda),\ C(\lambda)]$$
$$=\frac{\operatorname{zs}(1-e)K-eE}{\varkappa'^2},\quad \frac{e(E-\varkappa'^2K)-\operatorname{zn}(1-e)K}{\varkappa^2\varkappa'^2},\quad \frac{e(K-E)-\operatorname{zn}eK}{\varkappa^2}.\quad(8)$$

27. An ocean film, covering the surface of a Jacobian rotating ellipsoid, would have the depth inversely as the local gravity, or directly as p the perpendicular on the tangent plane, neglecting its self-gravitation; and so it would resemble the electrical coating or charge on the insulated ellipsoid.

But on Maclaurin's theorem, if the matter of the ellipsoid, § 22, was cut out and condensed on the surface, this film would have the same exterior potential as the solid ellipsoid if the thickness was inversely as p, as if a thin focoid of uniform density.

Maclaurin's theorem is equivalent then to saying that the external potential of the solid ellipsoid is unaltered if a confocal cavity is excavated, and the stuff condensed uniformly in the remaining focoidal shell.

Denoting by l, m, n, the direction cosines of the perpendicular p and p_1 on parallel tangent planes of the confocal ellipsoids defined by λ and λ_1,

$$p^2=(\lambda+\alpha^2)l^2+(\lambda+\beta^2)m^2+(\lambda+\gamma^2)n^2=\lambda+\alpha^2l^2+\beta^2m^2+\gamma^2n^2,$$
$$p_1^2=\lambda_1+\alpha^2l^2+\beta^2m^2+\gamma^2n^2; \tag{1}$$

and defining the thickness of the focoidal shell by the distance between the two parallel tangent planes

$$p_1^2-p^2=\lambda_1-\lambda,\quad p_1-p=\frac{\lambda_1-\lambda}{p_1+p}\ ; \tag{2}$$

reducing for a thin shell to a thickness

$$dp=\frac{d\lambda}{2p}. \tag{3}$$

In Thomson and Tait's "Natural Philosophy" (T and T', § 494 d,) the problem is considered in the reverse way by taking a P. F.

$$V=\tfrac{1}{2}\left(1-\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}-\frac{z^2}{\gamma^2}\right),\ \text{inside the ellipsoid;}\ V=0\ \text{in exterior space;} \tag{4}$$

so that V is continuous, but its variation is discontinuous in crossing the ellipsoid; thence the appropriate density ρ is determined by Laplace's equation,

$$4\pi G\rho=-\nabla^2V=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2},\ \text{inside the ellipsoid, but outside}=0; \tag{5}$$

and, in crossing the surface, the superficial density σ is given by

$$4\pi G\sigma=\frac{dV}{dp}=\frac{dV}{dx}\frac{px}{\alpha^2}+\ldots\ldots=-\frac{px^2}{\alpha^4}\ldots\ldots=-\frac{1}{p}. \tag{6}$$

Thus the focoidal film or shell of superficial density $\sigma=\dfrac{1}{4\pi Gp}$ will have the same exterior potential as the solid ellipsoid of density

$$\rho=\frac{1}{4\pi G}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right),$$

and the same mass; since, integrating over the surface S, and volume V,

$$\int\frac{dS}{4\pi Gp}=\frac{1}{4\pi G}\int\frac{l\,dydz+m\,dzdx+n\,dxdy}{p}$$
$$=\frac{1}{4\pi G}\int\left(\frac{xdydz}{\alpha^2}+\frac{ydzdx}{\beta^2}+\frac{zdxdy}{\gamma^2}\right)$$
$$=\frac{V}{4\pi G}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)=V\rho=M. \tag{7}$$

Thence for the focoid shell

$$\text{outside, } W=\tfrac{3}{2}GM\int_{\lambda}^{\infty}\left(1-\frac{x^2}{\psi+\alpha^2}-\frac{y^2}{\psi+\rho^2}-\frac{z^2}{\psi+\gamma^2}\right)\frac{d\psi}{P(\psi)} \qquad (8)$$

$$\text{inside, } W=\tfrac{3}{2}GM\int_{0}^{\infty}\left(1-\frac{x^2}{\psi+\alpha^2}-\frac{y^2}{\psi+\rho^2}-\frac{z^2}{\psi+\gamma^2}\right)\frac{d\psi}{P(\psi)}$$

$$-\tfrac{3}{2}GM\frac{\alpha\beta\gamma}{\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2}\left(1-\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}-\frac{z^2}{\gamma^2}\right). \qquad (9)$$

So that if the solid ellipsoid (1), § 22, is not homogeneous, but stratified in confocals, the exterior potential is unaltered.

But if the strata are similar homothetic ellipsoids, then

$$W=2\pi G\alpha\beta\gamma\int_{\lambda,0}^{\infty}f\left(1-\frac{x^2}{\psi+\alpha^2}-\frac{y^2}{\psi+\beta^2}-\frac{z^2}{\psi+\gamma^2}\right)\frac{d\psi}{P(\psi)}, \qquad (10)$$

where the density is given by

$$\rho=f'\left(1-\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}-\frac{z^2}{\gamma^2}\right). \qquad (11)$$

Thus, if there is a cavity of semi-axes $k(\alpha,\beta,\gamma)$ in a thick homeoid, the potential in the interior is given by

$$W=2\pi G\alpha\beta\gamma\,[f(1-k^2)-f(0)]\int_{0}^{\infty}\frac{d\psi}{P(\psi)}. \qquad (12)$$

Consult memoirs on this subject by Ferrers and Dyson, in the *Quarterly Journal of Mathematics*, XIV, p. 1, 1877, and XXV, p. 259, 1890.

28. We resume here the interpretation of the terms in the potential W in (9), § 22, as we think it would strike a Maxwell or Clifford, in an investigation of the physical meaning of them, to show reason why these terms depending on the matter inside the ellipsoid and on the surface should arise in an integration extending through exterior space, beyond the attracted point away to infinity.

In his "Biographical Introduction" to Clifford's "Lectures and Essays," Sir Frederick Pollock cites his reminiscence of a walk with Clifford at Cambridge, "when Clifford explained to him in words the inner meaning of Ivory's theorem, and its geometrical conditions, omitting all the formidable apparatus of coordinates and integrals, where the chain of symbolic proof seemed artificial and dead, and failed to satisfy the reason where it compelled the understanding."

Clifford's line of argument is probably the same as that incorporated later in T and T′, § 532, depending on Ivory's theorem of corresponding points.

In an associated problem of hydrodynamics or induced magnetism, we can take the function $\frac{dW}{dx}$ as a new P. F., and give a physical interpretation.

According to Maxwell (E. and M. II, § 437), $\frac{dW}{dx}$ for any attracting solid will give the magnetic potential or velocity function of the body, only however, when $\frac{dW}{dx}$ is a linear function of the coordinates x, y, z, within the body, and W is then a quadratic function in the interior.

The only case with which we are acquainted in which W is a quadratic function of the coordinates within the body is that in which the body is bounded by a complete surface of the second degree.

But in the case of a lens, bounded by portions of a spherical surface, W is not a quadratic function, so that $\frac{dW}{dx}$ will not give a velocity function for the movement of the surface, or a magnetic potential for uniform magnetization in that direction.

29. Begin by considering the hydrodynamical interpretation of the component $\frac{dW}{dx}$ of the attraction of the ellipsoid, by taking an equivalent velocity function

$$\phi = xA(\lambda),\text{ or more generally, } x[A(\lambda) - A(\lambda_1)] = \int_\lambda^{\lambda_1} \frac{x}{\psi + a^2} \frac{d\psi}{P(\psi)}. \tag{1}$$

The up-gradient of ϕ in the direction of the normal of the confocal λ is then, writing A, A_1 for $A(\lambda)$, $A(\lambda_1)$,

$$\frac{d\phi}{dp} = \frac{dx}{dp}(A - A_1) + x\frac{dA}{dp} = l(A - A_1) + 2px\frac{dA}{d\lambda}$$

$$= l\left[A - A_1 + 2(\lambda + a^2)\frac{dA}{d\lambda}\right] = 2l\surd(\lambda + a^2)\frac{d}{d\lambda}[\surd(\lambda + a^2)(A - A_1)], \tag{2}$$

or with

$$\frac{dA}{d\lambda} = -\frac{1}{\lambda + a^2}\frac{1}{P(\lambda)},\quad 2(\lambda + a^2)\frac{dA}{d\lambda} = -\frac{2}{P(\lambda)} = -A - B - C, \tag{3}$$

$$\frac{d\phi}{dp} = -l(A_1 + B + C) = -lu,\quad u = A_1 + B + C, \tag{4}$$

taking the down-gradient of ϕ as the velocity; and this shows that any confocal λ may be supposed to swim in the liquid for a moment, without distortion, with this velocity u parallel to Ox.

At infinity, B and $C = 0$, and $u = A_1$, so that $A_1 = 0$, $\lambda_1 = \infty$ is required to make the velocity there zero.

But in (4) the normal velocity is zero over the ellipsoid λ_2, where

$$B(\lambda_2)+C(\lambda_2)=-A(\lambda_1), \tag{5}$$

requiring λ_1+a^2 to be negative, and making in (1),

$$\phi=x[A(\lambda)+B(\lambda_2)+C(\lambda_2)]. \tag{6}$$

Thus, if there is an infinite stream with velocity $-U$ parallel to Ox, and the ellipsoid λ_2 is inserted, the velocity function is changed from Ux to

$$\phi=Ux\frac{A(\lambda)+B(\lambda_2)+C(\lambda_2)}{B(\lambda_2)+C(\lambda_2)}=Ux\left(\frac{A}{B_2+C_2}+1\right); \tag{7}$$

and with the stream reduced to rest, and the ellipsoid λ_2 advancing with velocity U,

$$\phi=Ux\frac{A}{B_2+C_2}. \tag{8}$$

And generally, in the space between the ellipsoids λ_3, λ_4,

$$\phi=x\frac{u_3(A+B_4+C_4)-u_4(A+B_3+C_3)}{B_3+C_3-B_4-C_4}, \tag{9}$$

with the ellipsoid λ_3, λ_4 advancing with velocity u_3, u_4; reducing with $u_3=u_4=U$ to $\phi=-Ux$.

30. To show how the continuity of the liquid requires that $A(\lambda)$ should have the form given in (1), § 29, consider the flow across the annular section $K-K_2$ made by $x=0$ of the ellipsoid λ, and an interior ellipsoid λ_2, moving with velocity u and u_2 along Ox.

Then $uK-u_2K_2$ is the rate of increase of volume between the two half ellipsoids, and this must be made good by filling up of the flow of the liquid across the annulus, with velocity $-A+A_1$, since $x=0$ makes $\frac{d\phi}{dx}=A-A_1$.

The integral equation of continuity is then

$$uK-u_2K_2+\int_{\lambda_2}^{\lambda}(A-A_1)\,dK=0, \tag{1}$$

and differentiating with respect to λ for the differential equation of continuity

$$\frac{d}{d\lambda}uK+(A-A_1)\frac{dK}{d\lambda}=0,\quad \frac{d}{d\lambda}(u+A-A_1)K-\frac{dA}{d\lambda}K=0. \tag{2}$$

With the value of u from (2), (3), (4), § 29,

$$u+A-A_1=-2(\lambda+a^2)\frac{dA}{d\lambda}, \tag{3}$$

$$2\frac{d}{d\lambda}\left[(\lambda+a^2)\frac{dA}{d\lambda}K\right]+\frac{dA}{d\lambda}K=0,\quad 2(\lambda+a^2)\frac{d}{d\lambda}\left(\frac{dA}{d\lambda}K\right)+3\frac{dA}{d\lambda}K=0, \tag{4}$$

and integrating
$$\frac{dA}{d\lambda}K(\lambda+\alpha^2)^{\frac{3}{2}}=\text{constant}; \tag{5}$$

so that with
$$K=\pi\sqrt{(\lambda+\beta^2\cdot\lambda+\gamma^2)}=\frac{\frac{1}{2}\pi P}{\sqrt{(\lambda+\alpha^2)}}, \tag{6}$$

$$\frac{dA}{d\lambda}=\frac{\text{constant}}{(\lambda+\alpha^2)P(\lambda)}, \tag{7}$$

as in (1), § 29.

31. When the ellipsoid of § 22 is of revolution, make the circle on AB the focal circle for an oblate spheroid by putting $\beta=\gamma$, as in (9), § 25, but for a prolate spheroid take AB as the focal line, and put $\beta=\alpha$, as in (10), § 25. Then in parallel columns:

Spheroid	Oblate, on axis Ox	Prolate, on axis Oy
$\lambda+\alpha^2$	$a^2\cot^2\omega=B^2$	$\frac{a^2}{\text{sh}^2\zeta}=B^2$
$\lambda+\beta^2$	$a^2\sec^2\omega$	$a^2\frac{\text{ch}^2\zeta}{\text{sh}^2\zeta}$
$\beta^2-\alpha^2$	a^2	a^2
$\frac{AB}{PA+PB}$	$\sin\omega$	$\text{th}\,\zeta$
$P(\lambda)$	$2a^3\frac{\cos\omega}{\sin^3\omega}$	$2a^3\frac{\text{ch}\,\zeta}{\text{sh}^3\zeta}$
$aD(\lambda)=\int_\lambda^\infty\frac{ad\psi}{P(\psi)}$	ω	ζ
$a^3A(\lambda)$	$\tan\omega-\omega$	$\frac{1}{4}\text{sh}2\zeta-\frac{1}{2}\zeta$
$a^3B(\lambda)$	$\frac{1}{2}\omega-\frac{1}{4}\sin 2\omega$	$\zeta-\text{th}\,\zeta$
$\frac{aW}{\frac{3}{2}GM}$	$\omega-\frac{x^2}{a^2}(\tan\omega-\omega)$ $-\frac{y^2+z^2}{a^2}(\frac{1}{2}\omega-\frac{1}{4}\sin 2\omega)$	$\zeta-\frac{x^2+z^2}{a^2}(\frac{1}{4}\text{sh}2\zeta-\frac{1}{2}\zeta)$ $-\frac{y^2}{a^2}(\zeta-\text{th}\,\zeta)$
At infinity	$\omega=\sin\omega=\frac{a}{r}$	$\zeta=\text{th}\,\zeta=\frac{a}{r}$
$a^3A(\infty)=a^3B(\infty)$	$\frac{1}{3}\omega^3$	$\frac{1}{3}\zeta^3$
$\frac{rW(\infty)}{\frac{3}{2}GM}$	$\frac{2}{3}$	$\frac{2}{3}$
$\frac{a^3N}{\frac{3}{2}GM}$	$xy^2(\frac{1}{2}\omega-\frac{1}{4}\sin 2\omega)+\frac{2}{3}x^3\tan^3\omega$	$x^2y(\frac{1}{4}\text{sh}^2\zeta-\frac{1}{2}\zeta)+\frac{2}{3}y^3\text{sh}^3\zeta$

The Hypergeometric Function in its Physical Applications.

32. Many of the functions required in a physical problem can be classed in a special case of the Hypergeometric (H. G.) Function, defined in Forsyth's "Differential Equations," Chapter VI.

Beginning with Euler's First and Second Integral, called the Gamma and Beta Function, defined by

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx=\frac{1}{n}\int_0^\infty e^{-\sqrt[n]{z}}dz, \tag{1}$$

$$B(m, n)=\int_0^1 s^{m-1}(1-s)^{n-1}ds=2\int_0^{\frac{1}{2}\pi}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}d\theta, \tag{2}$$

with $s=\sin^2\theta$, $1-s=\cos^2\theta$; they are connected by the relation

$$B(m, n)=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}, \tag{3}$$

so that the Gamma Function alone requires tabulation, given by Bertrand (Integral Calculus) as $\log\Gamma n$, and between 0 and 1 for n, since $\Gamma(n+1)=n\Gamma n$.

Proceeding with a generalization, the definite integral

$$F(\alpha, \beta, \gamma, x)=A\int_0^1 s^{\beta-1}(1-s)^{\gamma-\beta-1}(1-xs)^{-\alpha}ds \tag{4}$$

defines the Hypergeometric Function, where A is chosen so as to make the function unity when $x=0$, and the function reduces to a Beta Function; so that

$$A\int_0^1 s^{\beta-1}(1-s)^{\gamma-\beta-1}ds=1, \quad \frac{1}{A}=B(\beta, \gamma-\beta). \tag{5}$$

Introducing the Elliptic Function, and putting

$$s=\text{sn}^2v, \quad 1-s=\text{cn}^2v, \quad 1-xs=\text{dn}^2v, \quad x=\kappa^2, \tag{6}$$

$$F(\alpha, \beta, \gamma, x)=2A\int_0^K(\text{sn}v)^{2\beta-1}(\text{cn}v)^{2\gamma-2\beta-1}(\text{dn}v)^{-2\alpha+1}dv, \tag{7}$$

or with $v=eK$,

$$F(\alpha, \beta, \gamma, x)=2A\int_{e=0}^1(\text{sn}\,eK)^{2\beta-1}(\text{cn}\,eK)^{2\gamma-2\beta-1}(\text{dn}\,eK)^{-2\alpha+1}K\,de, \tag{8}$$

where β and $\gamma-\beta$ must be positive; and this is in a form analogous to Euler's Second Integral in (2).

Then F is a solution of the hypergeometric differential equation (Forsyth's D. E., Chapter VI),

$$x(1-x)\frac{d^2F}{dx^2}+[\gamma-(\alpha+\beta+1)x]\frac{dF}{dx}-\alpha\beta F=0. \tag{9}$$

Schwartz has shown that $F(\alpha, \beta, \gamma, x)$ is an algebraical function of the fourth element x in the special cases identified by Klein in his "Ikosahedron" as the polyhedral functions.

The XXIV transformations of the H. G. function in (8) are then obtained from the four arguments:

$$v+0, \quad K, \quad K+K'i, \quad K'i, \tag{10}$$

with Abel's six linear transformations for the modulus:

$$x, \quad 1-x, \quad \frac{1}{1-x}, \quad \frac{x}{x-1}, \quad \frac{x-1}{x}, \quad \frac{1}{x}, \tag{11}$$

a succession of the complementary and reciprocal modulus.

33. In the special case of $\alpha+\beta=1$, $\gamma=1$, then (9), § 32, becomes the D. E. of the zonal harmonic of order n, where

$$n(n+1)=-\alpha\beta=-N, \text{ suppose, } n+\tfrac{1}{2}=\surd(\tfrac{1}{4}-N), \tag{1}$$

and for any order n, integral, fractional, or complex, the zonal harmonic of $x=\sin^2\frac{1}{2}\theta$ is

$$P_n=F=2A\int_0^K\left(\frac{\operatorname{sn} v \operatorname{dn} v}{\operatorname{cn} v}\right)^{\surd(1-4N)=2n+1}dv=A\int_0^{2K}\left(\frac{1-\operatorname{cn} w}{1+\operatorname{cn} w}\right)^{\surd(\frac{1}{4}-N)}dw, \tag{2}$$

with $w=2v$, and a modular angle $\frac{1}{2}\theta$.

Should $\frac{1}{4}-N$ be negative, a Mehler function arises of order $p=\surd(N-\frac{1}{4})$, given by

$$F=\int_0^{2K}\cos\log\left(\frac{1-\operatorname{cn} w}{1+\operatorname{cn} w}\right)^p dw, \tag{3}$$

satisfying the D. E.

$$x(1-x)\frac{d^2F}{dx^2}+(1-2x)\frac{dF}{dx}-(p^2+\tfrac{1}{4})F=0. \tag{4}$$

If we put

$$\frac{1+\operatorname{cn} w}{1-\operatorname{cn} w}=e^\alpha, \quad \operatorname{cn} w=\operatorname{th}\tfrac{1}{2}\alpha, \quad \operatorname{sn} w=\operatorname{sech}\tfrac{1}{2}\alpha, \quad \operatorname{dn} w=\operatorname{sech}\tfrac{1}{2}\alpha\surd(\operatorname{ch}^2\tfrac{1}{2}\alpha-x), \tag{5}$$

$$-\operatorname{sn} w \operatorname{dn} w dw=\tfrac{1}{2}\operatorname{sech}^2\tfrac{1}{2}\alpha d\alpha, \quad dw=\frac{-\frac{1}{2}d\alpha}{\surd(\operatorname{ch}^2\frac{1}{2}\alpha-x)}, \tag{6}$$

then with $p=\surd(\frac{1}{4}-N)$, as in (2),

$$F=A\int_{-\infty}^{\infty}e^{p\alpha}\frac{\frac{1}{2}d\alpha}{\surd(\operatorname{ch}^2\frac{1}{2}\alpha-x)}=A\int_{0,\,-\infty}^{\infty,\,0}\frac{e^{p\alpha}d\alpha}{\surd(\operatorname{ch}^2\frac{1}{2}\alpha-x)}=A\int_0^\infty\frac{\operatorname{ch} p\alpha d\alpha}{\surd(\operatorname{ch}^2\frac{1}{2}\alpha-x)}, \tag{7}$$

changing in (3), where $p=\surd(N-\frac{1}{4})$, to

$$F=A\int_0^\pi\frac{\cos p\alpha d\alpha}{\surd(\operatorname{ch}^2\frac{1}{2}\alpha-x)}. \tag{8}$$

Hobson's expression (*Phil. Trans.*, 1896) for the Mehler function of order p, solution of (4); and so a H. G. function of imaginary order, with

$$\alpha=\tfrac{1}{2}+pi, \quad \beta=\tfrac{1}{2}-pi, \quad \gamma=1, \quad x=\sin^2\tfrac{1}{2}\theta. \tag{9}$$

The Tesseral Harmonic (Hobson, *Phil. Trans.*, 1896),

$$P_{n,m}(\mu)=\frac{1}{\Pi(m)}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}}F\{-n,\ n+1,\ 1+m,\ \tfrac{1}{2}(1-\mu)\}\tag{10}$$

can then be expressed by the definite integral

$$\int_0^K\left(\frac{\operatorname{sn}v\operatorname{dn}v}{\operatorname{cn}v}\right)^{2n+1}(\operatorname{cn}v)^{2m}dv,\tag{11}$$

or

$$\int_0^{2K}\left(\frac{1-\operatorname{cn}w}{1+\operatorname{cn}w}\right)^{n+\frac{1}{2}}\left(\frac{\operatorname{dn}w+\operatorname{cn}w}{1+\operatorname{dn}w}\right)^m dw.\tag{12}$$

34. Comparing the expansion, in ascending powers of $\frac{r}{a}$ or $\frac{a}{r}$, of the typical terms $\frac{1}{R}$ and R of the P. F. and S. F., where

$$R^2=a^2-2ar\mu+r^2,\tag{1}$$

$$\frac{1}{R}=\Sigma\left(\frac{r^n}{a^{n+1}}\text{ or }\frac{a^n}{r^{n+1}}\right)P_n(\mu),\tag{2}$$

$$R=\Sigma\left(\frac{r^{n+1}}{a^n}\text{ or }\frac{a^{n+1}}{r^n}\right)I_n(\mu),\tag{3}$$

then $P_n(\mu)$ is the zonal harmonic, and $I_n(\mu)$ the associated function for the S. F., and then in (2), § 4, and (2), § 6, with

$$x=\tfrac{1}{2}(1-\mu)=\sin^2\tfrac{1}{2}\theta,\quad x'=\tfrac{1}{2}(1+\mu)=\cos^2\tfrac{1}{2}\theta,$$

$$\frac{d}{d\mu}(1-\mu^2)\frac{dP_n}{d\mu}+n(n+1)P_n=0,\quad \frac{d}{dx}\left(xx'\frac{dP_n}{dx}\right)+n(n+1)P_n=0,\tag{4}$$

$$(1-\mu^2)\frac{d^2I_n}{d\mu^2}+n(n+1)I_n=0,\quad xx'\frac{d^2I_n}{dx^2}+n(n+1)I_n=0,\tag{5}$$

so that

$$\frac{dI_n}{d\mu}=P_n,\quad I_n=\int P_n d\mu=-\frac{1-\mu^2}{n(n+1)}\frac{dP_n}{d\mu}=\frac{1}{n(n+1)}\sin\theta\frac{dP_n}{d\theta}.\tag{6}$$

And when n is an integer,

$$P_n=\frac{1}{n!}\frac{d^n(xx')^n}{dx'^n},\quad I_n=\frac{2}{n!}\left(\frac{d}{dx'}\right)^{n-1}(xx')^n,\tag{7}$$

$$(2n+1)I_n=P_{n+1}-P_{n-1},\tag{8}$$

$$I_n=-P_{n+1}+2\mu P_n-P_{n-1},\tag{9}$$

$$n(n+1)I_n=-(1-\mu^2)\frac{dP_n}{d\mu}=(n+1)(P_{n+1}-\mu P_n)=n(\mu P_n-P_{n-1}),\tag{10}$$

and so on, as discussed by Sampson on the "Stokes Current Function," *Phil. Trans.*, 1890, where the expressions given for I_n by the H. G. formulas can be replaced by definite integrals of the elliptic function.

The relations between a P. F. V and its S. F. N

$$r\sin\theta\frac{dV}{dr}=\frac{dN}{rd\theta},\quad r\sin\theta\frac{dV}{rd\theta}=-\frac{dN}{dr} \tag{11}$$

are satisfied then by typical terms, such as

$$(Ar^n+Br^{-n-1})P_n(\mu)\text{ in }V,\text{ and }[nAr^{n+1}+(n+1)Br^{-n}]I_n(\mu)\text{ in }N; \tag{12}$$

and constant V and N represent orthogonal surfaces; and I_n may be replaced by $P_{n+1}-P_{n-1}$ in (8).

35. The Hicks toroidal function (*Phil. Trans.*, 1881–4) is of similar nature, defined by

$$P_m(u)=\int_0^\pi\frac{d\theta}{(\operatorname{ch}u+\operatorname{sh}u\cos\theta)^{m+\frac{1}{2}}}, \tag{1}$$

solution of the D. E.

$$\frac{d^2P}{du^2}+\coth u\frac{dP}{du}-(m^2-\tfrac{1}{4})P=0; \tag{2}$$

or writing C and S for ch u and sh u, and putting

$$C+1=2\operatorname{ch}^2\tfrac{1}{2}u=2x,\quad C-1=2\operatorname{sh}^2\tfrac{1}{2}u=2(x-1), \tag{3}$$

$$\begin{aligned}\frac{d^2P}{du^2}+\coth u\frac{dP}{du}&=\frac{d^2P}{du^2}+\frac{C}{S}\frac{dP}{du}=\frac{1}{S}\frac{d}{du}\left(S\frac{dP}{du}\right)\\&=\frac{d}{dC}(C^2-1)\frac{dP}{dC}=\frac{d}{dx}x(x-1)\frac{dP}{dx}\\&=-x(1-x)\frac{d^2P}{dx^2}-(1-2x)\frac{dP}{dx}=(m^2-\tfrac{1}{4})P,\end{aligned} \tag{4}$$

the H. G. D. E. with

$$\alpha+\beta=1,\quad\gamma=1,\quad\alpha\beta=-m^2+\tfrac{1}{4},\quad\alpha,\beta=\tfrac{1}{2}\pm m. \tag{5}$$

Then

$$\begin{aligned}&C+S\cos\theta=C+S-2S\sin^2\tfrac{1}{2}\theta=e^u\Delta^2\tfrac{1}{2}\theta=\frac{\operatorname{dn}^2v}{\gamma'},\\&\tfrac{1}{2}\theta=\operatorname{am}v=\operatorname{am}eGe^{-u}=\gamma',\quad C=\tfrac{1}{2}\left(\frac{1}{\gamma'}+\gamma'\right),\quad S=\tfrac{1}{2}\left(\frac{1}{\gamma'}-\gamma'\right),\end{aligned} \tag{6}$$

$$P_m(u)=\int_0^{\frac{1}{2}\pi}\frac{2d\frac{1}{2}\theta}{\left(\frac{\Delta\frac{1}{2}\theta}{\sqrt{\gamma'}}\right)^{2m+1}}=2\sqrt{\gamma'}\int_0^G\frac{dv}{\left(\frac{\operatorname{dn}v}{\sqrt{\gamma'}}\right)^{2m}}=2\sqrt{\gamma'}\int_0^G\left(\frac{\operatorname{dn}v}{\sqrt{\gamma'}}\right)^{2m}dv, \tag{7}$$

as the value is unaltered on changing v into $G-v$.

The sequence equation,

$$(2m+1)P_{m+1}-4mCP_m+(2m-1)P_{m-1}=0, \tag{8}$$

a difference equation with solution P_m, is then obtained by an integration between 0 and G of the differentiation

$$\frac{d}{dv}\frac{\gamma^2}{\gamma'}\operatorname{sn} v \operatorname{cn} v\left(\frac{\operatorname{dn} v}{\sqrt{\gamma'}}\right)^{2m-1}=(2m+1)\left(\frac{\operatorname{dn} v}{\sqrt{\gamma'}}\right)^{2m+2}$$
$$-4mC\left(\frac{\operatorname{dn} v}{\sqrt{\gamma'}}\right)^{2m}+(2m-1)\left(\frac{\operatorname{dn} v}{\sqrt{\gamma'}}\right)^{2m-2}. \tag{9}$$

Starting with

$$P_0=2\sqrt{\gamma'}F(\gamma),\ P_1=\frac{2E(\gamma)}{\sqrt{\gamma'}},\ \text{and then}\ P_2=\tfrac{2}{3}\left(\frac{1}{\gamma'}+\gamma'\right)\frac{2E(\gamma)}{\sqrt{\gamma'}}-\tfrac{2}{3}\sqrt{\gamma'}F(\gamma), \tag{10}$$

the sequence equation (8) will determine P_m for integral values of m.

36. In a quadric transformation of the toroidal function in (7), § 35, put

$$\kappa=\operatorname{th}\tfrac{1}{2}u=\frac{1-\gamma'}{1+\gamma'},\quad C=\operatorname{ch} u=\frac{1+\kappa^2}{1-\kappa^2},\quad S=\frac{2\kappa}{1-\kappa^2}, \tag{1}$$

$$C+S\cos\theta=\frac{1+\kappa\sin\psi}{1-\kappa\sin\psi},\quad \cos\theta=\frac{\sin\psi-\kappa}{1-\kappa\sin\psi},$$

$$\sin\theta=\frac{\kappa'\cos\psi}{1-\kappa\sin\psi},\quad \tan^2\tfrac{1}{2}\theta=\frac{1+\kappa}{1-\kappa}\cdot\frac{1-\sin\psi}{1+\sin\psi}, \tag{2}$$

$$\frac{-d\theta}{\sqrt{(C+S\cos\theta)}}=\frac{\kappa' d\psi}{\Delta(\psi,\kappa)}=\kappa' dw,\quad \psi=\operatorname{am}(w,\kappa),\quad w=(1-2c)K, \tag{3}$$

$$P_m(u)=\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left(\frac{1-\kappa\sin\psi}{1+\kappa\sin\psi}\right)^m\frac{\kappa' d\psi}{\Delta\psi}=\int_{-L}^{L}\left(\frac{1-\kappa\operatorname{sn} w}{1+\kappa\operatorname{sn} w}\right)^m\kappa' dw, \tag{4}$$

$$P_0=2\kappa' F(\kappa),\quad P_1=\frac{2E(\kappa)-\kappa'^2F(\kappa)}{\kappa'},\ \ldots, \tag{5}$$

$$\gamma'\operatorname{tn}^2 eG=\frac{1-\operatorname{sn}(1-2e)K}{1+\operatorname{sn}(1-2e)K},\quad \frac{\operatorname{dn}^2 eG}{\gamma'}=\frac{1+\kappa\operatorname{sn}(1-2e)K}{1-\kappa\operatorname{sn}(1-2e)K}. \tag{6}$$

This quadric transformation is illustrated geometrically on an ellipse of excentricity $\kappa=\operatorname{th}\frac{1}{2}u$, in the connection between θ the focal angle or true anomaly from perihelion, and ψ the minor excentric angle or anomaly; and then with $v=eG$, $w=(1-2e)K$, $\theta=2\operatorname{am} eG$, $\psi=\operatorname{am}(1-2e)K$.

It is also shown on figure 5, in the expression of the potential of an anchor ring or torus, discussed by Dyson, *Phil. Trans.*, 1893.

37. In the reduction of the toroidal of the second kind

$$Q_m(u)=\int_0^\infty\frac{d\theta'}{(C+S\operatorname{ch}\theta')^{m+\frac{1}{2}}}, \tag{1}$$

substitute

$$\operatorname{ch}\tfrac{1}{2}\theta' = \sec\phi, \quad \operatorname{sh}\tfrac{1}{2}\theta' = \tan\phi, \quad d\theta' = 2\sec\phi\, d\phi, \tag{2}$$

$$C + S\operatorname{ch}\theta' = \frac{\Delta^2\phi}{\gamma'\cos^2\phi}, \text{ to modulus } \gamma' = e^{-u}, \tag{3}$$

$$\frac{d\theta'}{\sqrt{(C+S\operatorname{ch}\theta')}} = \frac{2\sqrt{\gamma'}d\phi}{\Delta\phi} = 2\sqrt{\gamma'}dv', \quad \phi = \operatorname{am}(v', \gamma') = \operatorname{am} fG', \tag{4}$$

$$\begin{aligned} Q_m(u) &= \int_0^{\frac{1}{2}\pi}\left(\frac{\sqrt{\gamma'}\cos\phi}{\Delta\phi}\right)^{2m}\frac{2\sqrt{\gamma'}d\phi}{\Delta\phi} \\ &= 2\gamma'^{m+\frac{1}{2}}\int_0^{G'}[\operatorname{sn}(G-v')]^{2m}dv' = 2\gamma'^{m+\frac{1}{2}}\int_0^{G'}(\operatorname{sn} v')^{2m}dv'. \end{aligned} \tag{5}$$

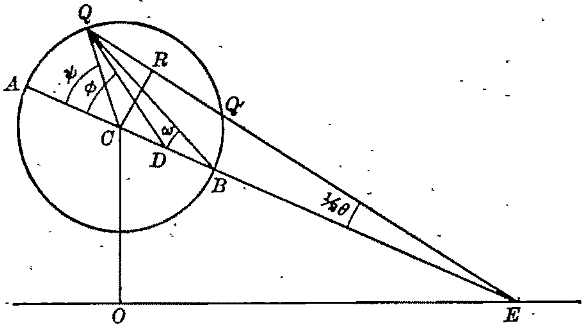

FIG. 5.

And with this toroidal in the form (this v not the same as in § 35, but employed with u, in the stereographic coordinates of A. J. M., § 24, p. 411), *i.e.* of the former volume.

$$Q_m(u) = \int_0^{\pi}\frac{\cos mv\,dv}{\sqrt{(2\cdot C+\cos v)}}, \tag{6}$$

$$\frac{dv}{\sqrt{(2\cdot C+\cos v)}} = \frac{\frac{1}{2}dv}{\sqrt{(\operatorname{ch}^2\frac{1}{2}u - \sin^2\frac{1}{2}v)}} = \frac{\varkappa' d\frac{1}{2}v}{\Delta(\frac{1}{2}v, \varkappa')} = \varkappa' dw',$$

$$\varkappa = \operatorname{sech}\tfrac{1}{2}u, \quad \varkappa = \operatorname{th}\tfrac{1}{2}u = \frac{1-\gamma'}{1+\gamma'}, \tag{7}$$

the complementary quadric transformation, with $\frac{1}{2}v = \operatorname{am}(w', \varkappa') = \operatorname{am} fK'$,

$$Q_m(u) = \int_0^{K'}\cos(2m\operatorname{am} w')\,dw'. \tag{8}$$

The connection between the reduction of (1) and (6) is made through

$$\cos\tfrac{1}{2}v = \frac{\sqrt{(C+S\operatorname{ch} v')}}{\operatorname{sh}\frac{1}{2}u + \operatorname{ch}\frac{1}{2}u\operatorname{ch} v'}, \quad \sin\tfrac{1}{2}v = \frac{\operatorname{ch}\frac{1}{2}u\operatorname{sh} v'}{\operatorname{sh}\frac{1}{2}u + \operatorname{ch}\frac{1}{2}u\operatorname{ch} v'}, \tag{9}$$

$$\frac{dv}{\sqrt{(2\cdot C+\cos v)}} = \frac{dv'}{\sqrt{(C+S\operatorname{ch} v')}}, \tag{10}$$

$$\operatorname{cn} fK' = \frac{\operatorname{cn} fG'\operatorname{dn} fG'}{1+\gamma'\operatorname{sn}^2 fG'}, \quad \operatorname{sn} fK' = \frac{(1+\gamma')\operatorname{sn} fG'}{1+\gamma'\operatorname{sn}^2 fG'}, \quad \operatorname{dn} fG' = \frac{1-\gamma'\operatorname{sn}^2 fG'}{1+\gamma'\operatorname{sn}^2 fG'}, \tag{11}$$

in the complementary quadric transformation to § 36.

Then Basset's function L_m ("Hydrodynamics" I, p. 107) defined, with $c=e^{-u}=\gamma'$, by

$$mL_m=\int_0^\pi \frac{\cos(m-1)v-\cos(m+1)v}{\sqrt{(2\cdot C+\cos v)}}\,dv = Q_{m-1}-Q_{m+1}, \tag{12}$$

$$(2m+3)L_{m+1}-4mCL_m+(2m-3)L_{m-1}=0. \tag{13}$$

A comparison can be made too with the expressions given in Todhunter's "Functions of Laplace, Lamé, and Bessel," Chapter IV.

38. For the potential V of a solid anchor ring of mass M at a point E on the axis (Dyson, *Phil. Trans.*, 1893; Routh, "Statics," II, p. 96), we take their expression:

$$\frac{V}{M}=\frac{2}{\pi}\int_0^\pi \frac{\sin^2\psi d\psi}{EQ}, \tag{1}$$

where on figure 5, $EC=A$, $CQ=a$, $OC=c$, $ACQ=\psi=2\omega$,

$$EQ^2=r^2=A^2+2aA\cos\psi+a^2=(A+a)^2\cos^2\omega+(A-a)^2\sin^2\omega, \tag{2}$$

and with $EA=A+a=r_1$, $EB=A-a=r_2$,

$$\frac{V}{M}=\frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\sin^2 2\omega d\omega}{\sqrt{(r_1^2\cos^2\omega+r_2^2\sin^2\omega)}}. \tag{3}$$

With r for variable, and writing R for $r_1^2-r^2\cdot r^2-r_2^2$,

$$\sin^2\omega=\frac{r_1^2-r^2}{r_1^2-r_2^2},\quad \cos^2\omega=\frac{r^2-r_2^2}{r_1^2-r_2^2},\quad \frac{d\omega}{\sqrt{(r_1^2\cos^2\omega+r_2^2\sin^2\omega)}}=\frac{dr}{\sqrt{R}}, \tag{4}$$

$$\frac{V}{M}=\frac{16}{\pi(r_1^2-r_2^2)^2}\int_{r_2}^{r_1}\sqrt{R}dr. \tag{5}$$

Integrating by parts,

$$\begin{aligned}\int\sqrt{R}dr&=r\sqrt{R}-\int[(r_1^2+r_2^2)r^2-2r^4]\frac{dr}{\sqrt{R}}\\&=r\sqrt{R}-2\int\sqrt{R}dr+(r_1^2+r_2^2)\int\frac{r^2dr}{\sqrt{R}}-2r_1^2r_2^2\int\frac{dr}{\sqrt{R}}\\&=\tfrac{1}{3}r\sqrt{R}+\tfrac{1}{3}(r_1^2+r_2^2)\int\frac{r^2dr}{\sqrt{R}}-\tfrac{2}{3}r_1^2r_2^2\int\frac{dr}{\sqrt{R}}.\end{aligned} \tag{6}$$

Between the limits r_2 and r_1,

$$\sqrt{R}=0,\quad \int\frac{r^2dr}{\sqrt{R}}=r_1\int\Delta\omega d\omega=r_1E(\gamma),\quad \int\frac{dr}{\sqrt{R}}=\frac{F(\gamma)}{r_1},\quad \gamma'=\frac{r_2}{r_1}, \tag{7}$$

$$\frac{V}{M}=\frac{16}{3\pi r_1}\,\frac{(r_1^2+r_2^2)r_1^2E(\gamma)-2r_1^2r_2^2F(\gamma)}{(r_1^2-r_2^2)^2}=\frac{16}{3\pi r_1}\,\frac{(1+\gamma'^2)E(\gamma)-2\gamma'^2F(\gamma)}{\gamma^4}. \tag{8}$$

Or with a quadric transformation, as in Routh, "Statics," II, p. 96, with D, E inverse points in the circle AQB, and putting, as in pendulum motion,

$$CEQ=CQD=\tfrac{1}{2}\theta,\quad CDQ=CQE=\phi=\psi-\tfrac{1}{2}\theta,$$

$$\frac{V}{M}=\frac{2}{\pi A^2}\int\cos^2\phi\sqrt{(A^2-a^2\sin^2\phi)}\,d\phi=\frac{4}{\pi A}\int_0^{\frac{1}{2}\pi}\cos^2\phi\,\Delta(\phi,\varkappa)\,d\phi,$$

$$\varkappa=\frac{a}{A}=\frac{1-\gamma'}{1+\gamma'},\qquad (9)$$

$$\frac{V}{M}=\frac{4}{\pi A}\int_{e=0}^{1}\operatorname{cn}^2 eK\operatorname{dn}^2 eK\,deK=\frac{4}{\pi A}\int\left(\frac{\operatorname{dn}^4 eK}{\varkappa^2}-\frac{\varkappa'^2\operatorname{dn}^2 eK}{\varkappa^2}\right)deK$$

$$=\frac{4}{\pi A}\int_0^K\left[\frac{\varkappa'^2}{\varkappa^2}\left(\frac{\operatorname{dn} v}{\sqrt{\varkappa'}}\right)^4-\frac{\varkappa'^3}{\varkappa^2}\left(\frac{\operatorname{dn} v}{\sqrt{\varkappa'}}\right)^2\right]dv=\frac{4}{\pi A}\,\frac{1}{2\sqrt{\varkappa'}}\left(\frac{\varkappa'^2}{\varkappa^2}P_2-\frac{\varkappa'^3}{\varkappa^2}P_1\right),\qquad (10)$$

in the sequence equation (8), § 35, making

$$\frac{V}{M}=\frac{4}{3\pi A}\,\frac{2(1+\varkappa'^2)E(\varkappa)-\varkappa'^3F(\varkappa)}{\varkappa^2},\qquad (11)$$

a quadric transformation of the expression in (8).

So also for U, the skin potential of the anchor ring, if R is the midpoint of QQ',

$$\frac{U}{M}=\int\frac{PR}{\pi A^2}\,d\phi=\frac{1}{\pi A^2}\int_0^{\pi}\sqrt{(A^2-a^2\sin^2\phi)}\,d\phi=\frac{2}{\pi A}\int_0^{\frac{1}{2}\pi}\Delta\phi\,d\phi=\frac{2E(\varkappa)}{\pi A}.\qquad (12)$$

39. As shown on figure 1 and in § 37,

$$e^{-u}=\gamma'=\frac{PB}{PA}=\frac{EB}{EA}=\frac{1-\varkappa}{1+\varkappa},\quad \varkappa=\frac{OE}{OA}=\operatorname{th}\tfrac{1}{2}u,$$

$$\tfrac{1}{2}v=DEp=DPB=\operatorname{am} fK',\quad DEP=\operatorname{am}(1-f)K',$$

$$EBP=\operatorname{am} 2fG',\quad DBP=\operatorname{am} 2(1-f)G';$$

$$\Omega=2\pi(1-f)-4K\operatorname{zn} fK'=2\pi(1-f)-2G\operatorname{zn} 2fG'-2G\gamma'\operatorname{sn} 2fG'.$$

The point P may be supposed to circulate round the circle on DE in pendulum motion, with velocity due to the level of Ox, or proportional to BP or AP.

Starting from E, where $f=0$, $\Omega=2\pi$, as P moves along the semicircle ESD, f grows from 0 to 1, and Ω diminishes from 2π to 0. As P continues to complete the circuit back to E, f grows from 1 to 2, and Ω is negative, decreasing from 0 to -2π.

Thus 4π must be added to Ω in crossing the disc AB to make a fresh start; or P moving the other way round, clockwise, 4π must be subtracted in passing through E, as twelve hours is subtracted in passing through XII o'clock.

The angle $\phi=\operatorname{am} fG'$ in § 37 is constructed by bisecting the angle EBP in fig. 1 by By, crossing Dp in y, and then drawing the coaxial circle touching Dp

at y, on diameter de on DE, with limiting points A and B, and yd, ye bisect AyB externally and internally. Then, with centre at a, if dr, es are the tangents at d, e, the line rs passes through B, and crosses the circle on FB in q, at the same level as a, because Fq at right angles to Bq bisects rs.

Then with $DBp = \operatorname{am} 2fG'$, $DBr = \operatorname{am} fG' = \phi$,

$$\gamma' = \frac{DB}{DA} = \frac{EB}{EA} = \frac{DB-EB}{DA-EA} = \frac{FB}{FE}, \quad Fa = FB \sin^2\phi = FE \cdot \gamma' \operatorname{sn}^2 fG';$$

and with $DEp = \tfrac{1}{2} v = \operatorname{am} fK'$,

$$\operatorname{dn} fK' = \frac{Bp}{BD} = \frac{py}{yD} = \frac{Ea}{aD} = \frac{EF-Fa}{DF+Fa} = \frac{1-\gamma' \operatorname{sn}^2 fG'}{1+\gamma' \operatorname{sn}^2 fG}.$$

Again, by the property of coaxial circles,

$$\frac{Dy}{DB} = \frac{py}{pB} = \frac{rd}{rB} = \sin\phi; \quad Dy = DF(1+\gamma') \operatorname{sn} fG',$$

$$\operatorname{sn} fK' = \sin \tfrac{1}{2} v = \sin Day = \frac{Dy}{Da} = \frac{(1+\gamma') \operatorname{sn} fG'}{1+\gamma' \operatorname{sn}^2 fG'};$$

and

$$ay^2 = aB \cdot aA = FB \operatorname{cn}^2 fG' \cdot FA \operatorname{dn}^2 fG' = FE^2 \operatorname{cn}^2 fG' \operatorname{dn}^2 fG',$$

$$\operatorname{cn} fK' = \cos DAy = \frac{ay}{Da} = \frac{\operatorname{cn} fG' \operatorname{dn} fG'}{1+\gamma' \operatorname{sn}^2 fG'}.$$

In a transformation with the stereographic coordinates (u, v) of the point P, where

$$y + ix = a \operatorname{th} \tfrac{1}{2}(u+iv), \quad y,\, x = \frac{a(\operatorname{sh} u,\ \sin v)}{\operatorname{ch} u + \cos v},$$

the circular arc APB, orthogonal to the circle DPE, will cross AB at the curvilinear angle $ABP = v = 2 \operatorname{am} fK' = DFp$, while the rectilinear angle $ABP = \operatorname{am} 2fG'$.

40. Shown on figure 2 and in § 36,

$$x = \frac{OE}{OB} = \frac{1-\gamma'}{1+\gamma'} = \operatorname{th} \tfrac{1}{2} u, \quad \tfrac{1}{2}\theta = \omega = ABQ = \operatorname{am} eG, \quad AQE = ABq = \operatorname{am} (1-e)G,$$

$$AQq' = \operatorname{am} (1+e)G; \quad \text{and} \quad QE = AE \operatorname{dn} eG, \quad Eq = AE \operatorname{dn} (1-e)G.$$

The point Q may be supposed to circulate round the circle on AB in pendulum motion, with velocity due to the level of F, or proportional to EQ or DQ.

When turned about Oy into planes at right angles, the P and Q circles of figure 1 and 2 are linked as a magnetic and electric circuit.

The quadric transformation of § 36 is shown by drawing the perpendicular OR, AX on Qq; and then, with $OER = \chi = \operatorname{am} 2eK$,

$$OR = OE \sin \chi = OQ \cdot \varkappa \operatorname{sn} 2eK, \quad QR = OQ \operatorname{dn} 2eK,$$
$$ER = OE \cos \chi = OQ \cdot \varkappa \operatorname{cn} 2eK, \quad RX = OQ \operatorname{cn} 2eK;$$

and in (6) § 36,

$$\gamma' \operatorname{tn}^2 eG = \frac{\operatorname{tn} eG}{\operatorname{tn}(1-e)G} = \frac{\tan ABQ}{\tan ABq}$$

$$= \frac{\tan AqQ}{\tan AQq} = \frac{QX}{Xq} = \frac{QR-RX}{QR+RX} = \frac{\operatorname{dn} 2eK - \operatorname{cn} 2eK}{\operatorname{dn} 2eK + \operatorname{cn} 2eK},$$

$$\frac{\operatorname{dn}^2 eG}{\gamma'} = \frac{\operatorname{dn} eG}{\operatorname{dn}(1-e)G} = \frac{QE}{Eq} = \frac{QR+RE}{QR-RE} = \frac{\operatorname{dn} 2eK + \varkappa \operatorname{cn} 2eK}{\operatorname{dn} 2eK - \varkappa \operatorname{cn} 2eK}.$$

But if L is taken in OE such that $EL = \varkappa' \cdot LO$, then $LR = LO \operatorname{dn} 2eK$; and if RL is produced to meet the circle on OE again in R',

$$OER' = ORR' = \operatorname{am}(1-2e)K;$$

and

$$LR \sin ORL = OL \cos \chi, \quad \sin ORR' = \frac{OL}{LR} \operatorname{cn} 2eK = \frac{\operatorname{cn} 2eK}{\operatorname{dn} 2eK} = \operatorname{sn}(1-2e)K = \sin \psi;$$

and the angle ψ of the results in § 36 is shown in figure 2 by the angle ORL.

Continuing the quadric transformation,

$$OLR = \operatorname{am} 4eL, \quad \lambda = \frac{1-\varkappa'}{1+\varkappa'} = \left(\frac{1-\sqrt{\gamma'}}{1+\sqrt{\gamma'}}\right)^2 = \frac{CL}{CE},$$

if C is the centre of the circle on OE; and so on.

41. Maxwell obtains the expression of U, Ω, Ω' for a point Q on the axis, in a series of powers of $z = CQ$; and thence infers the series for a point P off the axis by introducing the zonal harmonic $Q_i(\phi)$ of the appropriate order i, as a factor of each term.

His method can be extended to the determination in the A. J. M. of W, W' (§ 3), $\dfrac{dV}{db}$ and V (§ 6), $\dfrac{dV}{d\gamma}$ (§ 15) and v (§ 16) for a thin lens; and an identification made of the result for a point Q on the axis, and thence generally for P off the axis.

The complicated dissection and integrations can then be avoided, required in G. W. Hill's method, although the results of his method must serve as a guide to a form, intangible and invisible otherwise.

Beginning with $\Omega(Q)$, the conical angle subtended at Q by the spherical segment on its circular base AB,

$$\frac{\Omega(Q)}{2\pi}=1-\frac{QO}{QA}=1-\frac{c\cos\gamma-z}{QA}=1-\left(\cos\gamma-\frac{z}{c}\right)\frac{c}{QA}$$

$$=1-\left(\cos\gamma-\frac{z}{c}\right)\left(1+\Sigma Q_i\frac{z^i}{c^i}\right), \tag{1}$$

writing Q_i for the zonal harmonic $Q_i(\gamma)$,

$$\frac{\Omega(Q)}{2\pi}=1-\cos\gamma+(Q_0-Q_1\cos\gamma)\frac{z}{c}+(Q_1-Q_2\cos\gamma)\frac{z^2}{c^2}$$

$$+\ldots.+(Q_{i-1}-Q_i\cos\gamma)\frac{z^i}{c^i}\ldots., \tag{2}$$

and with $z=0$, $\Omega(C)=2\pi(1-\cos\gamma)$.

But at Q', inverse point of Q, and with the other aspect of the spherical surface,

$$\frac{\Omega(Q')}{2\pi}=\frac{Q'O}{Q'A}-1=\frac{z'-c\cos\gamma}{Q'A}-1$$

$$=(z'-c\cos\gamma)\left(\frac{1}{z'}+Q_1\frac{c}{z'^2}+\ldots.+Q_i\frac{c^i}{z'^{i+1}}+\ldots.\right)-1$$

$$=(Q_2-Q_1\cos\gamma)\frac{c^2}{z'^2}+\ldots.+(Q_{i+1}-Q_i\cos\gamma)\frac{c^{i+1}}{z'^{i+1}}+\ldots., \tag{3}$$

and then in Ω' or $\Omega(P')$ at P', replace $\frac{c^{i+1}}{z'^{i+1}}$ by $\frac{c^{i+1}}{r'^{i+1}}Q_i(\phi)=\frac{r^{i+1}}{c^{i+1}}Q_i(\phi)$, so that

$$\frac{\Omega' r'}{2\pi c}=\Sigma(Q_{i+1}-Q_i\cos\gamma)\frac{r^i}{c^i}Q_i(\phi). \tag{4}$$

This makes

$$\frac{\Omega c+\Omega' r'}{2\pi}=c(1-\cos\gamma)+c\Sigma(Q_{i-1}-2Q_i\cos\gamma+Q_{i+1})\frac{r^i}{c^i}Q_i(\phi) \tag{5}$$

which agrees with Maxwell's result for $\frac{P}{2\pi}$ (E. and M., § 694), because

$$\int_\mu^1 Q_i d\mu=\frac{1-\mu^2}{i(i+1)}\frac{dQ_i}{d\mu}=\frac{\mu Q_i-Q_{i+1}}{i}=\frac{Q_{i-1}-Q_{i+1}}{2i+1}=Q_{i-1}-2\mu Q_i+Q_{i+1}, \tag{6}$$

and $\mu=\cos\gamma$. Also

$$\frac{P(Q)}{2\pi}=\frac{a}{QA}=\sin\gamma\left(1+\Sigma Q_i\frac{z^i}{c^i}\right),\quad \frac{P}{2\pi}=\sin\gamma\Sigma Q_i\frac{r^i}{c^i}Q_i(\phi). \tag{7}$$

Next for W, the P. F. of the circular plate AB (A. J. M., § 3),

$$\frac{W(Q)}{2\pi G\sigma}=\int_0^a\frac{ydy}{\sqrt{(y^2+QO^2)}}=QA-QO=\frac{QA^2}{QA}-QO$$

$$=c^2\left(1-2\frac{z}{c}\cos\gamma+\frac{z^2}{c^2}\right)\frac{1}{c}\left(Q_0+Q_1\frac{z}{c}+\ldots.+Q_i\frac{z^i}{c^i}\ldots.\right)$$

$$-c\cos\gamma+z, \tag{8}$$

$$\frac{W(Q)}{2\pi G\sigma c}=1-\cos\gamma+(1-\cos\gamma)\,\frac{z}{c}+(Q_0-2Q_1\cos\gamma+Q_2)\,\frac{z^2}{c^2}$$
$$+\ldots\ldots+(Q_{i-2}-2Q_{i-1}\cos\gamma+Q_i)\,\frac{z^i}{c^i}\ldots\ldots, \qquad (9)$$

$$\frac{W(Q')}{2\pi G\sigma}=Q'A-Q'O=\frac{Q'A^2}{Q'A}+c\cos\gamma-z'$$
$$=z'^2\left(1-2\frac{c}{z'}\cos\gamma+\frac{c^2}{z'^2}\right)\frac{1}{z'}\left(Q_0+Q_1\frac{c}{z'}+\ldots\ldots+Q_i\frac{c^i}{z'^i}\ldots\ldots\right)$$
$$+c\cos\gamma-z'=(Q_0-2Q_1\cos\gamma+Q_2)\,\frac{c^2}{z'}$$
$$+\ldots\ldots+(Q_i-2Q_{i+1}\cos\gamma+Q_{i+2})\,\frac{c^{i+2}}{z'^{i+1}}\ldots\ldots, \qquad (10)$$

$$\frac{W(Q')z'}{2\pi G\sigma c^2}=\tfrac{1}{2}\sin^2\gamma+(Q_1-2Q_2\cos\gamma+Q_3)\,\frac{c}{z'}\ldots\ldots$$
$$+(Q_i-2Q_{i+1}\cos\gamma+Q_{i+2})\,\frac{c^i}{z'^i}\ldots\ldots \qquad (11)$$

$$\frac{W(Q)c+W(Q')z'}{2\pi G\sigma c^2}=\tfrac{1}{2}(1-\cos\gamma)(3+\cos\gamma)+\tfrac{1}{2}(1-\cos\gamma)^2(2+\cos\gamma)\,\frac{z}{c}\ldots\ldots$$
$$+(Q_{i-2}-2Q_{i-1}\cos\gamma+2Q_i-2Q_{i+1}\cos\gamma+Q_{i+2})\frac{z^i}{c^i}\ldots\ldots, \qquad (12)$$

$$\frac{Wc+W'r'}{2\pi G\sigma c^2}=\Sigma(Q_{i-2}-2Q_{i-1}\cos\gamma+2Q_i-2Q_{i+1}\cos\gamma+Q_{i+2})\frac{r^i}{c^i}Q_i(\phi). \qquad (13)$$

So far all these functions are P. F.'s, P, Ω, Ω', U, W, W'; but going back to A. J. M., § 3, for the expression of W in terms of Pa, QA, Ωb, here QA is a S. F., given we find by

$$QA=Pa-\Omega b-\frac{W}{G\sigma}=c\Sigma(Q_{i-2}-Q_{i-1}\cos\gamma)(P_{i-1}\cos\phi-P_i)\,\frac{r^i}{c^i}$$
$$=c\Sigma\frac{\sin^2\gamma\sin^2\phi Q'_{i-1}P'_{i-1}(\phi)}{(i-1)i}\;\frac{r^i}{c^i}. \qquad (14)$$

42. The expression in (4), § 6, A. J. M., p. 384, of the axial component of the attraction of the plano-convex lens is the equivalent of the potential of the base AB, coated with density σ, less the potential of the spherical surface, coated at E with density $\sigma\mu$, $\mu=\cos OCE$. It will serve too for the magnetic potential of the lens, or equivalent current sheet round the portion of the spherical surface.

The thin lens of § 15 may then be considered a spherical segment, coated with density $\sigma(\mu-\cos\gamma)$.

The result of (4), § 6, shows that the coating $\sigma\mu$ would have potential $U(\sigma\mu)$, which can be written

$$\frac{U(\sigma\mu)}{G\sigma}=\frac{W}{G\sigma}-\frac{1}{G\rho}\frac{dV}{db}=\frac{Wc+W'r'}{3G\sigma c}+\frac{U(\sigma)\cos\gamma}{3G\sigma}$$
$$=\tfrac{1}{3}\left(\frac{W}{G\sigma}+\Omega c\cos\gamma\right)+\tfrac{1}{3}\left(\frac{W'}{G\sigma}+\Omega' c\cos\gamma\right)\frac{r'}{c}. \qquad (1)$$

There is an opportunity for a geometrical interpretation on the lines of that given for $U(\sigma)$ in § 3, but a difficulty is to account for the factor $\frac{1}{3}$.

To verify this in Maxwell's manner at a point Q on the axis, we have to evaluate

$$\frac{U(\sigma\mu, Q)}{2\pi G\sigma} = \int_{\cos\gamma}^{1} \frac{\mu d\mu}{\sqrt{(c^2-2cz\cos\gamma+z^2)}} = \frac{1}{c}\int\left(1+\Sigma Q_i\frac{z^i}{c^i}\right)\mu d\mu;$$

requiring $\int_\mu^1 Q_i \mu d\mu$. (2)

Mr. J. R. Wilton gives me the general formula

$$\int_\mu^1 Q_i Q_j d\mu = \frac{(1-\mu^2)(Q_i' Q_j - Q_i Q_j')}{(i-j)(i+j+1)}. \tag{3}$$

But if $i=j$, we must make use of Hargreave's recurring formula (Whittaker, *Analysis*, p. 212).

$$(2i+1)Q_i^2-(2i-1)Q_{i-1}^2 = \frac{d}{d\mu}[(Q_i^2+Q_{i-1}^2)-2Q_iQ_{i-1}]. \tag{4}$$

We only require here the special case of $j=1$, and then make use of the ordinary formulas,

$$iQ_{i-1}-(2i+1)Q_i\mu+(i+1)Q_{i+1}=0, \tag{5}$$

$$(2i+1)\int_\mu^1 Q_i\, d\mu = Q_{i-1}-Q_{i+1} = \frac{(1-\mu^2)Q_i'}{i(i+1)}, \tag{6}$$

$$(2i+1)\int_\mu^1 Q_i\mu d\mu = i\int Q_{i-1}d\mu + (i+1)\int Q_{i+1}d\mu; \tag{7}$$

and thence we find, after reduction,

$$\int_{\cos\gamma}^1 Q_i\mu d\mu = \tfrac{1}{3}(Q_{i-2}-Q_{i-1}\cos\gamma+2Q_i\sin^2\gamma-Q_{i+1}\cos\gamma+Q_{i+2}),$$

with $\int Q_0\mu d\mu = \frac{1}{2}\sin^2\gamma$, $\int Q_1\mu d\mu = \frac{1}{3}(1-\cos^3\gamma)$. (8)

But according to the preceding expressions in (5) and (13)

$$\begin{aligned}
&\frac{W(Q)c+W(Q')z'}{2\pi G\sigma c}+\frac{U(Q)\cos\gamma}{G\sigma}\\
&\quad=\tfrac{1}{2}c(1-\cos\gamma)(3+\cos\gamma)+\tfrac{1}{2}(1-\cos\gamma)(2+\cos\gamma+\cos^2\gamma)z\ldots\\
&\qquad+(Q_{i-2}-2Q_{i-1}\cos\gamma+2Q_i-2Q_{i+1}\cos\gamma+Q_{i+2})\frac{z^i}{c^{i-1}}\ldots\\
&\qquad+c\cos\gamma(1-\cos\gamma)+\tfrac{1}{2}z\cos\gamma\sin^2\gamma\ldots\\
&\qquad\qquad+\cos\gamma(Q_{i-1}-2Q\cos\gamma+Q_{i+1})\frac{z^i}{c^{i-1}}\ldots\\
&\quad=\tfrac{3}{2}c\sin^2\gamma+(1-\cos^3\gamma)z\ldots\\
&\qquad+(Q_{i-2}-Q_{i-1}\cos\gamma+2Q_i\sin^2\gamma-Q_{i+1}\cos\gamma+Q_{i+2})\frac{z^i}{c^{i-1}}\ldots,
\end{aligned} \tag{9}$$

which adds the audit up, and so the identification is complete.

Then for v, the P. F. of the thin curved lens of § 15, treated as a spherical segment, coated with density $\sigma(\mu-\cos\gamma)$,

$$\frac{v}{G\sigma}=\frac{Wc+W'r'-2Uc\cos\gamma}{3G\sigma c},\quad \frac{v}{Gm}=\frac{Wc+W'r'-2Uc\cos\gamma}{3\pi G\sigma c^3(1-\cos\gamma)^2}, \tag{10}$$

if of mass $m=2\pi\sigma c^2\int_{\cos\gamma}^{1}(\mu-\cos\gamma)\,d\mu=\pi\sigma c^2(1-\cos\gamma)^2$.

For a flat lens, with $\gamma=0$, this expression for v takes an indeterminate form, and is best evaluated independently, as in § 16, by a dissection of concentric circles and radiating straight lines.

The P. F. W of the flat disc may be evaluated at the same time in this manner, and then

$$\frac{1}{G\sigma}\frac{dW}{d\theta}=\int_0^a\frac{y\,d\theta}{PQ'}=PQ-PO-IA\cos\theta, \tag{11}$$

$$I=\int_0^a\frac{dy}{PQ'}=\text{ch}^{-1}\frac{PQ}{PZ}-\text{ch}^{-1}\frac{PO}{PZ}=\text{sh}^{-1}\frac{c+A\cos\theta}{PZ}-\text{sh}^{-1}\frac{A\cos\theta}{PZ}, \tag{12}$$

$$PQ'^2=y^2+2Ay\cos\theta+A^2+b^2,\quad PQ^2=a^2+2Aa\cos\theta+A^2+b^2,$$
$$PZ^2=A^2\sin^2\theta+b^2,\quad PO^2=A^2+b^2, \tag{13}$$

$$\frac{dI}{d\theta}=-\frac{Aa\cos\theta+A^2+b^2}{PZ^2}\cdot\frac{A\sin\theta}{PQ}+\frac{PO\cdot A\sin\theta}{PZ^2} \tag{14}$$

43. The integral $\int I\,d\theta$ is intractable; it arises in the skin potential of the curved wall of a circular cylinder, and is shown graphically by a quadrature on the Mercator chart, as explained in the *Trans. American Math. Society*, October, 1907, §§ 53, 66.

But $\int IA\cos\theta\,d\theta$ is tractable, and integrating by parts

$$\int_0^{2\pi}IA\cos\theta\,d\theta=(IA\sin)_0^{2\pi}-\int\frac{dI}{d\theta}A\sin\theta\,d\theta$$
$$=\int\frac{Aa\cos\theta+A^2+b^2}{PZ^2}\cdot\frac{b^2d\theta}{PQ}-\int(Aa\cos\theta+A^2+b^2)\frac{d\theta}{PQ}$$
$$+PO\int\left(1-\frac{b^2}{PZ^2}\right)d\theta, \tag{15}$$

in which the first integral is $(2\pi-\Omega)b$, as shown in the *Trans. American Math. Society*, § 48, by a dissection of the circular area AB into sector elements $\frac{1}{2}a^2d\theta$, and then

$$\frac{W}{G\sigma}=\int PQ\,d\theta-2\pi PO+(2\pi-\Omega)b+QA-\frac{A^2+b^2}{a}\int\frac{d\theta}{PQ}+2\pi\cdot PO-2\pi b$$
$$=\frac{A^2+a^2+b^2}{a}P-2QA+(2\pi-\Omega)b+QA-\frac{A^2+b^2}{a}P-2\pi b$$
$$=Pa-QA-\Omega b; \tag{16}$$

as before.

The flat lens may be considered a disc, coated with density

$$\sigma\left(1-\frac{y^2}{a^2}\right)=\sigma\frac{AQ'\cdot Q'B}{OA^2},$$

and its potential is shown in § 16 to be expressible by the three elliptic integrals, of the I, II, and III kind, P, Q, and Ω.

A formula of integration by parts will show that a similar expression is obtained for a density $\sigma\left(1-\frac{y^2}{a^2}\right)^n$, or otherwise for a density varying as some even power of y.

But for an odd power, the intractable integral $\int I d\theta$ arises, in addition to the elliptic integrals.

A formula of reduction is obtained by the integration of the relation

$$\frac{d}{dy}(y^nPQ')=[(n+1)y^{n+1}+(2n+1)y^nA\cos\theta+ny^{n-1}(A^2+b^2)]\frac{1}{PQ'}. \quad (17)$$

In the definite integral the algebraical part vanishes at both limits.

Mr. Bromwich has expressed the results in a series in the *Proc. London Math. Society*, September, 1912.

On the electrified disc, and on a flattened oblate spheroid, the density is $\sigma\left(1-\frac{y^2}{a^2}\right)^n$, with $n=-\frac{1}{2}, \frac{1}{2}$; and the result for the potential is non-elliptic and given already. A similar formula of reduction will give the result for a density $\sigma\left(1-\frac{y^2}{a^2}\right)^{i-\frac{1}{2}}$, where i is an integer; and the evaluation should attract a careful worker, when the need arises in a physical problem.

So also for the S. F. of these coatings of superficial density, to be worked out as an exercise.

STAPLE INN, LONDON, *October* 12, 1915.

On the Asymptotic Character of Functions Defined by Series of the Form $\Sigma c_n g(x+n)$.*

By R. D. Carmichael.

Introduction.

In a previous memoir † I have laid the foundations of a general theory of series of the forms

$$\Omega(x)=\sum_{n=0}^{\infty} c_n g(x+n), \tag{1}$$

$$\overline{\Omega}(x)=\sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)}, \tag{1'}$$

where $c_0, c_1, c_2, \ldots.$ are constants and $g(x)$ is a function of x having the asymptotic character

$$g(x) \sim h(x)\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots.\right), \qquad h(x)=x^{P(x)}e^{Q(x)}, \tag{2}$$

$P(x)$ and $Q(x)$ being polynomials which we write in the form

$$P(x)=\mu_0+\mu_1 x+\mu_2 x^2+\ldots.+\mu_k x^k, \qquad \mu_k \neq 0 \text{ if } k>0;$$
$$Q(x)=\alpha_0+\alpha_1 x+\alpha_2 x^2+\ldots.+\alpha_m x^m, \qquad \alpha_m \neq 0 \text{ if } m>0.$$

(In case $k=0$ we assume that $m>1$.) In *I* we required that the asymptotic relation (2) should be valid only for x approaching infinity in a positive sense along any line whatever parallel to the axis of reals. But for the application of series $\Omega(x)$ and $\overline{\Omega}(x)$ to a study of the properties of functions in the neighborhood of a singular point, it is convenient to place a stronger condition on $g(x)$ and to require, in fact, that the asymptotic relation (2) shall be valid when x is confined in any way to a sector V including the positive axis of reals in its interior. Furthermore, we shall suppose that the singularities of $g(x)$ in the finite plane are isolated points, and that $g(x)$ is analytic in V at every point for which $|x|$ is not less than a given constant K.

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* Presented to the American Mathematical Society (at Chicago), April 21, 1916.

† *Transactions of the American Mathematical Society,* Vol. XVII (1916), pp. 207–232. This paper will be referred to as *I*. For the most part we employ here the notation of the earlier memoir.

We employ σ to denote α_m or μ_k according as m is or is not greater than k and write

$$\sigma=\sigma_0+\sigma_1\sqrt{-1},$$

where σ_0 and σ_1 are real. If $\lambda[\mu]$ is a constant determined as in Theorem XII of *I*, then the region of convergence [absolute convergence] of $\Omega(x)$ is bounded by the straight line $R(\sigma x)=\lambda[R(\sigma x)=\mu]$ and lies on that side of this line for which $R(\sigma x)<\lambda[R(\sigma x)<\mu]$. According to Theorem XVI of *I*, a given function $f(x)$ can not have more than one expansion in a series $\Omega(x)$ for a given $g(x)$, provided that $R(\sigma)$ is negative; but such an expansion is not necessarily unique when the latter condition is not satisfied. Accordingly, we shall now further restrict $g(x)$ by requiring that $R(\sigma)$ shall be negative. We shall also suppose that non-exceptional points for both $\Omega(x)$ and $\overline{\Omega}(x)$ exist in every strip parallel to the line $R(\sigma x)=0$.

The general object of this paper is to determine the asymptotic character of a function $\Omega(x)$ defined by a series of the form (1). In § 1 a somewhat extended discussion results in the fundamental Theorem I by which the asymptotic character of the function $\Omega(x)$ in relation to $g(x)$ is determined. This result is quite satisfying from the point of view of its elegance and simplicity. For the important case when $k=1$ and $m=0$ or 1, Theorem I may be stated in the somewhat more convenient form of Theorem II. The character of the convergence of the series $\overline{\Omega}(x)$ with respect to uniformity is treated in § 2, the result being stated as Theorem III.

Guided by the result contained in Theorem I, I introduce in § 3 an extension of the notion of asymptotic representation. In § 4 it is shown that a special case of this extended notion includes as a special case the notion of asymptotic representation in the sense of Poincaré. In § 5 a further treatment is given of this new asymptotic representation in the case when $k=1$, $m=0$ or 1 and $\sigma=-1$. It is shown in this case that the notion is equivalent to that of Poincaré, and explicit formulae are obtained exhibiting this equivalence. Finally, in § 6, an illustrative example is given to emphasize the fact that an $\overline{\Omega}$-series is capable of representing conveniently the asymptotic character of certain functions near infinity, while at the same time it yields also the essential properties of these functions in the finite plane.

§ 1. *Asymptotic Character of the Function $\Omega(x)$ in Relation to $g(x)$.*

At the close of *I* we saw incidentally that the function $\Omega(x)$ defined by the series in equation (1) has the properties expressed in the relations

$$\lim\{g(x+s)\}^{-1}\{\Omega(x)-\sum_{n=0}^{s}c_n g(x+n)\}=0, \qquad s=0,1,2,\ldots, \tag{3}$$

provided that the limits are taken for x approaching infinity in a positive sense along the axis of reals. We now make inquiry as to what less stringent restrictions may be placed on the approach of x to infinity without destroying the general validity of relations (3). Naturally we make the general restriction that x shall remain in V as it approaches infinity.

In the first place, if relations (3) are to be valid in the special case when c_1 is different from zero and all the coefficients $c_2, c_3, \ldots$ in (1) are equal to zero, it is necessary that the ratio $g(x+1)/g(x)$ shall approach zero as x approaches infinity in the manner to be specified. Moreover, if the region over which x may vary in approaching infinity is such that $x+n$ is in the region when x is in the region, n being a positive integer, then it is easy to see that the foregoing condition is also sufficient in all cases in which the series in (1) has only a finite number of coefficients different from zero, since

$$\frac{g(x+n)}{g(x+s)}=\frac{g(x+s+1)}{g(x+s)}\cdot\frac{g(x+s+2)}{g(x+s+1)}\cdot\ldots\cdot\frac{g(x+n)}{g(x+n-1)} \tag{4}$$

when n is any integer greater than the integer s. Accordingly, we are led to ascertain first the conditions under which $g(x+1)/g(x)$ shall approach zero as x approaches infinity.

We may write

$$\frac{g(x+1)}{g(x)}\sim e^{[P(x+1)-P(x)]\log x}e^{Q(x+1)-Q(x)}e^{P(x+1)\log\left(1+\frac{1}{x}\right)}\left(1-\frac{a_1}{x^2}+\ldots\right),$$

the relation being valid for x approaching infinity in V. From this it follows readily that $g(x+1)/g(x)$ will approach zero when and only when

$$e^{P_1(x)\log x+Q_1(x)}$$

approaches zero, x being confined to V and $P_1(x)$ and $Q_1(x)$ being polynomials with the values

$$\begin{aligned}
P_1(x)&=P(x+1)-P(x),\\
Q_1(x)&=Q(x+1)-Q(x)+\frac{1}{x}p_1(x)-\frac{1}{2x^2}p_2(x)\\
&\qquad+\frac{1}{3x^3}p_3(x)-\ldots+(-1)^{k-1}\frac{1}{kx^k}p_k(x),
\end{aligned}$$

where $p_j(x)$ denotes the sum of the terms of degree j or greater in the function $P(x+1)$ when arranged as a polynomial in x. Moreover, if we write

$$\frac{g(x+1)}{g(x)}=e^{P_1(x)\log x+Q_1(x)}\phi(x), \tag{5}$$

then $\phi(x)$ approaches unity as x approaches infinity in V.

The leading term in the polynomial $P_1(x)$ obviously is of degree $k-1$ and has the coefficient $k\mu_k$. If $m>k$, the leading term in the polynomial $Q_1(x)$ is of degree $m-1$ and has the coefficient $m\alpha_m$. Now, if we put $x=\rho e^{i\theta}$ where ρ is positive and write

$$R\{P_1(x)\log x+Q_1(x)\}=\pi_1(\rho)\log\rho+\pi_2(\rho),$$

we see that $\pi_1(\rho)$ and $\pi_2(\rho)$ are polynomials in ρ with coefficients involving θ. If $k\geqq m$, so that μ_k is denoted by σ, then the coefficient of the highest power of ρ in $\pi_1(\rho)$ is $k(\sigma_0\cos t\theta-\sigma_1\sin t\theta)$, where $t=k-1$. If $k<m$, so that α_m is denoted by σ, then the coefficient of the highest power of ρ in $\pi_2(\rho)$ is $m(\sigma_0\cos t\theta-\sigma_1\sin t\theta)$, where $t=m-1$. If θ is held fixed and ρ is allowed to approach $+\infty$, then the absolute value of the exponential function in (5) approaches zero or infinity according as $\sigma_0\cos t\theta-\sigma_1\sin t\theta$ is less than zero or greater than zero; and when the last quantity is equal to zero, the absolute value of the exponential quantity in (5) may approach infinity or may approach zero owing to different possible forms of $P_1(x)$ and $Q_1(x)$.

From these considerations we are led to restrict x to the largest sector V_1 which includes the positive axis of reals, is in V and is such that

$$\sigma_0\cos t\theta-\sigma_1\sin t\theta\leqq-\varepsilon \tag{6}$$

for every x in V_1, ε being a positive quantity as small as one pleases, and t being equal to $m-1$ or $k-1$ according as m is or is not greater than k. It is clear that $g(x+1)/g(x)$ approaches zero as x approaches infinity in V_1. (In case $k=1$ and $m=0$ or 1, so that $t=0$, it is obvious that ε may be chosen so that (6) is satisfied for all values of θ; in this case V_1 coincides with V.)

Returning now to a consideration of the series $\Omega(x)$, let us denote its convergence number by λ so that the boundary of its region of convergence is the straight line $R(\sigma x)=\lambda$. Let λ_1 be any number less than λ, and confine x so that $R(\sigma x)\leqq\lambda_1$. Denote by $\overline{V}$ the maximum region which is common to V_1 and the half-plane $R(\sigma x)\leqq\lambda_1$. Obviously $\overline{V}$ contains the positive axis of reals in its interior, except possibly for a segment of finite length adjacent to the point 0.

Let x_0 be a non-exceptional point for the series $\Omega(x)$ subject to the condition that

$$\lambda_1<R(\sigma x_0)<\lambda.$$

Put

$$u_n=c_n g(x_0+n),\quad v_n=\frac{g(x+n)}{g(x_0+n)},\quad n\geqq l,$$

the integer l being chosen so that $g(x_0+n)$ is different from zero for every n not less than l. Then, if we write

$$U_{l-1}=0; \quad U_n=u_l+u_{l+1}+\ldots\ldots+u_n, \quad n \geqq l,$$

we have

$$\begin{aligned}\sum_{n=l}^{m} c_n g(x+n) &= \sum_{n=l}^{m} u_n v_n \\ &= \sum_{n=l}^{m} (U_n-U_{n-1}) v_n \\ &= -\sum_{n=l}^{m} U_n (v_{n+1}-v_n) + U_m v_{m+1}.\end{aligned}$$

From the convergence of $\Omega(x_0)$ it follows that U_n approaches a finite limit as n increases indefinitely. If x is held fixed while m increases indefinitely, it is clear that V_m approaches zero. (Compare (10) of *I* and the discussion immediately following.) Hence it follows from the last equation that a constant M exists (independent of x) such that for every non-exceptional x in the half-plane $R(\sigma x) \leqq \lambda_1$ we have

$$\left| \sum_{n=l}^{\infty} c_n g(x+n) \right| \leqq M \sum_{n=l}^{\infty} |v_{n+1}-v_n|, \tag{7}$$

the latter series being certainly convergent as we saw in *I*.

In view of (1) relations (3) may be expressed in the form

$$\lim \sum_{n=s+1}^{\infty} c_n \frac{g(x+n)}{g(x+s)} = 0, \qquad s=0, 1, 2, \ldots\ldots \tag{8}$$

Let s_1 be a fixed integer and let s be any integer less than s_1. Then we may write

$$\sum_{n=s+1}^{\infty} c_n \frac{g(x+n)}{g(x+s)} = \sum_{n=s+1}^{s_1} c_n \frac{g(x+n)}{g(x+s)} + \frac{g(x+s_1)}{g(x+s)} \sum_{n=s_1+1}^{\infty} c_n \frac{g(x+n)}{g(x+s_1)}.$$

Now, suppose that x approaches infinity in $\overline{V}$ in such wise that (8) is valid for the particular value s_1 of s. Then from the last relation it follows readily that (8) is also valid, for such approach of x to infinity, whenever s is less than s_1. Therefore, to prove all the relations (8), and hence all the relations (3), valid for a given approach of x to infinity in $\overline{V}$, it is sufficient to prove them valid for those values of s which are greater than some given number.

Employing the result of the preceding paragraph and comparing (7) and (8) we see that all the relations (3) will be satisfied provided that x approaches infinity in $\overline{V}$ in such a way that

$$\lim \frac{1}{g(x+s)} \sum_{n=s+1}^{\infty} |v_{n+1}-v_n| = 0 \tag{9}$$

for all values of the integer s greater than a conveniently chosen positive integer s_1. We choose s_1 so that it is not less than l. It is obvious that (9) may be replaced by the following equivalent relations:

$$\lim \sum_{n=s+1}^{\infty} \left| \frac{g(x_0+s)}{g(x+s)} \cdot \frac{g(x+n)}{g(x_0+n)} \left\{ \frac{g(x+n+1)}{g(x_0+n+1)} \cdot \frac{g(x_0+n)}{g(x+n)} - 1 \right\} \right| = 0, \quad s > s_1. \tag{10}$$

We are now in position to prove the following theorem:

THEOREM I. *Let $\Omega(x)$ be a function defined by a series* (1) *whose convergence number λ is not $-\infty$. Let t denote $m-1$ or $k-1$ according as m is or is not greater than k. Let $\overline{V}$ denote the greatest region of the x-plane common to V, the half-plane $R(\sigma x)\leqq\lambda_1$ and that sector $\sigma_0\cos t\theta-\sigma_1\sin t\theta\leqq-\varepsilon$ which includes the positive axis of reals, where λ_1 is any number less than λ, and ε is any positive number however small and θ is defined by the relation $x=\rho e^{i\theta}$, ρ being positive. Then if x approaches infinity in $\overline{V}$ we have*

$$\lim \{g(x+s)\}^{-1}\{\Omega(x)-\sum_{n=0}^{s} c_n g(x+n)\}=0, \qquad s=0,1,2,\ldots\ldots$$

If in (10) we replace g in part by h it is easy to see from the foregoing results that the proof of this theorem will be complete if we show that

$$\lim \frac{h(x_0+s)}{h(x_0+s+1)}\cdot\frac{h(x+s+1)}{h(x+s)}\sum_{n=s+1}^{\infty}\left|\frac{h(x_0+s+1)h(x+n)}{h(x+s+1)h(x_0+n)}\right| \cdot\left|\frac{g(x+n+1)g(x_0+n)}{g(x+n)g(x_0+n+1)}-1\right|=0, \qquad s>s_1, \qquad (11)$$

the limit being taken for x approaching infinity in $\overline{V}$; and hence if we show that the sum of the series in (11) is bounded.

For this series in (11) we use for the moment the symbol $T(x,s)$. By $\overline{T}(x,s)$ we denote the series

$$\overline{T}(x,s)=\sum_{n=s+1}^{\infty}\left|\frac{h(x_0+s+1)h(x+n)}{h(x+s+1)h(x_0+n)}\right|\cdot\left\{\left|\frac{g(x+n+1)g(x_0+n)}{g(x+n)g(x_0+n+1)}\right|+1\right\}.$$

Then the series $T(x,s)$ is term by term less than or equal to the series $\overline{T}(x,s)$.

In case k or m is greater than unity it will now be shown that $\overline{T}(x,s)$ is bounded as x approaches infinity in $\overline{V}$; and thus the proof of the theorem for this case will be completed.

Let x_1 be a non-exceptional point for the series $\Omega(x)$ such that $\lambda_1<R(\sigma x_1)<R(\sigma x_0)$. Then the series $\overline{T}(x,s)$ converges whatever the value of s (greater than s_1). Therefore, in order to complete the proof of the theorem in this case it is sufficient (compare equation (4)) to show that a number X exists such that

$$\frac{h(x+j+1)}{h(x_1+j+1)}\cdot\frac{h(x_1+j)}{h(x+j)} \qquad (12)$$

is in absolute value less than 1 for every integer j, provided that $|x|>X$ and x is in $\overline{V}$. This we shall now prove without the restriction that k or m is greater than unity.

If we denote the expression in (12) by E and employ the notation $P_1(x)$ and $Q_1(x)$ introduced in the earlier part of this section, then we may write:

$$E=\frac{(x+j)^{P_1(x+j)}\left(1+\dfrac{1}{x+j}\right)^{P(x+j+1)}}{(x_1+j)^{P_1(x_1+j)}\left(1+\dfrac{1}{x_1+j}\right)^{P(x_1+j+1)}}\,e^{Q(x+j+1)-Q(x_1+j+1)-Q(x+j)+Q(x_1+j)};$$

or

$$E=(x+j)^{P_1(x+j)}(x_1+j)^{-P_1(x_1+j)}e^{Q_1(x+j)-Q_1(x_1+j)}S\equiv\overline{E}S,$$

where S is a quantity which approaches unity if either x or j approaches infinity, or if both x and j approach infinity in an independent manner, x remaining in $\overline{V}$. We may write

$$\overline{E}=\frac{|x+j|^{P_1(x+j)}\,e^{iP_1(x+j)\arg(x+j)}}{|x_1+j|^{P_1(x_1+j)}e^{iP_1(x_1+j)\arg(x_1+j)}}\,e^{Q_1(x+j)-Q_1(x_1+j)}.$$

We may write $P_1(x+j)$ as a polynomial in j in the form[*]

$$P_1(x+j)=\pi_0(x)+\pi_1(x)j+\ldots\ldots+\pi_{k-1}(x)j^{k-1}.$$

We introduce the notation

$$\overline{P}_1(x+j)=|\pi_0(x)|+|\pi_1(x)|j+\ldots\ldots+|\pi_{k-1}(x)|j^{k-1}.$$

Now, since x must remain in $\overline{V}$ it is easy to see that $|x+j|/|x_1+j|$ is bounded away from zero,[†] however x and j vary, provided that $|x|$ is sufficiently large. Hence a constant M exists such that

$$|\overline{E}|\leqq|x+j|^{R[P_1(x+j)]+\overline{P}_1(x_1+j)}e^{M[\overline{P}_1(x+j)+\overline{P}_1(x_1+j)]}e^{R[Q_1(x+j)-Q_1(x_1+j)]}\tag{13}$$

provided that $|x|$ is sufficiently large.

It is convenient now to separate into cases owing to the relative magnitude of k and m.

First, let us suppose that $k\geqq m$. Then (as we have seen) the leading term of $P_1(x)$, considered as a polynomial in x, is $k\sigma x^{k-1}$. With this in hand, let us consider $P_1(x+j)$ as a polynomial in j whose coefficients are arranged

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[*] The treatment in the text applies strictly only when $k>0$; but with the understanding that $P_1(x)\equiv0$ and $\overline{P}_1(x)\equiv0$ when $k=0$ the formulae employed are valid.

[†] This is obvious in case σ_1 is zero or j is bounded. When σ_1 is not zero and j is not bounded it may be proved as follows: Take j greater than $|x_1|$. Then the ratio in consideration is greater than $\frac{1}{4}$ if $u\geqq-\frac{1}{2}j$. When u is negative the relation $R(\sigma x)\leqq\lambda_1$ (which is satisfied when x is in $\overline{V}$) may be written in the form $\sigma_0 u-\sigma_1 v\leqq\lambda_1$ and obviously leads to the relation

$$-u\leqq\left|\frac{\sigma_1}{\sigma_0}\right||v|+\left|\frac{\lambda_1}{\sigma_0}\right|,$$

since σ_0 is negative. If $u<-\frac{1}{2}j$ we have

$$|v|>\frac{1}{2}\left|\frac{\sigma_0}{\sigma_1}\right|j-\left|\frac{\lambda_1}{\sigma_1}\right|,$$

whence we conclude at once to the desired result when $u<-\frac{1}{2}j$. This completes the proof.

as polynomials in ρ, where ρ is defined as in the theorem. It is clear that the highest power of ρ in the coefficient of each power of j has a coefficient whose real part is negative, since

$$\sigma_0 \cos s\theta - \sigma_1 \sin s\theta \leqq -\varepsilon$$

for each value of s in the set $s=0, 1, 2, \ldots, t$. Now, for any given positive number K_1, however large, it is clear that a constant X_1 exists such that $|x+j| > e^{K_1}$ for every j and every x such that $|x| \geqq X_1$. Hence, if we replace $|x+j|$ in (13) by e^{K_1}, having properly chosen K_1 as a quantity independent of x and j, and write the resulting relation in the form

$$|\overline{E}| < e^{\pi(\rho, j)},$$

it is clear that $\pi(\rho, j)$ is a polynomial in j whose coefficients are all negative provided that ρ is greater than an appropriately chosen constant X_2. Hence $|\overline{E}| < 1$ for every j provided that $|x| \geqq X_2$. Hence the expression in (12) has the desired property of being less than 1 in absolute value for every j greater than s_1, provided only that $|x|$ is greater than an appropriately chosen constant X.

In the case in which $k < m$ the same conclusion may readily be obtained in a similar manner. In this case the polynomial $Q_1(x+j)$ plays the leading rôle and in a corresponding way, since the leading term in $Q_1(x)$, considered as a polynomial in x, is $m\sigma x^{m-1}$. In carrying out the argument it is convenient to replace $|x+j|$ by $e^{K_2(j+|u|+|v|)}$ and to observe that the latter quantity is greater than the former, however small the constant K_2 may be, provided only that $|x|$ is sufficiently large. It is easy to supply the detailed argumentation requisite here.

This completes the proof of the theorem for the case in which k or m is greater than unity.

For the case which remains we have $k=1$ and $m=0$ or 1.[*] Moreover, we have seen that it is sufficient to know that the series $T(x, s)$ is bounded for x approaching infinity in $\overline{V}$. But the sum of any finite number of terms of $T(x, s)$ approaches a finite value as x thus approaches infinity. Omitting $r-1$ terms of $T(x, s)$ we see that it is sufficient to our purpose to know that the quantity

$$\sum_{n=s+r}^{\infty} \left| \frac{h(x_0+s+1)\,h(x+n)}{h(x+s+1)\,h(x_0+n)} \right| \cdot \left| \frac{g(x+n+1)\,g(x_0+n)}{g(x+n)\,g(x_0+n+1)} - 1 \right| \tag{14}$$

is bounded as x approaches infinity in $\overline{V}$. The positive integer r which appears in this expression may be chosen to suit our convenience.

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[*] If λ_1 were chosen so as to be less than $\lambda + 1/R(\sigma)$ and the definition of $\overline{V}$ were modified accordingly, the theorem could be proved in this case just as in the preceding. But for the stronger form of the theorem a modified argument is necessary.

For the case now in consideration $h(x)$ has the special value

$$h(x) = x^{\mu_0+\sigma x} e^{\alpha_0+\beta x}.$$

Then, from the relation between $h(x)$ and $g(x)$, it follows that $g(x+1)/g(x)$ may be written in the form

$$\frac{g(x+1)}{g(x)} = x^{\sigma} e^{\sigma+\beta}\left(1+\frac{\xi(x)}{x}\right), \tag{15}$$

where the function $\xi(x)$ is bounded as x approaches infinity in $\overline{V}$. By means of this relation we readily have

$$\frac{g(x+n+1)g(x_0+n)}{g(x+n)g(x_0+n+1)} - 1 = \frac{(x+n)^{\sigma}}{(x_0+n)^{\sigma}} - 1 + \frac{(x+n)^{\sigma}}{(x_0+n)^{\sigma}} \cdot \frac{r(x+n)-r(x_0+n)}{1+r(x_0+n)},$$

where $r(x) = \xi(x)/x$. From these relations and the fact that $|(x+n)^{\sigma}/(x_0+n)^{\sigma}|$ is bounded it is easy to see that the quantity in (14) is bounded provided that the same is true of each of the quantities

$$\left.\begin{aligned} &\sum_{n=s+r}^{\infty} \left| \frac{h(x_0+s+1)h(x+n)}{h(x+s+1)h(x_0+n)} \right| \cdot \left| \frac{(x+n)^{\sigma}}{(x_0+n)^{\sigma}} - 1 \right|, \\ &\sum_{n=s+r}^{\infty} \left| \frac{h(x_0+s+1)h(x+n)}{h(x+s+1)h(x_0+n)} \right| \cdot \frac{1}{n}. \end{aligned}\right\} \tag{16}$$

That the latter of these two expressions is bounded may be proved by the method employed in the preceding case in the derivation of a similar result. Hence it remains to be shown that the first expression in (16) is bounded as x approaches infinity in $\overline{V}$.

Now the first series in (16) may be written in the form

$$\sum_{n=s+r}^{\infty} \left| \frac{h(x_0+s+r)h(x+n)}{h(x+s+r)h(x_0+n)} \right| \cdot \left| \frac{h(x_0+s+1)h(x+s+r)}{h(x+s+1)h(x_0+s+r)} \right| \cdot \left| \frac{(x+n)^{\sigma}}{(x_0+n)^{\sigma}} - 1 \right|.$$

Then from the form of $h(x)$ it is easy to see that the sum of this series is bounded provided that the same is true of the sum of the series

$$\sum_{n=s+r}^{\infty} \left| \frac{h(x_0+s+r)h(x+n)}{h(x+s+r)h(x_0+n)} \right| \cdot |S(x,n)|, \tag{17}$$

where $S(x,n)$ denotes the quantity

$$S(x,n) = (x-x_0)^{(r-1)\sigma}\left\{\frac{(x+n)^{\sigma}}{(x_0+n)^{\sigma}} - 1\right\} = (x-x_0)^{(r-1)\sigma}\left\{\left(1+\frac{x-x_0}{x_0+n}\right)^{\sigma} - 1\right\}.$$

Now, if $|x-x_0| < |x_0+n|$ and x is in $\overline{V}$ it is easy to see through the application of the binomial theorem to the last expression for $S(x,n)$ that a constant M_1 exists such that

$$|S(x,n)| < \frac{M_1}{n}.$$

Again, the quantity enclosed in braces in the first expression for $S(x, n)$ is bounded, however x and n may vary provided only that $|x|$ is sufficiently large, since under such conditions $|x+n|/|x_0+n|$ is bounded away from zero (as we saw above). Hence, if $|x-x_0| \geqq |x_0+n|$ and x is in $\overline{V}$, it follows readily that a constant M_2 exists such that

$$|S(x, n)| < \frac{M_2}{n},$$

provided that the integer r is such that $r-1 \geqq -1/R(\sigma)$. We choose r in such manner as to satisfy this relation.

Combining these two inequalities for $|S(x, n)|$ we see that a constant M_3 exists such that $|S(x, n)| < M_3/n$ for every x and n provided only that n and $|x|$ are sufficiently large and x is in $\overline{V}$. Hence the sum of the series in (17) is bounded provided that the same is true of the sum of the series

$$\sum_{n=s+r}^{\infty} \left| \frac{h(x_0+s+r)\,h(x+n)}{h(x+s+r)\,h(x_0+n)} \right| \cdot \frac{1}{n}.$$

That this sum is bounded may be proved by the method employed in the preceding case in the derivation of a similar result.

This completes the proof of Theorem I.

By $\omega(x)$ and $\overline{\omega}(x)$ we denote the series

$$\omega(x) = \sum_{n=0}^{\infty} c_n g_1(x+n), \tag{18}$$

$$\overline{\omega}(x) = \sum_{n=0}^{\infty} c_n \frac{g_1(x+n)}{g_1(x)}, \tag{18'}$$

where $g_1(x)$ is that special case of $g(x)$ in which $k=1$ and $m=0$ or 1. For the special case $\overline{\omega}(x)$ of the series $\overline{\Omega}(x)$ the foregoing theorem may readily be put in the following interesting form:

THEOREM II. *Let $\overline{\omega}(x)$ be a function defined by a series (18′) whose convergence number λ is not $-\infty$. Let $\overline{V}$ denote the greatest region of the x-plane common to V and the half-plane $R(\sigma x) \leqq \lambda_1$, λ_1 being a constant which is less than λ. Then, if x approaches infinity in $\overline{V}$, we have*

$$\lim x^{-s\sigma} \left\{ \overline{\omega}(x) - \sum_{n=0}^{s} c_n \frac{g_1(x+n)}{g_1(x)} \right\} = 0, \qquad s = 0, 1, 2, \ldots\ldots \tag{19}$$

For the proof of the theorem it is sufficient to observe that

$$\frac{g_1(x+s)}{x^{s\sigma} g_1(x)}$$

approaches a finite quantity as x approaches infinity in $\overline{V}$, a result which follows readily from (15), or directly from the asymptotic form of $g_1(x)$.

For an important range of cases the region $\overline{V}$ in Theorem II may be replaced by the half-plane $R(\sigma x) \leqq \lambda_1$, namely, those cases in which the rays bounding the sector V (except for their common point zero) lie entirely within the half-plane $R(\sigma x) \geqq 0$. In these cases relations (19) are valid for x approaching infinity in any half-plane lying within the half-plane of convergence of the series $\overline{\omega}(x)$. In particular, this is the case when $g_1(x)$ is the first principal solution of the difference equation

$$f(x+1) = A(x) f(x),$$

in which $A(x)$ is a function which is analytic at infinity and has there a zero of the first order. The series $\overline{\omega}(x)$, defined by means of such a function $g_1(x)$, are of prime importance as we shall show in a later paper. For the special case when $A(x) = 1/x$ the series $\overline{\omega}(x)$ is the factorial series. Theorem II for the special case of factorial series is due to Nörlund.* The demonstration employed by Nörlund in the special case does not seem to be applicable for deriving the general theorems here obtained.

§ 2. *Uniformity of Convergence of the Series* $\overline{\Omega}(x)$.

From relation (7) it follows that constants N_1 and N_2 exist such that

$$\left.\begin{aligned}\left|\sum_{n=s+1}^{\infty} c_n \frac{g(x+n)}{g(x)}\right| &\leqq N_1 \sum_{n=s+1}^{\infty} \left|\frac{g(x+n)}{g(x)\,g(x_0+n)}\right| \cdot \left|\frac{g(x+n+1)\,g(x_0+n)}{g(x+n)\,g(x_0+n+1)} - 1\right| \\ &\leqq N_2 \sum_{n=s+1}^{\infty} \left|\frac{h(x+n)\,h(x_0+r)}{h(x+r)\,h(x_0+n)}\right| \cdot \left|\frac{h(x+r)\,h(x_0)}{h(x)\,h(x_0+r)}\right| \\ &\qquad \cdot \left|\frac{g(x+n+1)\,g(x_0+n)}{g(x+n)\,g(x_0+n+1)} - 1\right|\end{aligned}\right\} \quad (20)$$

for every integer s greater than 1, r being a positive integer less than s. When $k=1$ and $m=0$ or 1, one may show, by the method employed in the latter part of the proof of Theorem I in the preceding paragraph, that a constant N_3 exists such that

$$\left|\sum_{n=s+1}^{\infty} c_n \frac{g(x+n)}{g(x)}\right| \leqq N_3 \sum_{n=s+1}^{\infty} \left|\frac{h(x+n)\,h(x_0+r)}{h(x+r)\,h(x_0+n)}\right| \cdot \frac{1}{n}, \tag{21}$$

provided that $r > 1/R(\sigma)$ and s is greater than r.

Now, let x_1 be a non-exceptional point for the series $\overline{\Omega}(x)$ such that $\lambda_1 < R(\sigma x_1) < R(\sigma x_0)$. Denote by (20_1) the relation obtained from (20) by replacing the last expression in absolute value signs by the sum of the absolute values of its terms. Then the series in (20_1) dominates that in (20). But if either k or m is greater than unity the series in (20_1) converges when x is replaced by x_1 and this resulting series dominates (20_1) itself, at least if $|x|$ is sufficiently large and x is in $\overline{V}$, as one sees through aid of the fact that the quantity in (12) is less than 1 in absolute value when $|x|$ is sufficiently large. If $k=1$ and $m=0$ or 1, the series in (21) converges when x is replaced by x_1, and as before, the resulting series dominates that in (21) itself, at least if $|x|$ is sufficiently large and x is in $\overline{V}$. Hence, in either case, there exists a convergent series of constant terms $\gamma_1+\gamma_2+\gamma_3+\ldots$, such that we have the term by term inequality

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* *Acta Mathematica,* Vol. XXXVII (1914), pp. 327–387.

$$\sum_{n=s+1}^{\infty}\left|c_n\frac{g(x+n)}{g(x)}\right|<\gamma_{s+1}+\gamma_{s+2}+\ldots, \tag{22}$$

provided that $s>1$ and $s>r+1$, at least if $|x|$ is sufficiently large and x is in $\overline{V}$.

Now, in Theorem III of *I*, we saw that $\overline{\Omega}(x)$ converges uniformly in any closed domain D which lies within its region of convergence and contains no points which are exceptional for this series, or are limit points of points which are exceptional for this series. From this result and relation (22), we conclude at once to the following theorem:

THEOREM III. *Let S be any region which lies in $\overline{V}$* and does not contain either in its interior or on its boundary a point which is exceptional for the series $\overline{\Omega}(x)$ or is a limit point of points which are exceptional for the series $\overline{\Omega}(x)$. The series $\overline{\Omega}(x)$ converges uniformly in S.*

This theorem for the special case of factorial series is due to Nörlund (*l. c.*). Nörlund's method of demonstration in the special case does not seem to be applicable for deriving the general theorem here obtained.

§ 3. *Extension of the Notion of Asymptotic Representation.*

In the complex x-plane let D be any region which extends to infinity, as for instance a sector bounded by two rays from zero to infinity. A function $f(x)$ is said to have in D the Poincaré asymptotic power series representation

$$f(x)\sim c_0+\frac{c_1}{x}+\frac{c_2}{x^2}+\ldots, \tag{23}$$

provided that

$$\lim x^s\left\{f(x)-\left(c_0+\frac{c_1}{x}+\ldots+\frac{c_s}{x^s}\right)\right\}=0, \qquad s=0,1,2,\ldots,$$

the limit in each case being taken for x approaching infinity in D. The power series in (23) may be either convergent or divergent.

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* For the definition of the region $\overline{V}$ see Theorem I.

In analogy with this definition and guided by the result contained in Theorem I, it is natural to introduce an extended notion of asymptotic representation. Accordingly, we shall say that a function $f(x)$ has in D the asymptotic Ω-representation

$$f(x) \sim \sum_{n=0}^{\infty} c_n g(x+n), \tag{24}$$

with respect to $g(x)$ provided that

$$\lim \{g(x+s)\}^{-1}\{f(x) - \sum_{n=0}^{s} c_n g(x+n)\} = 0, \qquad s=0, 1, 2, \ldots,$$

the limit in each case being taken for x approaching infinity in D. Similarly we shall say that a function $f(x)$ has in D the asymptotic $\bar{\Omega}$-representation

$$f(x) \sim \sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)}, \tag{25}$$

with respect to $g(x)$ provided that

$$\lim \frac{g(x)}{g(x+s)} \left\{ f(x) - \sum_{n=0}^{s} c_n \frac{g(x+n)}{g(x)} \right\} = 0, \qquad s=0, 1, 2, \ldots,$$

the limit being taken as before. For the purpose of these definitions it is not necessary that the series in (24) and (25) shall converge in D; in fact, they may be divergent for all values of x.

That the foregoing definition of asymptotic $\bar{\Omega}$-representation includes the asymptotic power series representation of Poincaré, as a special case may be seen by giving to $g(x)$ the value e^{-x^2}, and in the resulting asymptotic series (25) replacing e^{2x} by z; for the series in (25) then becomes a descending power series in z, and the factor $g(x)/g(x+s)$ is equal to the product of z^s by a quantity which is independent of z and different from zero. But this relation between the two types of asymptotic representation is relatively unimportant when compared with that which is brought to light in the next section.

It is clear that the asymptotic relations (24) and (25) are most delicate when k and m, the degrees of the polynomials $P(x)$ and $Q(x)$, are least. Consequently, it is natural to treat first the special case in which $k=1$ and $m=0$ or 1. It will indeed be seen that this case is of leading importance in the theory. Moreover, it is the case which is associated in the most valuable way with the notion of asymptotic representation from the point of view of Poincaré. Accordingly, we devote the next three sections to a treatment of this case.

§ 4. *On a Special Case of Asymptotic $\bar{\Omega}$-Representation in Relation to the Asymptotic Representation of Poincaré.*

In what follows we shall assume that the region D defined in § 3 lies in V.

In the special case when $g(x)$ has the value $g_1(x)$, where

$$g_1(x) \sim x^{\mu_0+\sigma x} e^{\alpha_0+\beta x}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots.\right), \tag{26}$$

the limits defining the asymptotic relation

$$f(x) \sim \sum_{n=0}^{\infty} c_n \frac{g_1(x+n)}{g_1(x)} \tag{27}$$

are equivalent to the limits

$$\lim x^{-s\sigma}\left\{f(x)-\sum_{n=0}^{s} c_n \frac{g_1(x+n)}{g_1(x)}\right\}=0, \qquad s=0,1,2,\ldots., \tag{28}$$

the limit in each case being taken for x approaching infinity in D. This follows readily from the fact that $g_1(x+s)x^{-s\sigma}/g(x)$ approaches a finite non-zero quantity as x approaches infinity in V.

We shall now obtain another set of limits equivalent to those in (28). For this purpose we observe that we have a Poincaré asymptotic representation of the form

$$\frac{g_1(x+n)}{g_1(x)} \sim x^{n\sigma}\left(\alpha_{0n}+\frac{\alpha_{1n}}{x}+\frac{\alpha_{2n}}{x^2}+\ldots.\right),$$

where $\alpha_{0n}, \alpha_{1n}, \ldots.$ are a set of numbers dependent only on n and the constants which enter into the asymptotic formula (26) for $g_1(x)$. Thence, we see that relations (28) are equivalent to the following:

$$\lim x^{-s\sigma}\left\{f(x)-\sum_{n=0}^{s} c_n x^{n\sigma}\left(\alpha_{0n}+\frac{\alpha_{1n}}{x}+\ldots.+\frac{\alpha_{n_s,n}}{x^{n_s}}\right)\right\}=0, \qquad s=0,1,2,\ldots. \tag{29}$$

Here the quantities n_s are integers dependent only on n and s. It is clear that a suitable value of n_s is any integer not less than $(s-n)(|\sigma_0|+|\sigma_1|)$. In particular, if $\sigma=-1$ we may take $n_s=s-n$; and this we do.

For the special case in which $\sigma=-1$ it is easy to see that relations (29) are equivalent to the Poincaré asymptotic formula

$$f(x) \sim \beta_0+\frac{\beta_1}{x}+\frac{\beta_2}{x^2}+\ldots., \tag{30}$$

where

$$\beta_\nu = c_0\alpha_{\nu 0}+c_1\alpha_{\nu-1,1}+c_2\alpha_{\nu-2,2}+\ldots.+c_\nu\alpha_{0\nu}.$$

Hence, relation (27) with $\sigma=-1$ gives rise to the Poincaré asymptotic representation (30). In the next section we shall show that these two relations are indeed equivalent, and shall exhibit the explicit formulae which put in evidence this equivalence. Thus we shall see that relation (27) contains (30) as a special case and shall make clear the importance of the former in view of the fundamental character of the latter.

Again, if we give to σ the rational value $-p/q$, where p and q are relatively prime positive integers, it is clear that relations (29) lead to that generalization of (30) in which x is replaced by a *q-th* root of x. In general there are necessary relations among the resulting quantities β; but if $p=1$ it is easy to see that they may have any values whatever when the quantities $c_0, c_1, c_2, \ldots.$ are arbitrary.

If we give to σ an irrational real value and derive the formula corresponding to (30), we shall find that the powers of x which enter into it are not only not of integral index, but also that these indices are not integral multiples of any number whatever. This brings out the fact that (27) affords an essential generalization of (30).

On account of the leading importance of the case in which $\sigma=-1$, this case alone will be treated in the remainder of the paper.

§ 5. *Explicit Formulae Connecting Relations* (27) *and* (30) *when* $\sigma=-1$.

In the special case when $\sigma=-1$ we shall denote $g_1(x)$ by $\bar{g}(x)$, so that we have

$$\bar{g}(x) \sim x^{\mu_0-x} e^{\alpha_0+\beta x}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots.\right). \tag{31}$$

We have seen that a relation of the form

$$\frac{\bar{g}(x+n)}{\bar{g}(x)} \sim x^{-n}\left(\beta_{0n}+\frac{\beta_{1n}}{x}+\frac{\beta_{2n}}{x^2}+\ldots.\right) \tag{32}$$

exists, where $\beta_{0n}, \beta_{1n}, \ldots.$ are a set of numbers independent of x. We shall now determine the values of these numbers β_{ij} in terms of $\mu_0, \alpha_0, \beta, a_1, a_2, \ldots..$

Setting

$$\bar{h}(x)=x^{\mu_0-x} e^{\alpha_0+\beta x},$$

we have

$$\frac{\bar{h}(x+n)}{\bar{h}(x)}=x^{-n} e^{(\beta-1)n}\left(1+\frac{\gamma_{1n}}{x}+\frac{\gamma_{2n}}{x^2}+\ldots.\right), \tag{33}$$

where the constants $\gamma_{1n}, \gamma_{2n}, \ldots.$ are readily determined consecutively by expanding the function in the second member of the relation

$$1+\frac{\gamma_{1n}}{x}+\frac{\gamma_{2n}}{x^2}+\ldots.=\left(1+\frac{n}{x}\right)^{\mu_0-n} e^{\frac{n^2}{2x}-\frac{n^3}{3x^2}+\frac{n^4}{4x^3}-\ldots..}$$

If we multiply (32) through by $\bar{g}(x)$, replace $\bar{g}(x+n)$ and $\bar{g}(x)$ by their asymptotic values from (31) and reduce the result by means of (33) we readily obtain the following relation:

$$e^{(\beta-1)n}\left(1+\frac{\gamma_{1n}}{x}+\frac{\gamma_{2n}}{x^2}+\ldots.\right)\left(1+\frac{a_1}{x+n}+\frac{a_2}{(x+n)^2}+\ldots.\right)$$
$$\sim\left(\beta_{0n}+\frac{\beta_{1n}}{x}+\ldots.\right)\left(1+\frac{a_1}{x}+\ldots.\right). \tag{34}$$

But

$$(x+n)^{-k}=x^{-k}\left(1+\frac{n}{x}\right)^{-k}$$
$$=x^{-k}\left(1-\frac{k}{1!}\cdot\frac{n}{x}+\frac{k(k+1)}{2!}\cdot\frac{n^2}{x^2}-\frac{k(k+1)(k+2)}{3!}\cdot\frac{n^3}{x^3}+\ldots.\right);$$

whence we have

$$1+\frac{a_1}{x+n}+\frac{a_2}{(x+n)^2}+\ldots.\sim 1+\frac{\bar{a}_1}{x}+\frac{\bar{a}_2}{x^2}+\ldots. \tag{35}$$

where

$$\bar{a}_k=a_k-(k-1)na_{k-1}$$
$$+\frac{(k-1)(k-2)}{2!}n^2a_{k-2}-\frac{(k-1)(k-2)(k-3)}{3!}n^3a_{k-3}+\ldots.. \tag{36}$$

Replacing the second series in (34) by its value from (35) and equating coefficients of like powers of x we have the following recurrence relations for determining $\beta_{0n}, \beta_{1n}, \ldots.$:

$$\left.\begin{array}{c}\beta_{0n}=e^{(\beta-1)n},\\ \beta_{1n}+a_1\beta_{0n}=e^{(\beta-1)n}(\bar{a}_1+\gamma_{1n}),\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ \sum\limits_{i=0}^{k}a_i\beta_{k-i,n}=e^{(\beta-1)n}\sum\limits_{i=0}^{k}\bar{a}_{k-i}\gamma_{in},\end{array}\right\} \tag{37}$$

where $a_0=c_0=\bar{a}_0=1$. It is evident that these relations uniquely determine the quantities $\beta_{0n}, \beta_{1n}, \ldots..$

From this it follows readily that the asymptotic $\bar{\Omega}$-representation

$$f(x)\sim\sum_{n=0}^{\infty}c_n\frac{\bar{g}(x+n)}{\bar{g}(x)} \tag{38}$$

leads to the Poincaré asymptotic representation

$$f(x)\sim\beta_0+\frac{\beta_1}{x}+\frac{\beta_2}{x^2}+\ldots., \tag{39}$$

where

$$\beta_\nu=c_0\beta_{\nu 0}+c_1\beta_{\nu-1,1}+c_2\beta_{\nu-2,2}+\ldots.+c_\nu\beta_{0\nu}. \tag{40}$$

On the other hand, if (39) is given then from (40) the constants $c_0, c_1, c_2, \ldots.$ may be uniquely determined (since $\beta_{0\nu}\neq 0$); and hence, conversely, relation (39) leads to relation (38).

This is sufficient to establish the complete equivalence between the Poincaré asymptotic relation (39) and the asymptotic $\overline{\Omega}$-representation (38). Later we shall point out an important advantage which the latter has over the former in that the series in (38) is convergent (in a half-plane) for a large class of functions $f(x)$ for which the corresponding series in (39) is everywhere divergent.

In an important class of cases the asymptotic character of the function $\rho(x)$,

$$\rho(x)=\frac{\bar{g}(x+1)}{\bar{g}(x)},$$

is known directly and has a simple form. (It can of course be computed in the general case.) In case this asymptotic series for $\rho(x)$ is convergent, and actually represents $\rho(x)$, one has readily the descending power series expansion of $\rho'(x)/\rho(x)$. It is easy to see that it has the form

$$\frac{\rho'(x)}{\rho(x)}=-\left(\frac{1}{x}+\frac{\mu_{20}}{x^2}+\frac{\mu_{30}}{x^3}+\ldots.\right), \tag{41}$$

where $\mu_{20}, \mu_{30}, \ldots.$ are constants depending on the constants in $\bar{g}(x)$. In the important special case in which $\bar{g}(x)=1/\Gamma(x)$ it is easy to see that (41) takes the simple form $\rho'(x)/\rho(x)=-1/x$.

The constants β_{in} may be determined readily in terms of the constants $\mu_{20}, \mu_{30}, \ldots.$; and the results thus obtained will sometimes be in more convenient form than those obtained by solving equations (37) for β_{in}. For effecting this determination let us write

$$y_n\equiv\frac{\bar{g}(x+n)}{\bar{g}(x)}=\rho(x)\rho(x+1)\ldots.\rho(x+n-1);$$

whence we have

$$\frac{y_n'}{y_n}=\frac{\rho'(x)}{\rho(x)}+\frac{\rho'(x+1)}{\rho(x+1)}+\ldots.+\frac{\rho'(x+n-1)}{\rho(x+n-1)}.$$

From (41) it is clear that we have an expansion of the form

$$\frac{\rho'(x+j)}{\rho(x+j)}=-\left(\frac{1}{x}+\frac{\mu_{2j}}{x^2}+\frac{\mu_{3j}}{x^3}+\ldots.\right).$$

Moreover, since

$$(x+j)^{-l}=x^{-l}\left(1-\frac{l}{1!}\cdot\frac{j}{x}+\frac{l(l+1)}{2!}\cdot\frac{j^2}{x^2}-\frac{l(l+1)(l+2)}{3!}\cdot\frac{j^3}{x^3}+\ldots.\right),$$

it is easy to see that μ_{ij} has the value

$$\mu_{ij}=\mu_{i0}-j\mu_{i-1,0}\frac{i-1}{1!}+j^2\mu_{i-2,0}\frac{(i-1)(i-2)}{2!}-\ldots. \\ +(-1)^{i-2}j^{i-2}\mu_{20}(i-1)+(-1)^{i-1}j^{i-1}. \tag{42}$$

Hence we have
$$\frac{y_n'}{y_n}=-\left(\frac{n}{x}+\frac{\varepsilon_{1n}}{x^2}+\frac{2\varepsilon_{2n}}{x^3}+\frac{3\varepsilon_{3n}}{x^4}+\dots\right),$$
where
$$i\varepsilon_{in}=\sum_{j=0}^{n-1}\mu_{i+1,j}. \tag{43}$$
Therefore,
$$\frac{\bar{g}(x+n)}{\bar{g}(x)}\equiv y_n=x^{-n}\beta_{0n}e^{\frac{\varepsilon_{1n}}{x}+\frac{\varepsilon_{2n}}{x^2}+\dots\dots}$$
By comparison with (32) we have
$$\frac{1}{\beta_{0n}}\left(\beta_{0n}+\frac{\beta_{1n}}{x}+\frac{\beta_{2n}}{x^2}+\dots\right)=1+\left(\frac{\varepsilon_{1n}}{x\,!}+\frac{\varepsilon_{2n}}{x^2}+\dots\right)$$
$$+\frac{1}{2\,!}\left(\frac{\varepsilon_{1n}}{x}+\frac{\varepsilon_{2n}}{x^2}+\dots\right)^2+\dots\dots$$
Thence we see that
$$\frac{\beta_{in}}{\beta_{0n}}=1+\varepsilon_{in}+\frac{1}{2\,!}\Sigma\varepsilon_{j_1n}\varepsilon_{j_2n}+\frac{1}{3\,!}\Sigma\varepsilon_{j_1n}\varepsilon_{j_2n}\varepsilon_{j_3n}+\dots, \tag{44}$$
where in each case the summation is to be taken for varying subscripts j such that their sum in each case shall be i.

Relations (42), (43), (44) serve to express β_{in} in terms of $\mu_{20}, \mu_{30}, \dots\dots$ Thence through (40) we have the relation existing between the constants in the asymptotic formulae (38) and (39).

In case $\rho(x)$ is a rational function the constants β_{in} may be determined in a very simple manner. For the sake of simplicity in the formulae we shall assume that the poles of $\rho(x)$ are of the first order, and that no two of its polar points differ in affixes by an integer. Denoting the poles of $\rho(x)$ by $p_1, p_2, \dots, p_h$ we have a partial fraction expansion of the form
$$y_n\equiv\rho(x)\rho(x+1)\dots\rho(x+n-1)=\sum_{l=1}^{h}\sum_{j=0}^{n-1}\frac{\eta_{lj}}{x-p_l+j}; \tag{45}$$
whence we see readily that
$$y_n=\sum_{l=1}^{h}\sum_{j=0}^{n-1}\eta_{lj}\left(\frac{1}{x}+\frac{p_l-j}{x^2}+\frac{(p_l-j)^2}{x^3}+\dots\right).$$
Therefore,
$$\beta_{in}=\sum_{l=1}^{h}\sum_{j=0}^{n-1}\eta_{lj}(p_l-j)^{n+i-1}. \tag{46}$$

As an example take the case in which $\rho(x)=1/x$. Then we have
$$\frac{1}{x(x+1)\dots(x+n-1)}=\frac{1}{x}-\frac{1}{1\,!}\frac{1}{x+1}$$
$$+\frac{1}{2\,!}\frac{1}{x+2}-\dots+(-1)^{n-1}\frac{1}{(n-1)\,!}\frac{1}{x-n+1},$$
and
$$\beta_{in}=(-1)^{n+i-1}\sum_{j=0}^{n-1}(-1)^j\frac{1}{j\,!}j^{n+i-1}. \tag{47}$$

That the two asymptotic formulae (38) and (39) are equivalent in the special case in which $\rho(x)=1/x$ has been shown by Nielsen,[*] who obtained formulae equivalent to (47) and (40) for the expression of this equivalence.

§ 6. *Illustrative Example.*

Let us consider the function $f(x)$ defined by the series

$$f(x)=\sum_{n=0}^{\infty} c_n \frac{\bar{g}(x+n)}{\bar{g}(x)}, \tag{48}$$

where the constants c_n are subject to the condition

$$|c_n| < M(n!)^{1-\epsilon},$$

where M and ϵ are positive constants. This series is term by term less in absolute value than the series

$$\sum_{n=0}^{\infty} M(n!)^{1-\epsilon}\left|\frac{\bar{g}(x+n)}{\bar{g}(x)}\right|.$$

That the latter series converges for every non-exceptional value of x is readily shown by the Cauchy ratio test. Hence the series in (48) converges for every non-exceptional value of x. Therefore, it defines a function $f(x)$ which is analytic at every point x which is non-exceptional for the series in (48). At the exceptional points for this series $f(x)$ will in general have singularities. In particular, if $\rho(x)$ is a rational function such that its poles are of the first order and no two of its polar points differ in affix by an integer, then $f(x)$ in general will have an infinite number of poles of the first order having infinity as their sole limit point. In such a case as this the power series in (39) diverges for every value of x. It yields the asymptotic character of $f(x)$, but does not yield anything else directly. On the other hand, the series in (48) also yields the asymptotic character of $f(x)$ and—what is much more—it affords a convergent representation of $f(x)$ at all points which are non-exceptional for the series in (48); and these latter points are in general points of singularity of $f(x)$. Moreover, the series furnishes in general a ready means for investigating the character of the singularities of $f(x)$.

This example illustrates an important result which I expect to present in full in a subsequent paper. It turns out that important classes of functions exist, each function of which has an essential singularity at infinity, but nevertheless admits a convergent $\bar{\Omega}$-expansion. Moreover, as one sees from the present paper, such an $\bar{\Omega}$-expansion exhibits directly those properties of the function which in many investigations one desires first of all to derive.

UNIVERSITY OF ILLINOIS, *April*, 1916.

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[*] Nielsen, *Annales Scientifiques de l'École Normale Supérieure*, Series 3, Vol. XXI (1904), pp. 449–458.

Possible Characteristic Operators of a Group.

BY G. A. MILLER.

§ 1. *Introduction.*

A characteristic operator of a group G is an operator which corresponds to itself in every possible automorphism of G. Every possible group contains at least one characteristic operator, viz., the identity. A necessary and sufficient condition that an abelian group involves another characteristic operator is that it contains one and only one operator of order 2. Hence we may confine our attention to non-abelian groups in the study of possible characteristic operators. In what follows we shall therefore assume that G is non-abelian.

The characteristic operators of G clearly constitute an abelian characteristic subgroup of G. The main object of the present article is to prove that it is possible to construct a non-abelian group which has for its characteristic operators all the operators of an arbitrary abelian group. By means of the fact that at least one such non-abelian group can be constructed, it is easy to prove that the number of such groups is always infinite. In fact, such an infinite system can be obtained directly by forming the direct products of one such group and simple groups of composite orders.

§ 2. *Group Containing a Characteristic Operator Whose Order is an Arbitrary Power of a Prime Number.*

Let p represent any prime number of the form $1+kq^{\alpha}$, α being an arbitrary positive integer. The transitive substitution group K of degree p and of order $p(p-1)$ contains a subgroup of order pq^{α} which involves no invariant operator except identity. To obtain the desired group in the form of a regular substitution group, we shall first construct an intransitive substitution group H by establishing a $(1, 1)$ correspondence between q^{α} regular cyclic groups of order pq^{β}, $\beta \leqq \alpha$. The q^{α} transitive constituents of H are conjugate under the powers of a substitution t of order q^{α} which is commutative with every substitution of H, q being a prime number.

Let s be a substitution of order q^{α} which is commutative with t and with the substitutions of order q^{β} contained in H, but transforms the substitutions of order p contained in H into a power which belongs to exponent q^{α} mod p. That is, the group generated by s and a substitution of order p contained in H is simply isomorphic with K, and we may assume that the group generated by s and H involves q^{α} transitive constituents on the letters of H. The product of st and a substitution of order q^{β} in one of the transitive constituents of H is of order $q^{\alpha+\beta}$, and the q^{α}*-th* power of this product is the substitution of order q^{β} contained in H.

The group G generated by H and this product has for its central the cyclic group of order q^{β}, and the central quotient group of G is simply isomorphic with K. Since G involves only one subgroup of order p, and since all the operators of G which transform one of the generating operators of this subgroup into its n*-th* power must also transform each of its other operators into the same power, it results that in every possible automorphism of G the operators which transform its operators of order p into a particular power must correspond to themselves. In particular, the operators of order $q^{\alpha+\beta}$ in G which transform each of its operators of order p into the same power must correspond to themselves in each of these automorphisms.

The operators of order $q^{\alpha+\beta}$ in G whose q^{α}*-th* power is the same operator of order q^{β} in H clearly transform the operators of order p in H into one or more powers, which are distinct from the powers into which these operators of order p are transformed by those operators of order $q^{\alpha+\beta}$ whose q^{α}*-th* power is another operator of order q^{β} in H. From this it results directly that the operators of order q^{β} contained in the central of G are characteristic operators of G. Since these operators generate the central of G, this central must be composed of characteristic operators of G.

Since q is an arbitrary prime number, and β is an arbitrary positive integer, we have established the lemma that *it is possible to construct a non-abelian group which has an operator of any desired prime power order as a characteristic operator.* In fact, the existence of such a characteristic operator of order q^{β} is independent of p and α provided they satisfy the stated conditions, and hence it results from the preceding arguments that any one of a multiply infinite number of different groups may be used for G.

§ 3. *General Case.*

To prove that it is possible to construct a group G which has all the operators of an arbitrary abelian group G' as characteristic operators, it is only necessary to prove that a set of independent generators of G' are characteristic operators of a certain group G. We shall first consider the case when the order of each of the independent generators of G' is a power of q, and we shall let q^β represent one of the largest of these orders.

Let $s_1, s_2, \ldots, s_r$ represent a set of independent generators of G' and find r distinct prime numbers $p_1, p_2, \ldots, p_r$ of the form $1+kq^\beta$. According to the preceding section we can construct r distinct groups $G_1, G_2, \ldots, G_r$ such that the characteristic operators of these groups are generated by $s_1, s_2, \ldots, s_r$, respectively, and that the prime numbers of the form $1+kq^\beta$ which divide their orders are $p_1, p_2, \ldots, p_r$, respectively. The orders of the r groups $G_1, G_2, \ldots, G_r$ may be assumed to be

$$p_1q^\beta,\ p_2q^\beta,\ \ldots,\ p_rq^\beta$$

multiplied by the orders of $s_1, s_2, \ldots, s_r$, respectively.

The direct product of $G_1, G_2, \ldots, G_r$ contains a characteristic cyclic subgroup of order $p_1, p_2, \ldots, p_r$. In any automorphism of this direct product G_1 corresponds to a subgroup whose operators are commutative with each of the operators in the cyclic subgroup of order $p_2, p_3, \ldots, p_r$ contained in this direct product. To the operators of order $q^{\alpha'+\beta}$ in G_1, α' being the order of s_1, there must therefore correspond operators whose q^β power is the same as the q^β power of the corresponding operators of G_1. Hence s_1 is a characteristic operator of this direct product. As s_1 is an arbitrary independent generator of G', the direct product under consideration has all the operators of G' for its characteristic operators.

When the order of G' is not a power of a prime number, G' is the direct product of characteristic subgroups whose orders are such powers. By constructing groups whose characteristic operators constitute these characteristic subgroups, and then forming the direct product of these groups, we obtain a group whose characteristic operators are the operators of G'. This completes a proof of the theorem

It is possible to construct non-abelian groups whose characteristic operators constitute an arbitrary abelian group.

Linear Differential Equations in Infinitely Many Variables.[*]

BY WILLIAM L. HART.

§ 1. *Introduction.*

In the study of finite systems of ordinary differential equations, linear systems

$$\frac{dx_i}{dt}=\sum_{j=1}^{n} k_{ij}(t)x_j, \qquad (i=1, 2, \ldots, n), \tag{1}$$

are found to have many interesting properties, a number of which are connected with the notion of fundamental sets of solutions of (1). In the present paper the infinite system of ordinary differential equations

$$\frac{dx_i}{dt}=\sum_{j=1}^{\infty} k_{ij}(t)x_j, \qquad (i=1, 2, \ldots), \tag{2}$$

is considered and certain analogous properties are discussed where the notion of fundamental sets of solutions for (2) is defined in an appropriate manner.

Throughout the discussion it is assumed that the $k_{ij}(t)$ in (2) are power series in the complex variable t converging for $|t|\leqq r$. The matrix of elements $(k_{ij}(t))_{i,j=1,2,\ldots}$ is supposed to be of the limited type, in the sense defined by Hilbert. The set of complex numbers $\xi=(x_1, x_2, \ldots)$ is supposed to represent a point in Hilbert complex space; *i. e.*, if $\bar{x}_i$ represents the conjugate to x_i, then $\Sigma_{i=1}^{\infty} \bar{x}_i x_i$ converges.

In § 2 the system of notation used in the paper is explained. There is then given a brief outline of the results from the theory of limited linear and bi-linear forms in infinitely many variables which are used in the later work. In § 3 the homogeneous system (2) is considered. Under certain assumptions as to the limited character of the matrix $(k_{ij}(t))$ the existence of an analytic solution $(x_1(t), x_2(t), \ldots)$ taking on, for $t=t_0$, a given set of values $(a_1, a_2, \ldots)$, is established and an analytical representation for it obtained in terms of an auxiliary matrix.

In § 4 the adjoint system to the equations (2),

$$\frac{dx_i}{dt}=-\sum_{j=1}^{\infty} k_{ji}(t)x_j, \tag{3}$$

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[*] Presented to the American Mathematical Society, December 27, 1916.

is considered. As a result of the conditions satisfied by system (2), the system (3) is of the same type as (2). It is found that, if $\xi_1(t)=(x_{11}(t), x_{21}(t), \ldots .)$ and $\xi_2(t)=(x_{12}(t), x_{22}(t), \ldots .)$ are solutions of (2) and (3), respectively, then $\Sigma_{i=1}^{\infty} x_{i1}(t) x_{i2}(t)$ is a constant.

In § 5 the notion of fundamental sets of solutions of system (2) is defined in a manner suggested by that followed in the finite case if, in the latter, where there enters the idea of the non-vanishing of the determinant of the fundamental set, there is put the concept of a limited matrix possessing a unique limited reciprocal matrix. The existence of fundamental sets is established and certain of their properties are discussed. In this connection the adjoint system (3) proves to be useful because the reciprocal matrix of a fundamental set of solutions of (2) is a matrix of solutions of the system (3).

In § 6 the non-homogeneous system is considered which results from (2) on adding to the *i-th* equation the term $k_{i0}(t)$. The general solution of the modified system is obtained as the sum of a particular solution and the general solution of the system (2). On using a method analogous to that of variation of parameters, a particular solution is obtained by a quadrature with the aid of a fundamental set of solutions of (2).

Infinite systems of differential equations have been considered by H. Von Koch,* F. R. Moulton,† E. H. Moore,‡ J. F. Ritt,§ T. H. Hildebrandt,‖ and the author.¶ The general systems of Von Koch, Moulton and the author do not include the system (2) of the present paper. J. F. Ritt considered a single differential equation of infinite order which is a problem of a different type. Moore and Hildebrandt considered problems in general analysis including as special instances a certain infinite system of the type (2), in the field of reals. The results of Moore were not of the nature of those obtained in the present paper. The results of Hildebrandt generalize the theorems on finite systems in such a way that the notion of the determinant of the fundamental sets of solutions is retained, whereas in the present paper the matrices of the fundamental sets need not possess determinants.

§ 2. *Limited Bi-Linear Forms.*

In the work of the present paper it will be convenient to adopt the system of notation which is explained in the following paragraphs. The same type of notation has long been used by E. H. Moore [*] in his work on Integral Equations. In its general form the notation is like that met in vector and matrix algebra.

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* *Öfversigt af Kongliga Vetenskaps Akademiens Förhandlingar*, Vol. LVI (1899), pp. 395–411.

† *Proceedings of the National Academy of Sciences*, Vol. I (1915), p. 350.

‡ *Atti dei IV Congresso Internazionale dei Matematici*, Vol. II (Roma, 1908), p. 98.

§ *Transactions of the American Mathematical Society*, Vol. XVIII (1917), p. 27.

‖ *Ibid.*, Vol. XVIII (1917), p. 73.

¶ *Ibid.*, Vol. XVIII (1917), p. 125.

An infinite set of values $(x_1, x_2, \ldots.)$ will be represented by a single Greek letter, for example ξ. Then, the point, or vector, ξ in the space of infinitely many dimensions is considered as a function of the variable i whose range is $I=(1, 2, \ldots.)$; that is, $\xi(i)=x_i$. Points $(x_1, x_2, \ldots.)$ will be represented either by ξ, η, γ or α, with perhaps an added notation.

An infinite matrix $(k_{ij})_{i,j=1,2,\ldots.}$ in which i denotes the row, j the column, will be represented by a single Greek letter such as $\varkappa$. Then $\varkappa$ is considered as a function of the two indices i and j for which $\varkappa(i, j)=k_{ij}$. Any Greek letter except ξ, η, γ or α will in the future represent a matrix.

A Greek letter with an added notation represents the point or matrix obtained by adding the given notation to each element of the point or matrix given by the unannotated Greek letter. For example,

$$\xi'=(x'_1, x'_2, \ldots.), \qquad \xi(t)=\{x_1(t), x_2(t), \ldots.\},$$
$$\varkappa(t)=(k_{ij}(t))_{i,j=1,2,\ldots.}, \qquad \varkappa_n=(k_{ijn})_{i,j=1,2,\ldots.},$$
$$\frac{d\xi}{dt}=\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots.\right), \qquad \frac{d\varkappa}{dt}=\left(\frac{dk_{ij}}{dt}\right)_{i,j=1,2,\ldots.}.$$

In the subsequent work infinite series enter to a great extent. On using what is seen to be analogous to vector notation, let there be introduced the abbreviation

$$\sum_{i=1}^{\infty} x_i y_i = S\xi\eta, \qquad [\xi=(x_1, x_2, \ldots.),\ \eta=(y_1, y_2, \ldots.)].$$

The S is a summation operation giving what may be called the scalar product of the two points or vectors ξ and η. In similar fashion, the point η whose coördinates y_i are given by $y_i=\Sigma_{j=1}^{\infty} k_{ij}x_j$ and the point α whose coordinates z_i are given by $z_i=\Sigma_{j=1}^{\infty} x_j k_{ji}$, will be represented by, respectively,

$$\eta=S\varkappa\xi, \quad \alpha=S\xi\varkappa.$$

The operation S thus produces a point from a matrix and a point. Let $\varkappa'$ and $\varkappa''$ be two matrices; the product matrix λ is defined by

$$\lambda=(\sum_{g=1}^{\infty} k'_{ig}k''_{gj})_{i,j=1,2,\ldots.}=S\varkappa'\varkappa'',$$

or, S produces a matrix from two given matrices. The meaning of such expressions as $S\xi S\varkappa\eta$, $S\varkappa S\varkappa'\varkappa''$, $S(S\varkappa\xi)\eta$, is then evident. The expressions $S^2\xi\varkappa\eta$, $S^2\varkappa'\varkappa\xi$ are defined by, respectively,

$$S^2\xi\varkappa\eta = \sum_{i,j=1}^{\infty} x_i k_{ij} y_j = \lim_{n=\infty} \sum_{i,j=1}^{n} x_i k_{ij} y_j,$$
$$S^2\varkappa'\varkappa\xi = (\sum_{i,j=1}^{\infty} k'_{1i}k_{ij}x_j, \sum_{i,j=1}^{\infty} k'_{2i}k_{ij}x_j, \ldots.).$$

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[*] *Proceedings of the Fifth International Congress* (Cambridge), Vol. I, p. 230.

In the following discussion, "$\Sigma_{i,j=1}^{\infty}$" will always mean the limit indicated in the first of the expressions above.

It will sometimes be convenient to use the expressions considered in the preceeding paragraph when the sums involved run only from 1 to m. Then write S_m instead of S:

$$S_m\xi\eta = \sum_{i=1}^{m} x_i y_i, \quad S_m x\xi = \left(\sum_{i=1}^{m} k_{1i}x_i, \sum_{i=1}^{m} k_{2i}x_i, \ldots\right), \text{ etc.}$$

A stroke over a letter (*e. g.*, $\bar{x}$, $\bar{\xi}$) will always represent the conjugate complex number to the quantity bearing the notation.

DEFINITION 1. *A point ξ belongs to Hilbert complex space in case there exists*

$$S\bar{\xi}\xi = \sum_{i=1}^{\infty} \bar{x}_i x_i, \qquad [\xi = (x_1, x_2, \ldots)].$$

The number $\sqrt{S\bar{\xi}\xi}$ will be called the modulus of ξ and will be denoted by $M\xi$.

The proofs of certain of the properties of limited linear and bi-linear forms in infinitely many variables listed below are found in a paper* by E. Hellinger and O. Toeplitz. In the future the notation "*H. T. page* ——" refers to a page in this article. Hellinger and Toeplitz in most cases give the detailed proofs only for the case of real values of the variables, but as they remark (H. T. page 305), the extension to complex values is a simple matter.

PROPERTY 1 (*H. T. page* 295). *Suppose there exists a number l such that, for all points $\eta = (y_1, y_2, \ldots)$ of Hilbert space there holds, for every m,*

$$|S_m\xi\eta| \leqq lM\eta.$$

Then it follows that ξ belongs to Hilbert space and that $M\xi \leqq l$.

PROPERTY 2. *Let $\xi(u)$ and $\eta(u)$, where u is a complex variable belonging to a region R in the u-plane, belong to Hilbert space for every u. Suppose that*

$$S\bar{\xi}(u)\xi(u) \leqq l$$

for all u, while $S\bar{\eta}(u)\eta(u)$ converges uniformly for all u. Then

$$S\xi(u)\eta(u) = \sum_{i=1}^{\infty} x_i(u)y_i(u)$$

converges absolutely uniformly for all u of R.

This property is established with the aid of the Schwarz† inequality. For,

$$\sum_{i=n+1}^{\infty} |x_i(u)| \cdot |y_i(u)| \leqq \sqrt{\sum_{i=n+1}^{\infty} \bar{x}_i(u)x_i(u)}\sqrt{\sum_{i=n+1}^{\infty} \bar{y}_i(u)y_i(u)} \leqq l\sqrt{\sum_{i=n+1}^{\infty} \bar{y}_i(u)y_i(u)},$$

which approaches zero uniformly for all u as $n \doteq \infty$.

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* *Mathematische Annalen*, Vol. LXIX (1910), p. 289.

† H. T. page 293.

DEFINITION 2. *A matrix* $\varkappa = (k_{ij})_{i,j=1,2,\ldots}$ *is limited* l *in case for every pair of Hilbert points* ξ *and* η *and for every* m,

$$|S_m S_m \xi \varkappa \eta| = \left|\sum_{i=1}^{m} \sum_{j=1}^{m} x_i k_{ij} y_j\right| \leq l M\xi M\eta.$$

If $\varkappa$ is an infinite matrix, the conjugate matrix $\bar{\varkappa}$ and the transposed matrix $\varkappa'$ are defined by

$$\bar{\varkappa} = (\bar{k}_{ij})_{i,j=1,2,\ldots}, \quad \varkappa' = (k_{ji})_{i,j=1,2,\ldots}.$$

PROPERTY 3 (*H. T. page* 310). *Suppose that* $\varkappa$ *is limited* l. *Then* $\varkappa'$ *and* $\bar{\varkappa}$ *are also limited* l.

PROPERTY 4 (*H. T. page* 297). *Let* $\varkappa$ *be limited* l. *Then each row and each column of* $\varkappa$ *constitutes a point, belonging to Hilbert space, whose modulus is at most* l.

PROPERTY 5 (*H. T. page* 299). *If* $\varkappa$ *is limited* l *then, for every pair of Hilbert points* ξ *and* η, *the following infinite sums are defined and equal:*

$$S\xi S\varkappa\eta, \quad S(S\xi\varkappa)\eta, \quad S^2\xi\varkappa\eta.$$

PROPERTY 6 (*H. T. page* 300). *Let* $\varkappa'$ *and* $\varkappa''$ *be limited* l' *and* l'', *respectively. Then the product matrix* $\lambda = S\varkappa'\varkappa''$ *is limited* $l'l''$.

DEFINITION 3. *Let* $\varkappa$ *and* λ *be two limited matrices. The matrix* λ *is a right-hand, or a left-hand reciprocal of* $\varkappa$ *according as*

$$S\varkappa\lambda = \delta \quad or \quad S\lambda\varkappa = \delta,$$

where δ *is the unit matrix* $(d_{ij})_{i,j=1,2,\ldots}$, $d_{ii} = 1$, $d_{ij} = 0\,(i \neq j)$.

PROPERTY 7 (*H. T. page* 311). *If the matrix* $\varkappa$ *is limited and possesses both right and left-hand reciprocals, it follows that there exists uniquely a right-hand and a left-hand reciprocal, and moreover, they are equal to each other.*

PROPERTY 8 (*H. T. page* 312). *Let* $\varkappa$ *be a limited matrix and suppose that there exists a unique right (left) hand reciprocal* λ. *Then* λ *is also the unique left (right) hand reciprocal of* $\varkappa$.

In case $\varkappa$ possesses a unique left-hand and right-hand reciprocal λ, this matrix will be called simply the reciprocal of $\varkappa$. By direct verification it is established that there holds.

PROPERTY 9. *Suppose that* $\varkappa$ *is limited and possesses a reciprocal* λ. *Then for every two points* ξ *and* η *of Hilbert space, the two infinite systems of linear equations*

$$\xi = S\varkappa\eta, \quad \eta = S\lambda\xi$$

are equivalent.

§ 3. *The Homogeneous System.*

The subject for consideration is the system of equations (2) which in the notation of § 2 becomes

$$\frac{d\xi}{dt} = S\varkappa(t)\xi, \tag{4}$$

where $\varkappa(t) = (k_{ij}(t))_{i,j=1,2,\ldots}$ is a matrix each of whose elements is a power series in the complex variable t converging for $|t| \leqq r$. Throughout the discussion it will be assumed that $\varkappa(t)$ satisfies the hypothesis

(H_1). *The coefficients* $\varkappa_n = (k_{ijn})_{i,j=1,2,\ldots}$ *in the matrix equation*

$$\varkappa(t) = \sum_{n=0}^{\infty} \varkappa_n t^n,$$

are such that $\varkappa_n$ *is limited* l_n, *and moreover,*

$$\sum_{n=0}^{\infty} l_n r^n$$

converges. Therefore there exists a number N *such that* $l_n \leqq N r^n$.

PROPOSITION 1. *The matrix* $\varkappa(t)$ *is limited for* $|t| \leqq r$ *and its limit may be taken as the quantity* $l(t)$ *defined by*

$$l(t) = \sum_{n=0}^{\infty} l_n |t|^n.$$

To establish this result, recall the definition of a limited matrix. Let ξ and η be two points of Hilbert space. Then it follows that

$$\begin{aligned} |S_m S_m \xi \varkappa(t) \eta| &= |S_m S_m \xi (\sum_{n=0}^{\infty} \varkappa_n t^n) \eta| = |\sum_{n=0}^{\infty} t^n (S_m S_m \xi \varkappa_n \eta|, \\ &\leqq M\xi M\eta (\sum_{n=0}^{\infty} l_n |t|^n) = l(t)\,(M\xi M\eta). \end{aligned} \tag{5}$$

It is seen that (5) states the limited character of $\varkappa(t)$ and that $l(t)$ may be taken as the limit.

PROPOSITION 2. *For every* i *the series*

$$\sum_{g=1}^{\infty} \overline{k}_{ig}(t) k_{ig}(t) \tag{6}$$

converges uniformly for all $|t| \leqq r$. *Similarly, for every* g *the same condition is satisfied by the series*

$$\sum_{i=1}^{\infty} \overline{k}_{ig}(t) k_{ig}(t). \tag{7}$$

From the fact that $k_{ig}(t) = \Sigma_{n=0}^{\infty} k_{ign} t^n$ it is seen that

$$\sum_{g=m_1}^{m_2} \overline{k}_{ig}(t) k_{ig}(t) = \sum_{n,h=0}^{\infty} \bar{t}^n t^h (\sum_{g=m_1}^{m_2} \overline{k}_{ign} k_{igh}) \ll \sum_{n,h=0}^{\infty} l_h l_n \bar{t}^n t^h, \tag{8}$$

where the dominance relation is a consequence of Property (4) in view of the assumption of (H_1) as to the matrices $\varkappa_n$. As a result of (8) it may be stated that for every assigned number $l>0$ an integer n_l can be determined so that for all m_1 and m_2 and for $|t|\leqq r$,

$$\left|\sum_{n,\,h=n_l}^{\infty} \bar{t}^n t^h \left(\sum_{g=m_1}^{m_2} \bar{k}_{ign} k_{igh}\right)\right| \leqq \frac{l}{2}. \tag{9}$$

Because of the convergence for all n and h of the series $\Sigma_{g=1}^{\infty}\bar{k}_{ign}k_{igh}$, an integer m_l can be chosen so large that, for $m_1\geqq m_l$, $m_2\geqq m_l$, and for $|t|\leqq r$,

$$\left|\sum_{n,\,h=0}^{n_l} \bar{t}^n t^h \left(\sum_{g=m_1}^{m_2} \bar{k}_{ign} k_{igh}\right)\right| \leqq \frac{l}{2}. \tag{10}$$

From (9) and (10) it follows that the series in (8) is in absolute value at most l for $|t|\leqq r$, which establishes the proposition.

PROPOSITION 3. *Let t_0 be a point in the region $|t|\leqq r$. If $\lambda(t-t_0)$ represents the matrix obtained from $\varkappa(t)$ by expanding each element of $\varkappa(t)$ as a series in $(t-t_0)$, then the coefficient matrices λ_n in*

$$\lambda(s)=\sum_{n=0}^{\infty}\lambda_n s^n, \qquad (s=t-t_0),$$

are limited by numbers l'_n, and $\Sigma_{n=0}^{\infty} l'_n r_0^n$ converges, where r_0 is any value satisfying $r_0<r-|t_0|$.

In other words, Proposition (3) states that the analytic continuation of $\varkappa(t)$ to the point t_0 satisfies, for $|s|\leqq r_0$, conditions which are precisely similar to those satisfied by $\varkappa(t)$ for $|t|\leqq r$.

Let the notation for the analytic continuations of the elements of $\varkappa(t)$ be

$$k_{ij}(t)=\sum_{n=0}^{\infty} h_{ijn}s^n,$$

so that $\lambda(s)=\Sigma_{n=0}^{\infty}\lambda_n s^n$, where $\lambda_n=(h_{ijn})_{i,j=1,2,\ldots}$. In order to show that the λ_n are limited and that the limits satisfy the convergence condition of the proposition, recall Definition 2. Let ξ and η be two points of Hilbert space. Then

$$S_m S_m \xi\varkappa(t)\eta=\sum_{n=0}^{\infty} C_n t^n, \qquad (C_n=S_m S_n \xi\varkappa_n\eta). \tag{11}$$

From (H_1) it follows that

$$|C_n|\leqq l_n M\xi M\eta,$$

and, hence,

$$\sum_{n=0}^{\infty} C_n t^n \ll (M\xi M\eta)\sum_{n=0}^{\infty} l_n t^n, \tag{12}$$

where the right side converges for $|t|\leqq r$. Consequently, if l'_n represents the coefficient of s^n in the series formed by continuing the series on the right side of (12) to the point $t=|t_0|$, the analytic continuation of the series on the left to the point $t=t_0$ satisfies

$$\sum_{n=0}^{\infty} C_n t^n = \sum_{n=0}^{\infty} B_n s^n \ll \Big(\sum_{n=0}^{\infty} l'_n s^n\Big) M\xi M\eta, \qquad (|s|<r-|t_0|). \tag{13}$$

But, $B_n=S_m S_m \xi\lambda_n\eta$, and, hence, it has been established that

$$|S_m S_m \xi \lambda_n \eta| \leqq l'_n M\xi M\eta.$$

From (13) it is seen that $\Sigma_{n=0}^{\infty} l'_n r_0^n$ converges if $r_0<r-|t_0|$.

As a consequence of Proposition (3) the theorems stated below, where the hypotheses refer to the initial point $t=0$, apply to the case $t=t_0$, if r is changed to r_0 and l_n to l'_n.

§ 4. *The Existence Theorem for the System* (4).

The fundamental theorem concerning the solution of (4) involves functions $\xi(t)$ of a type which it will be convenient to represent by the class $\mathfrak{M}$.

DEFINITION (4). *A function $\xi(t)$ belongs to the class $\mathfrak{M}$ in case there exists a $w\,(0<w\leqq r)$ so that, for $|t|\leqq w$, $\xi(t)$ belongs to Hilbert space, and if the coordinates $x_i(t)$ of $\xi(t)$ are convergent power series, $x_i(t)=\Sigma_{n=0}^{\infty} x_{in} t^n$, or*

$$\xi(t)=\sum_{n=0}^{\infty} \xi_n t^n, \qquad [\xi_n=(x_{1n}, x_{2n}, \ldots\ldots)]. \tag{14}$$

Moreover, it is possible to interchange infinite summations in the following expression and to write

$$Sx(t)\xi(t)=S\Big(\sum_{n=0}^{\infty} x_n t^n\Big)\Big(\sum_{m=0}^{\infty} \xi_m t^m\Big)=\sum_{n,\,m}^{0,\,\infty} t^{n+m} S x_n \xi_m. \tag{15}$$

A function $\xi(t)$ will be said to belong to the class $\mathfrak{M}$ at t_1 if it satisfies conditions corresponding to those just enumerated with $x(t)$ replaced by its analytic continuation to the point t_1.

PROPOSITION 4. *Suppose that $\xi(t)$ is a function satisfying* (14) *and that, instead of* (15), *there holds*

$$M\xi(t)\leqq K<\infty, \qquad (|t|\leqq w). \tag{16}$$

Then it follows that $\xi(t)$ satisfies the condition (15).

Let the proposition be established with the aid of Property (2). Identify the $\xi(u)$ and $\eta(u)$ of that result with $\xi(t)$ and the point $(k_{i1}(t), k_{i2}(t), \ldots\ldots)$, respectively. In view of Proposition (2) and since $M\xi(t)\leqq K$, it follows that, as a consequence of Property (2), the series

$$\sum_{j=1}^{\infty} k_{ij}(t) x_j(t) = \sum_{j=1}^{\infty} \Big(\sum_{n=0}^{\infty} k_{ijn} t^n\Big)\Big(\sum_{m=0}^{\infty} x_{jm} t^m\Big)$$

converges uniformly. Therefore it may be rearranged as a power series in t; in other words $\xi(t)$ satisfies (15).

The existence of a solution of (4) taking on a given set of values for $t=0$ is established in

THEOREM I. *Let the system* (4) *satisfy* (H_1) *and suppose that* ξ_0 *is a point in Hilbert space. Then, among all functions of the class* $\mathfrak{M}$ *there exists uniquely for* $|t|<r$, *a function* $\xi(t)$ *which is a solution of* (4) *satisfying* $\xi(0)=\xi_0$. *Furthermore, the coefficients* ξ_n *in the expansion of* $\xi(t)$,

$$\xi(t)=\sum_{n=0}^{\infty}\xi_n t^n,$$

are given by $\xi_n=S\lambda_n\xi_0$ *where the matrices* λ_n *are the coefficients in the expansion of a certain matrix* $\lambda(t)=\Sigma_{n=0}^{\infty}\lambda_n t^n$ *satisfying the following conditions for* $|t|<r$:

The matrices λ_n *are limited* b_n *and the series*

$$l'(t)=\sum_{n=0}^{\infty}b_n|t|^n \tag{17}$$

converges for $|t|<r$.

Moreover, $\xi(t)$ *can be written in the form*

$$\xi(t)=S\lambda(t)\xi_0,$$

and the series giving the respective components $x_i(t)$ *of* $\xi(t)$ *converge in* t, *uniformly with respect to the index* i.

The theorem will be established by the method of undetermined coefficients. Suppose there exists a solution $\xi(t)$ of (4) belonging to the class $\mathfrak{M}$ and satisfying $\xi(0)=\xi_0$. Then it follows that

$$\xi(t)=\sum_{n=0}^{\infty}\xi_n t^n,\qquad \frac{d\xi(t)}{dt}=\sum_{n=1}^{\infty}n\xi_n t^{n-1}.$$

Hence there is obtained

$$\begin{aligned}\sum_{n=1}^{\infty}n\xi_n t^{n-1}&=S\varkappa(t)\Big(\sum_{n=0}^{\infty}\xi_n t^n\Big)=S\Big(\sum_{n=0}^{\infty}\varkappa_n t^n\Big)\Big(\sum_{m=0}^{\infty}\xi_m t^m\Big)\\&=\sum_{n,\,m=0}^{\infty}(S\varkappa_n\xi_m)t^{n+m}=\sum_{g=0}^{\infty}C_g t^g,\qquad \Big(C_g=\sum_{\substack{n,\,m=0\\ n+m=g}}^{g}S\varkappa_n\xi_m\Big),\end{aligned}\tag{18}$$

where the interchange of S and the infinite summations with respect to m and n is seen to be legitimate because $\xi(t)$ belongs to the class $\mathfrak{M}$. The equation (18) serves to determine uniquely the coefficients ξ_n; there results

$$\xi_0=\xi_0,\quad n\xi_n=C_{n-1},\qquad (n=1,2,\ldots),\tag{19}$$

which gives

$$\left.\begin{aligned}
&\xi_1 = S\varkappa_0\xi_0 = S\lambda_1\xi_0, && (\lambda_1 = \varkappa_0),\\
&2\xi_2 = S\varkappa_1\xi_0 + S^2\varkappa_0\lambda_1\xi_0 = 2S\lambda_2\xi_0, && \left(\lambda_2 = \frac{\varkappa_1 + S\varkappa_0\lambda_1}{2}\right),\\
&\dots\dots\dots\dots\dots\dots\dots\dots, && \dots\dots\dots\dots\dots,\\
&n\xi_n = S\varkappa_{n-1}\xi_0 + \dots\dots \\
&\qquad + S\varkappa_0\xi_{n-1} = nS\lambda_n\xi_0, && \left(\lambda_n = \frac{\varkappa_{n-1} + S\varkappa_{n-2}\lambda_1 + \dots\dots + S\varkappa_0\lambda_{n-1}}{n}\right).
\end{aligned}\right\} \quad (20)$$

In order to prove the theorem it will first be shown that the series giving the components $x_i(t)$ of the function $\xi(t) = \Sigma_{n=0}^{\infty}\xi_n t^n$, resulting from the coefficients ξ_n found in (20), converge absolutely for $|t| < r$ and uniformly with respect to the index i.

Recall that the matrix $\varkappa_n$ is limited $l_n \leqq N/r^n$; in the work below it will be assumed that the limit is equal to N/r^n. Then, because of Definition 2, it follows that

$$|\xi_1| \leqq NM\xi_0 = b_1 M\xi_0, \qquad (b_1 = N).$$

From Property (6) of limited matrices it is seen that λ_2 is limited b_2,

$$b_2 = \frac{N}{2}\left(N + \frac{1}{r}\right),$$

and hence $|\xi_2| \leqq b_2 M\xi_0$. On proceeding in this manner it can be established that every matrix λ_n is limited b_n, where

$$b_n = \frac{1}{n}\left(\frac{N}{r^{n-1}} + b_1\frac{N}{r^{n-2}} + \dots\dots + b_{n-1}N\right),$$

$$b_{n+1} = \frac{1}{n+1}\left(\frac{N}{r^n} + b_1\frac{N}{r^{n-1}} + \dots\dots + b_{n-1}\frac{N}{r} + b_n N\right).$$

It follows that there is the relation

$$b_{n+1} = \frac{b_n}{n+1}\left(\frac{n}{r} + N\right),$$

and that $|\xi_n| \leqq b_n M\xi_0$.

LEMMA 1. *The series $\Sigma_{n=0}^{\infty} b_n t^n$ converges for $|t| < r$.*

To see this consider the absolute value q_n of the ratio of the *n-th* to the $(n-1)$-*th* term:

$$q_n = \frac{b_n}{b_{n-1}}|t| = \left(\frac{n-1}{n}\right)\left(\frac{N_r}{n-1} + 1\right)\left|\frac{t}{r}\right|.$$

Since for $|t| < r$ $\lim_{n=\infty} q_n < 1$, the lemma is established. Moreover, since

$$\xi(t) = \sum_{n=0}^{\infty}\xi_n t^n = \sum_{n=0}^{\infty}(S\lambda_n\xi_0)t^n \ll \xi_0 + M\xi_0\left(\sum_{n=1}^{\infty} b_n t^n\right), \qquad (\lambda_0 = 1), \qquad (21)$$

it follows that the series for each component $x_i(t)$ of $\xi(t)$ converges just as fast and in the same region as the series in Lemma 1.

Now define a matrix $\lambda(t)$ formally by the equation

$$\lambda(t)=\sum_{n=0}^{\infty}\lambda_n t^n, \qquad \{\lambda(t)=(m_{ij}(t))_{i,j=1,2,\ldots}\}. \tag{22}$$

The series in (22) (one for each component of $\lambda(t)$) converge, because of Lemma (1), since $|\lambda_n|\leqq b_n$. The matrix $\lambda(t)$ therefore satisfies conditions corresponding to those imposed on $\varkappa(t)$ in hypothesis (H_1). Hence $\lambda(t)$ is limited $l'(t)$ where $l'(t)$ has the definition (17). From the fact that $\lambda(t)$ is limited uniformly for $|t|\leqq r_1<r$, it follows that $S\lambda(t)\xi_0$ converges uniformly and, therefore,

$$S\lambda(t)\xi_0=S(\sum_{n=0}^{\infty}\lambda_n t^n)\xi_0=\sum_{n=0}^{\infty}(S\lambda_n\xi_0)t^n=\xi(t).$$

To complete the proof of the theorem it remains to show that $\xi(t)$ satisfies (15). First consider

COROLLARY (1). *The modulus of the function $\xi(t)$ satisfies the equation*

$$M\xi(t)=\sqrt{S\bar{\xi}(t)\xi(t)}\leqq l'(t)M\xi_0.$$

It has been seen that $\xi(t)=S\lambda(t)\xi_0$. Hence, if α is a point in Hilbert space, it follows from Definition (2) that, since $\lambda(t)$ is limited $l'(t)$,

$$|S\alpha\xi(t)|=|S\alpha(S\lambda(t)\xi_0)|\leqq M\alpha(l'(t)M\xi_0).$$

Hence, in view of Property (1) it follows that $M\xi(t)$ has the upper bound stated in the corollary.

It is now easily verified that $\xi(t)$ satisfies the hypothesis of Proposition (4) and therefore belongs to the class $\mathfrak{M}$. This completes the proof of the theorem.

COROLLARY (2). *The solution $\xi(t)$ has the property that the two series*

$$M\xi(t)=\sum_{i=1}^{\infty}\overline{x}_i(t)x_i(t), \quad M\left(\frac{d\xi(t)}{dt}\right)=\sum_{i=1}^{\infty}\overline{\frac{dx_i(t)}{dt}}\,\frac{dx_i(t)}{dt}, \tag{23}$$

converge uniformly for $|t|\leqq r_1$ for all $r_1<r$.

There is obtained

$$\begin{aligned}|\sum_{i=n}^{m}\overline{x}_i(t)x_i(t)|&=|\sum_{i=n}^{m}(\sum_{h,j=0}^{\infty}\overline{x}_{ij}\bar{t}^{j}x_{ih}t^{h})|\\&\leqq\sum_{h,j=0}^{\infty}|t|^{h+j}|\sum_{i=n}^{m}\overline{x}_{ij}x_{ih}|\ll(M\xi_0)^2(\sum_{h,j=0}^{\infty}b_h b_j|t|^{h+j}),\end{aligned}$$

where the dominance relation is a consequence of the fact that $\xi_n=S\lambda_n\xi_0$. Then, to prove the uniform convergence of $S\bar{\xi}(t)\xi(t)$ one would proceed from this point on precisely as in the work leading to (9) and (10) of Proposition (2).

The proof of the uniform convergence of the other series of the corollary is of similar nature.

The corollary to Theorem I and the Theorem II which follow are of importance in the consideration of fundamental sets of solutions.

COROLLARY (3). *Let $\theta(t)$ be a matrix $(x_{ij}(t))$ in which every column constitutes a solution of* (4). *Then, if all these solutions belong to the class $\mathfrak{M}$, the matrix satisfies the equation*

$$\theta(t) = S\lambda(t)\theta(0). \tag{24}$$

The equation (24) is an obvious result of the condition which holds for every column of $\theta(t)$.

THEOREM II. *Let $\theta(t)$ be a matrix in which each column constitutes a solution of* (4) *belonging to the class $\mathfrak{M}$. Suppose that $\theta(0)$ is limited by the quantity $L(0)$. Then $\theta(t)$ is limited $L(t)$,*

$$L(t) = L(0)l'(t). \tag{25}$$

To establish this result refer to Corollary (3) to Theorem I. It is seen from Property (6) of § 2 that, as a consequence of equation (24), $\theta(t)$ is limited by the product of the limits of $\lambda(t)$ and $\theta(0)$, which is the result stated in the theorem.

§ 4. *The Adjoint System.*

In the consideration of a finite system of linear differential equations the solutions of the adjoint system are found to have certain interesting properties in relation to the solution of the given system. On carrying over to infinite systems the classical definition of adjoint systems, there is obtained

DEFINITION 5. *The system of linear differential equations adjoint to the system* (4) *is the system*

$$\frac{d\xi}{dt} = -Sx'(t)\xi, \tag{26}$$

where $x'(t)$ is the transposed matrix of $x(t)$.

The hypothesis (H_1) was symmetrical in its conditions on the rows and columns of $x(t)$ so that, as a consequence, system (26) is of the same type as (4). A useful relation between the solutions of (4) and (26) is

THEOREM III. *Let $\xi_1(t)$ and $\xi_2(t)$ be two functions belonging to the class $\mathfrak{M}$ for $|t| \leqq r_1 < r$ which are, respectively, solutions of the equations* (4) *and* (26). *Then*

$$\frac{d}{dt}\{S\xi_1(t)\xi_2(t)\} = 0.$$

On formally differentiating $S\xi_1\xi_2$, there is obtained

$$\frac{d}{dt}\{S\xi_1(t)\xi_2(t)\}=S\left(\frac{d\xi_1}{dt}\right)\xi_2+S\xi_1\left(\frac{d\xi_2}{dt}\right). \tag{27}$$

Assume for the present that the term by term differentiation is legitimate. Then, on substituting the values of $d\xi_1/dt$ and $d\xi_2/dt$, it follows that

$$\begin{aligned}\frac{d}{dt}\{S\xi_1(t)\xi_2(t)\}&=S\{S\varkappa(t)\xi_1(t)\}\xi_2(t)-S\xi_1(t)\{S\varkappa'(t)\xi_2(t)\}\\&=S^2\xi_2(t)\varkappa(t)\xi_1(t)-S\xi_1(t)\{S\xi_2(t)\varkappa(t)\}\\&=S^2\xi_2(t)\varkappa(t)\xi_1(t)-S^2\xi_2(t)\varkappa(t)\xi_1(t)=0,\end{aligned}$$

where the changes in the forms of expressions are justified by Property (5) of § 2.

To justify the term by term differentiation in (27), recall Property (2) and consider proving that the infinite series on the right in (27) converge uniformly. For the first term, identify the $\eta(u)$ and $\xi(u)$ of Property (2) with $\xi_2(t)$ and $d\xi_1/dt$, respectively. The hypothesis concerning η is satisfied because of Corollary (2) to Theorem I. Since

$$\frac{d\xi_1(t)}{dt}=S\varkappa(t)\xi_1(t)=S\varkappa(t)\{S\lambda(t)\xi_0\}, \qquad (\xi_1(0)=\xi_0),$$

it follows that the modulus of $d\xi_1/dt$ is at most $l(t)l'(t)M\xi_0$ which is uniformly bounded for $|t|\leqq r_1$. Hence the first term on the right in (27) converges uniformly and, similarly, it is established that the second term has the same property. Consequently, the term by term differentiation leading to (27) was legitimate and the proof of the theorem is complete.

§ 5. *Fundamental Sets of Solutions.*

In considering finite systems of linear differential equations the notion of fundamental sets of solutions is of great importance. In the definition of such sets, the determinant of the matrix formed by the set plays a primary rôle. On replacing the idea of the non-vanishing of this determinant by the notion of a limited matrix possessing a unique limited reciprocal matrix there is immediately suggested, as an analogue for (4) of the classical definition of fundamental sets,

DEFINITION 6. *A matrix of solutions $\theta(t)$ of (4) will be termed a fundamental set of solutions at $t=t_0$ in case*

$\theta(t_0)$ is a limited matrix, and in case (28)

there exists a unique limited reciprocal matrix for $\theta(t_0)$. (29)

In establishing the fact that a set of solutions, fundamental at t_0, is fundamental for all $|t|<r$, there will be needed

THEOREM IV. *Let $\xi(t)$ be the solution of* (4) *defined by the initial condition*

$$\xi(t_0)=\xi_0, \qquad (|t_0|<r). \tag{30}$$

Then the analytic continuation of $\xi(t)$ to the point $t=0$ can be obtained, and the expansion there converges for all $|t|<r$.

This result is obtained by recalling Proposition (3). The condition (30) defines a solution in the neighborhood of t_0, and this solution can then be continued to the point $t=0$ by successive applications of Proposition (3) and Theorem I.

THEOREM V. *If $\theta(t)$ is a fundamental set at $t=0$, and if ξ_0 is a point in Hilbert space, then there exists a point η of Hilbert space such that the solution $\xi(t)$ of* (4) *defined by the initial condition $\xi(0)=\xi_0$ is given by $\xi(t)=S\theta(t)\eta$.*

Let η be determined so that $S\theta(0)\eta=\xi_0$, where, since $\theta(0)$ has a unique reciprocal $\phi(0)$, it follows from Property (9) that $\eta=S\phi(0)\xi_0$. Consequently, there is obtained

$$\xi(t)=S\lambda(t)\xi_0=S\lambda(t)\,(S\theta(0)\eta)=S(S\lambda(t)\theta(0))\eta=S\theta(t)\eta,$$

where the interchange of summations is effected by means of Property (5).

The definition given for fundamental sets of solutions is seen to have content because the elements of a matrix of solutions $\theta(t)=(x_{ij}(t))$ can be defined by the initial conditions

$$x_{ij}(0)=d_{ij}, \qquad (d_{ii}=1;\ d_{ij}=0,\ i\neq j). \tag{31}$$

For $t=0$, $\theta(t)$ reduces to the unit matrix which is limited and possesses a unique limited reciprocal matrix. As a matter of fact, the matrix $\lambda(t)$ of Theorem I can easily be shown to be the matrix defined by the conditions (31).

With the aid of Theorem III consider the proof of the following theorem which states an important property of fundamental sets of solutions of (4).

THEOREM IV. *Let $\theta(t)$ be a matrix of solutions of* (4) *which is fundamental at $t=t_0$. Then, the analytic continuation of $\theta(t)$ to the point t_1 is fundamental at t_1 for all $|t_1|<r$.*

In the first place, it is easily seen that $\theta(t_1)$ is limited for all $|t_1|<r$, because $\theta(t)$ can be continued to the point $t=0$, and remains limited at every point reached in the continuation. Theorem II then shows that $\theta(t)$ is limited $L(0)l'(t)$ where $L(0)$ is the limit of the matrix $\theta(0)$. It remains to prove that $\theta(t_1)$ possesses a unique reciprocal matrix.

Let $\theta^{-1}_{(t_0)}$ be the reciprocal matrix of $\theta(t_0)$. Define a matrix $\phi(t)$ of solutions of the adjoint system (26) by the initial condition $\phi'(t_0)=\theta^{-1}_{(t_0)}$, where ϕ' represents the transposed matrix of ϕ. As a consequence of Theorem III it follows that in a sufficiently small neighborhood of t_0 the ϕ and θ satisfy

$$\frac{d}{dt}(S\phi'(t)\theta(t))=0.$$

Hence, in this neighborhood the product matrix $S\phi'(t)\theta(t)$ must remain constant and, therefore, is always equal to the unit matrix since this is the value it assumes for $t=t_0$. Since $\phi'(t)$ is limited, $\theta(t)$ therefore possesses a limited left-hand reciprocal $\theta^{-1}_{(t)}=\phi'(t)$ for all t in the restricted neighborhood of t_0. By analytic continuation of the matrices ϕ and θ this property is seen to hold for all $|t|<r$. It is easily seen that the existence of two distinct left-hand reciprocals of $\theta(t)$ for any value of t would imply a similar condition at $t=t_0$. Hence there exists a unique left-hand reciprocal $\theta^{-1}(t)=\phi'(t)$ for all $|t|<r$. By Property (8) of § 2 it follows that ϕ' is also the unique right-hand reciprocal of $\theta(t)$ for $|t|<r$.

§ 6. *The Non-Homogeneous System.*

Consider the infinite system

$$\frac{d\xi}{dt}=Sx(t)\xi+\gamma(t), \tag{32}$$

where the function $\gamma(t)$ satisfies the conditions specified in hypothesis

(H_2). *The components $y_i(t)$ of $\gamma(t)$ are analytic in t and are regular for $|t|\leqq r$. Furthermore, the coefficients γ_n in the expansion for $\gamma(t)$,*

$$\gamma(t)=\sum_{n=0}^{\infty}\gamma_n t^n,$$

are points in Hilbert space for which $M\gamma_n\leqq b_n$ where $\Sigma_{n=0}^{\infty}b_n r^n$ exists.

It is easily verified that the solutions of the systems (32) and (4) are related in the fashion stated in

THEOREM V. *Let $\xi_1(t)$ and $\xi_2(t)$ be two solutions of* (32) *belonging to Hilbert space for a certain range of values $|t|\leqq r_1<r$. Then $\xi_3=\xi_2-\xi_1$ is a solution of the system* (4).

PROPOSITION 5. *The function $\gamma(t)$ belongs to Hilbert space for every value of $|t|\leqq r$, and $M\gamma(t)\leqq\Sigma_{n=0}^{\infty}b_n|t|^n$.*

To show this, recall Property (1); it is seen that, if α is a point in Hilbert space, then

$$|S_m\alpha\gamma(t)|=|\sum_{n=0}^{\infty}(S_m\alpha\gamma_n)t^n|\leqq M\alpha(\sum_{n=0}^{\infty}b_n|t|^n). \tag{33}$$

From (33) it follows that $\gamma(t)$ is a Hilbert point for $|t| \leqq r$ and that the modulus $M\gamma(t)$ satisfies the inequality of the proposition.

PROPOSITION 6. *The series*

$$M\gamma(t) = \sum_{i=1}^{\infty} \overline{y}_i(t) y_i(t) \tag{34}$$

converges uniformly for $|t| \leqq r$.

It is verified that

$$\sum_{i=h}^{k} \overline{y}_i(t) y_i(t) \leqq \sum_{n,\, m=0}^{\infty} |t|^{n+m} \left| \sum_{i=h}^{k} \overline{y}_{in} y_{im} \right|, \tag{35}$$

where $\gamma_n = (y_{1n}, y_{2n}, \ldots.)$. As a consequence of hypothesis (H_2) it is seen that the series on the right in (35) is dominated by the series

$$\sum_{n,\, m=0}^{\infty} r^{n+m} (b_n b_m).$$

A discussion similar to that used in the consideration of Proposition 2, § 3, then establishes the uniform convergence of (34).

PROPOSITION 7. *The series giving the components* $y_i(t)$ *of* $\gamma(t)$ *converge in t uniformly with respect to the index i.*

This result follows from the fact that the series for $y_i(t)$,

$$y_i(t) = \sum_{n=0}^{\infty} y_{in} t^n,$$

is dominated by $\Sigma_{n=0}^{\infty} b_n t^n$.

PROPOSITION 8. *Let the notation for the analytic continuation of* $\gamma(t)$ *to the point* t_0 *be*

$$\gamma(t) = \alpha(s) = \sum_{n=0}^{\infty} \alpha_n s^n, \qquad (s = t - t_0;\ |s| < r - |t_0|).$$

Then it follows that the α_n *are points in Hilbert space with moduli* $M\alpha_n \leqq a_n$, *where* $\Sigma_{n=0}^{\infty} a_n |s|^n$ *converges for all* $|s| < r - |t_0|$.

The proposition states that the analytic continuation of $\gamma(t)$ satisfies conditions similar to those imposed on γ in (H_2). The proof will not be given since it would be identical in method with that given for Proposition (3).

The existence theorem for system (32) will be established by a method analogous to the classical method of *variation of parameters* used for a finite system of differential equations. In the proof use will be made of the results obtained in the preceding articles with reference to fundamental sets of solutions and concerning the adjoint system. In view of Theorem V it is necessary to exhibit only one particular solution of (32). The existence of a solution taking on the initial value $\xi(0) = 0$ is established in

THEOREM VI. *Let hypotheses* (H_1) *and* (H_2) *be satisfied. Then there exists uniquely a function* $\xi(t)$ *of the class* $\mathfrak{M}$, *for* $|t|<r$, *satisfying the equation* (32) *and assuming the initial value* $\xi(0)=0$.

Let $\theta(t)$ be a fundamental set of solutions of the homogeneous system (4), and let $\phi(t)$ be the reciprocal of $\theta(t)$. Recall the analytic character of the matrix ϕ which follows from the fact that it constitutes a matrix of solutions of the adjoint system (26). Consider the infinite system of linear equations

$$\xi=S\theta(t)\eta \tag{36}$$

as giving a transformation of the variable ξ in (32) from ξ to η. Then

$$\frac{d\xi}{dt}=S\frac{d\theta(t)}{dt}\eta+S\theta(t)\frac{d\eta}{dt}, \tag{37}$$

provided that the term by term differentiation was legitimate. On substituting (36) and (37) in (32) there is obtained

$$S\frac{d\theta}{dt}\eta+S\theta\frac{d\eta}{dt}=S\chi(t)\left(S\theta(t)\eta\right)+\gamma(t)=S\{S\chi(t)\theta(t)\}\eta+\gamma(t), \tag{38}$$

where the interchange on the right is justified by Property (5) of § 2. Since

$$\frac{d\theta(t)}{dt}=S\chi(t)\theta(t),$$

equation (38) reduces to

$$S\theta(t)\frac{d\eta}{dt}=\gamma(t). \tag{39}$$

The equation (39) becomes, in view of Property (9) of § 2,

$$\frac{d\eta}{dt}=S\phi(t)\gamma(t). \tag{40}$$

The equation (40) has the solution

$$\eta(t)=\int_0^t S\phi(t)\gamma(t)\,dt, \tag{41}$$

where the integration may be taken along the straight line in the t-plane leading from $t=0$ to $t=t$. In case it can be established that the η of (41) belongs to Hilbert space for $|t|<r$, and is of such a nature that the term by term differentiation in (37) was legitimate, it then follows that $\xi(t)=S\theta(t)\eta(t)$ is a solution of (32). To complete the proof it would remain to show that $\eta(t)$ is a function of the class $\mathfrak{M}$.

Let Property (2) of § 2 be applied to $S\phi(t)\gamma(t)$. Every row of $\phi(t)$ is a solution of the adjoint system (26) and hence, in view of Corollary (1) to Theorem I as applied to (26), the modulus of each row is uniformly bounded for $|t|\leqq r_1<r$. Therefore, since $S\bar{\gamma}(t)\gamma(t)$ converges uniformly, it follows that $S\phi(t)\gamma(t)$ converges uniformly, and hence represents an analytic function.

Consequently, every coordinate of $\eta(t)$ is an analytic function regular for $|t|<r$. Moreover, if α is a point in Hilbert space,

$$|S_m\alpha\eta(t)|\leqq\left|\int_0^t S_m\alpha\{S\phi(t)\gamma(t)\}dt\right|\leqq M\alpha\left|\int_0^t M\gamma(t)L(t)dt\right|,$$

where $L(t)$ is the limit of the matrix $\phi(t)$. As a result of Property (1) of § 2 it follows that $\eta(t)$ is a Hilbert point for $|t|<r$, and its modulus is uniformly bounded for $|t|\leqq r_1 (r_1<r)$ in view of the similar property which holds for $M\gamma(t)$ and $L(t)$.

To justify the equation (37) for the $\eta(t)$ of (41) it must be shown that the two series on the right in (37) converge uniformly. From (40) it is seen that the modulus of $d\eta/dt$ is at most $M\gamma(t)L(t)$ which is uniformly bounded for $|t|\leqq r_1<r$. As a result of Corollary (2) to Theorem I, the two infinite series

$$S\left(\overline{\frac{d\theta'(t)}{dt}}\right)\left(\frac{d\theta(t)}{dt}\right),\ S\bar{\theta}'(t)\theta(t),$$

converge uniformly since every column of θ is a solution of (4). Hence it is seen that Property (2) of § 2 may be applied to each of the series in (37) to establish their uniform convergence.

On substituting (41) in (36) it follows that

$$\xi(t)=S\theta(t)\eta(t)=S\theta(t)\left[\textstyle\int_0^t S\phi(t)\gamma(t)dt\right] \tag{42}$$

is a solution of (32) which satisfies $\xi(0)=0$. Since $M\eta(t)$ was uniformly bounded for $|t|\leqq r_1<r$ it follows that $M\xi(t)$ will also be uniformly bounded. It then is seen that, as a consequence of Proposition (4), the function $\xi(t)$ belongs to the class $\mathfrak{M}$.

The solution given by (42) is the only solution of (32) for which $\xi(0)=0$ because, if there were another solution $\xi_1\neq\xi$, it would follow that the difference $\xi_2=\xi-\xi_1$ would be a solution of (4) for which $\xi_2(0)=0$, and therefore $\xi_2(t)=0$ for $|t|<r$.

The existence of a solution of (32) could also be established directly by the method of undetermined coefficients, but the proof of the result is much more difficult than the corresponding proof for the system (4).

Various problems are suggested in connection with systems (4) and (32) by the many known properties of finite systems of linear differential equations. Moreover, systems in which the variables are real and the coefficients are real and continuous may be considered under conditions analogous to those of (H_1) and (H_2).

HARVARD UNIVERSITY, *February* 1, 1917.

On the Asymptotic Value of Sums of Power Residues.[*]

By Howard H. Mitchell.

1. In this paper certain limits will be obtained for the sum of all the positive integers less than a given prime m, the indices of which, with respect to a primitive root of m have the same residue, mod 2ν, where 2ν is a divisor of $m-1$. If $(m-1)/2\nu$ is even, ind $(m-r) \equiv$ ind r, mod 2ν, and hence the value of a sum of the type mentioned is $m(m-1)/4\nu$. We shall therefore suppose that $(m-1)/2\nu$ is odd.

By means of the limits obtained for these sums it will be found that if S_i denotes any one of them, and if ν remains fixed while m ranges over the primes that are congruent to $2\nu+1$, mod 4ν,[†] then

$$\lim_{m \doteq \infty} \left[\frac{S_i}{\frac{m(m-1)}{4\nu}} \right] = 1.$$

Perhaps the most interesting result that has been established concerning the sums of the type considered is that if m be a prime of the form $4n+3$ the sum of the quadratic non-residues of m that lie between 0 and m always exceeds the sum of the residues. From this it follows that the number of the residues between 0 and $m/2$ exceeds the number of the non-residues. These results were established by Dirichlet in connection with his investigation of the class number of quadratic forms.[‡]

Stern has shown that if m is a prime of the form $8n+3$, the sum of the quadratic non-residues between 0 and m is less than twice the sum of the residues, whereas if it has the form $8n+7$, the sum of the non-residues is less than three times the sum of the residues.[§] The latter result, however, is trivial, since in whatever way the integers from 1 to $m-1$ be divided into two sets containing $(m-1)/2$ numbers each, the sum of either set is less than three times the sum of the other.

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[*] Presented to the American Mathematical Society, December 28, 1916.

[†] Dirichlet has shown that in any arithmetical progression in which the first term and common difference have no common factor there is an infinity of primes (Dirichlet–Dedekind, *Zahlentheorie*, 4th edition, §§ 132–137).

[‡] *Ibid.*, § 104.

[§] *Journal für Mathematik*, Vol. LXXI (1870), pp. 152, 153; Bachman, *Encyklopädie der Mathematischen Wissenschaften*, Bd. I, p. 569.

The distribution of the quadratic residues is a problem that is closely related to the values of these sums, and one that has interested a number of writers.[*]

2. The author's results concerning the sums of the sort described are obtained by use of a certain type of Dirichlet's series. These series are special cases of those that were used in his proof of the infinity of primes in an arithmetical progression.[†] They were also employed by Kummer in the derivation of the expression for the class number of a cyclotomic realm.[‡]

Following the notation of Dirichlet-Dedekind,[§] we write

$$L^\circ(\alpha)=\sum_z \frac{\alpha^{\operatorname{ind} z}}{z},$$

where the sum is to be taken over all the positive integers z that are not divisible by a given prime m, α denotes any root of the equation $\alpha^\nu=-1$, where 2ν is a divisor of $m-1$, and ind z denotes the index of z, mod m, with respect to a particular primitive root.

In case $(m-1)/2\nu$ is odd, the sum of this series may be expressed in finite form as follows:[||]

$$L^\circ(\alpha)=-\frac{\pi i}{m^2}(\alpha,\theta)\phi(\alpha).$$

The symbol (α, θ) denotes the Lagrange resolvent function

$$(\alpha,\theta)=\sum_{r=1}^{m-1}\alpha^{\operatorname{ind} r}\theta^r,$$

where $\theta=e^{2\pi i/m}$. The symbol $\phi(\alpha)$ is defined by

$$\phi(\alpha)=-\sum_r r\alpha^{-\operatorname{ind} r},$$

where r assumes the same values.

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[*] Lebesgue, *Journal de Mathèmatiques*, Vol. VII (1842), p. 137; Götting, *Journal für Mathematik*, Vol. LXX (1869), p. 363; Stern, *loc. cit.*; Osborn, *Messenger of Mathematics*, Vol. XXV (1895), p. 45; Glaisher, *Quarterly Journal of Mathematics*, Vol. XXXIII (1902), pp. 319–328; *ibid.*, Vol. XXXIV (1903), pp. 1, 178; Karpinski, *Journal für Mathematik*, Vol. CXXVII (1904), p. 1; Holden, *Messenger of Mathematics*, Vol. XXXV (1905), pp. 102, 110; *ibid.*, Vol. XXXVI (1906), pp. 75, 126.

[†] *Loc. cit.;* also Werke, Vol. I, p, 313.

[‡] *Journal für Mathematik*, Vol. XL, p. 98.

[§] P. 625.

[||] Combining equations (58), (65), *loc. cit.*

We proceed to obtain an upper limit for the absolute value of $L^\circ(\alpha)$. In consequence of our assumption that $(m-1)/2\nu$ is odd, the series may be written in the form*

$$L^\circ(\alpha)=\sum_r \alpha^{\operatorname{ind} r}\left[\frac{1}{r}-\frac{1}{m-r}+\frac{1}{m+r}-\frac{1}{2m-r}+\ldots .\right],$$

where the sum is to be taken over the positive integers r that are less than $m/2$. The absolute value of $L^\circ(\alpha)$ is thus less than

$$1+\frac{1}{2}+\frac{1}{3}+\ldots .+\frac{1}{\frac{m-1}{2}}<\log\left(\frac{m+1}{2}\right)+C,$$

where C denotes Euler's constant, $C=.577+$.

The absolute value of (α,θ) is equal to $\sqrt{m}$. Hence the absolute value of $\phi(\alpha)$ is less than

$$\frac{m\sqrt{m}}{\pi}\left[\log\left(\frac{m+1}{2}\right)+C\right].$$

The function $\phi(\alpha)$ may be written in the form

$$\phi(\alpha)=-\sum_{i=0}^{2\nu-1} S_i\alpha^{-i},$$

where S_i denotes the sum of the positive integers r less than m for which $\operatorname{ind} r\equiv i$, mod 2ν. Since we are assuming that $\alpha^\nu=-1$, this may in turn be written

$$\phi(\alpha)=\sum_{i=0}^{\nu-1}(S_{i+\nu}-S_i)\,\alpha^{-i}.$$

From this we obtain

$$\nu(S_{i+\nu}-S_i)=\sum_\alpha \alpha^i\phi(\alpha),$$

where the sum is to be taken over all the roots of the equation $\alpha^\nu=-1$, and where i may have any one of the values $0, 1, 2, \ldots ., \nu-1$.

We therefore conclude that

$$|S_{i+\nu}-S_i|<\frac{m\sqrt{m}}{\pi}\left[\log\left(\frac{m+1}{2}\right)+C\right].$$

If a number r appears in the sum S_i, then $m-r$ will appear in the sum $S_{i+\nu}$, and hence

$$S_{i+\nu}+S_i=\frac{m(m-1)}{2\nu}.$$

The values of S_i and $S_{i+\nu}$ are thus included between the limits,

$$\frac{m(m-1)}{4\nu}\pm\frac{m\sqrt{m}}{2\pi}\left[\log\left(\frac{m+1}{2}\right)+C\right].$$

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* Cf. Glaisher, *Quarterly Journal of Mathematics*, Vol. XXXIII (1902), pp. 306, 307, 317.

From this it is clear that these sums obey an asymptotic law in that if ν is considered to remain fixed, and m to range over the primes that are congruent to $2\nu+1$, mod 4ν, then

$$\lim_{m=\infty}\left[\frac{S_i}{\frac{m(m-1)}{4\nu}}\right]=1.$$

3. Certain extensions of these results are possible in case m is composite. For example, if m is an odd integer of the form $4n+3$ not divisible by a square factor, and $\left(\frac{z}{m}\right)$ denotes the Jacobi symbol for quadratic residues, we have*

$$\sum_z\left(\frac{z}{m}\right)\frac{1}{z}=-\frac{\pi}{m\sqrt{m}}\sum_r\left(\frac{r}{m}\right)r,$$

where the sum on the left is to be taken over all the positive integers z that are prime to m, and the sum on the right over all the positive integers r that are less than m and prime to m.

From this relation we obtain in the same manner as above

$$-\Sigma\left(\frac{r}{m}\right)r<\frac{m\sqrt{m}}{\pi}\left[\log\left(\frac{m+1}{2}\right)+C\right],$$

from which we conclude that the sum of the positive integers r that are less than and prime to m, and for which $\left(\frac{r}{m}\right)=+1$ is included between the limits

$$\frac{m\phi(m)}{4}-\frac{m\sqrt{m}}{2\pi}\left[\log\left(\frac{m+1}{2}\right)+C\right],\quad\frac{m\phi(m)}{4},$$

and the sum of those for which $\left(\frac{r}{m}\right)=-1$ is included between the limits

$$\frac{m\phi(m)}{4},\quad\frac{m\phi(m)}{4}+\frac{m\sqrt{m}}{2\pi}\left[\log\left(\frac{m+1}{2}\right)+C\right],$$

where, as usual, $\phi(m)$ denotes the number of integers less than and prime to m.

If e is any fixed number less than 1, it may be shown that if m be taken sufficiently large, $\phi(m)>m^e$. From this it follows that

$$\frac{m\phi(m)}{4}$$

is an asymptotic value for either of the two sums.

A similar generalization may be given in the case of higher residues.

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* Dirichlet-Dedekind, § 103.

4. By means of the above result we may obtain an upper limit for the number of classes of quadratic forms of determinant $-m$, where m is an odd integer of the form $4n+3$ not divisible by a square factor. This number is given by

$$h=-\frac{1}{m}\left[2-\left(\frac{2}{m}\right)\right]\Sigma\left(\frac{r}{m}\right)r,$$ [*]

and it therefore satisfies the inequality,

$$h<\left[2-\left(\frac{2}{m}\right)\right]\frac{\sqrt{m}}{\pi}\left[\log\left(\frac{m+1}{2}\right)+C\right].$$

For large values of m this is a much lower limit than those obtained by Holden;[†] in fact it is evident that if e denotes any fixed number greater than 1/2,

$$\lim_{m=\infty}\left[\frac{h}{m^e}\right]=0.$$

Similar results may be obtained for the number of classes of ideals in the quadratic realm $k(\sqrt{-m})$.[‡]

By means of the upper limit found in § 2 for the absolute value of the function $\phi(\alpha)$ an upper limit may also be obtained for the so-called *first factor* of the class number of the cyclotomic realm determined by *m-th* roots of unity.[§] Kummer has stated, apparently without proof, that this first factor obeys an asymptotic law.[||]

An upper limit may also be obtained for the first factor of the class number of the realm of degree 2ν, determined by the 2ν periods formed from *m-th* roots of unity, where $(m-1)/2\nu$ is odd.[¶]

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[*] Dirichlet-Dedekind, § 104.

[†] *Messenger of Mathematics*, Vol. XXXV (1905), p. 106.

[‡] Hilbert, "Die Theorie der algebraischen Zahlkörper," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Vol. IV (1894), p. 320.

[§] Kummer, *Journal für Mathematik*, Vol. XL (1850), p. 110; Dirichlet-Dedekind, p. 630.

[||] *Journal de Mathématiques*, Vol. XVI (1851), p. 473. Cf. also H. J. S. Smith, "Report on the Theory of Numbers," *Works*, Vol. I, p. 114.

[¶] Kummer, *Journal für Mathematik*, Vol. XL (1850), p. 112.

On the Structure of Finite Continuous Groups.

By J. A. Bullard.

Introduction.

The structure of a finite continuous group is determined in part by the "characteristic equation of the group," that is, the characteristic equation of the general infinitesimal transformation of the adjoint group of the given group. It is the purpose of this paper to show how certain properties of a group of linear homogeneous transformations can be obtained at once from the characteristic equation of the general infinitesimal transformation of the group, and thus how the type of structure is in part determined immediately from the infinitesimal transformations of such a group without the determination of the adjoint group. In § 1 I show how to find the structural constants, and thus the roots of the characteristic equation, of the general infinitesimal transformation of the adjoint group of the general linear homogeneous group relative to a given subgroup; in § 2 I find similar results for the special linear homogeneous group relative to a given subgroup. These results lead to certain relations, given in § 3, between the characteristic equation of a group of linear homogeneous transformations and the characteristic equation of the general infinitesimal transformation of the given group. By the aid of these relations I establish theorems I–X by which certain properties of such a group can be at once determined from the general infinitesimal transformation of the group.

Section 1.

Let G denote the general linear homogeneous group in n variables. The symbols of the infinitesimal transformations of G are the n^2 differential operators

$$\varepsilon_{ij}=x_i\frac{\partial}{\partial x_j}, \qquad (i, j=1, 2, \ldots, n),$$

or any n^2 linearly independent linear homogeneous functions of these operators with constant coefficients. Let G_r denote any given subgroup of the general linear homogeneous group with r parameters, the symbols of whose infinitesimal transformations are the r given differential operators

$$\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_r, \text{ where } \mathscr{A}_h=\sum_{i,j=1}^{n} a_{hij}\varepsilon_{ij}, \qquad (h=1, 2, \ldots, r).$$

Then
$$\left\| \begin{matrix} a_{111}, a_{112}, \dots, a_{11n}, a_{121}, \dots, a_{1n1}, a_{1n2}, \dots, a_{1nn} \\ a_{211}, a_{212}, \dots, a_{21n}, a_{221}, \dots, a_{2n1}, a_{2n2}, \dots, a_{2nn} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{r11}, a_{r12}, \dots, a_{r1n}, a_{r21}, \dots, a_{rn1}, a_{rn2}, \dots, a_{rnn} \end{matrix} \right\| \neq 0,$$
and
$$(\mathscr{A}_i, \mathscr{A}_j) = \mathscr{A}_i\mathscr{A}_j - \mathscr{A}_j\mathscr{A}_i = \sum_{h=1}^{r} c_{jih}\mathscr{A}_h, \qquad (i, j = 1, 2, \dots, r).$$

We may take as the infinitesimal transformations of G the differential operators $\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_r$, together with any n^2-r other differential operators $\mathscr{A}_{r+1}, \mathscr{A}_{r+2}, \dots, \mathscr{A}_{n^2}$ defined as follows:
$$\mathscr{A}_{r+h} = \sum_{i,j=1}^{n} a_{r+h,i,j}\varepsilon_{ij}, \qquad (h = 1, 2, \dots, n^2-r),$$
provided these n^2 differential operators are independent, that is, provided the n^2 $\mathscr{A}$'s are linearly independent linear homogeneous functions of the ε's. In which case
$$\left| \begin{matrix} a_{i11}, a_{i12}, \dots, a_{i1n}, a_{i21}, \dots, a_{in1}, a_{in2}, \dots, a_{inn} \\ i = 1, 2, \dots, n^2 \end{matrix} \right| \neq 0.$$

I shall employ the following notation. The constituent of any matrix B in the p-*th* row and q-*th* column will be denoted by $(B)_{pq}$ and the determinant of B by $|B|$. Thus, if T denote the above matrix, $|T| \neq 0$ and $(T)_{i,(\mu-1)n+\nu} = a_{i\mu\nu}$, $(i = 1, 2, \dots, n^2;\ \mu, \nu = 1, 2, \dots, n)$. We shall now have, in Cayley's matrical notation,
$$(\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_{n^2}) = T(\varepsilon_{11}, \varepsilon_{12}, \dots, \varepsilon_{1n}, \varepsilon_{21}, \dots, \varepsilon_{n1}, \varepsilon_{n2}, \dots, \varepsilon_{nn}). \quad (1)$$

For $1 \leqq i \leqq n^2$ the equations of the infinitesimal transformation of G corresponding to $\mathscr{A}_i$, namely,
$$x'_\mu = x_\mu + \delta t \mathscr{A}_i x_\mu, \qquad (\mu = 1, 2, \dots, n),$$
may be written $(x'_1, x'_2, \dots, x'_n) = (1 + \delta t A_i)(x_1, x_2, \dots, x_n)$ where A_i is the matrix corresponding to $\mathscr{A}_i$ and
$$(A_i)_{\mu\nu} = a_{i\mu\nu}, \qquad (\mu, \nu = 1, 2, \dots, n). \quad (2)$$

For our present purpose we may substitute the matrices $A_1, A_2, \dots, A_{n^2}$ for the respective differential operators $\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_{n^2}$, regarding the former as the symbols of the infinitesimal transformations of G. Similarly, if e_{ij}, $(i, j = 1, 2, \dots, n)$, denote the matrix whose constituents are all zero except that in the i-*th* row and j-*th* column which is equal to 1, we may substitute

e_{ij}, $(i, j=1, 2, \ldots, n)$, for ε_{ij}. The general infinitesimal transformation of G_r is then represented by

$$A'=\sum_{h=1}^{r}\alpha_h\mathscr{A}_h=\sum_{h=1}^{r}\sum_{i,j=1}^{n}\alpha_h a_{hij}\varepsilon_{ij}, \text{ or by the matrix } A=\sum_{h=1}^{r}\alpha_h A_h,$$

where $(A')_{\mu\nu}=\sum_{i=1}^{r}\alpha_i a_{i\mu\nu}$, $(\mu, \nu=1, 2, \ldots, n)$. The general infinitesimal transformation of the general linear homogeneous group is represented by

$$A=\sum_{h=1}^{n^2}\alpha_h\mathscr{A}_h=\sum_{h=1}^{n^2}\sum_{i,j=1}^{n}\alpha_h a_{hij}\varepsilon_{ij}, \text{ or by the matrix } A=\sum_{h=1}^{n^2}\alpha_h A_h,$$

where $(A)_{\mu\nu}=a_{\mu\nu}=\sum_{i=1}^{n^2}\alpha_i a_{i\mu\nu}$, $(\mu, \nu=1, 2, \ldots, n)$.

We shall have then

$$(A_i, A_j)=\sum_{k=1}^{r}c_{ijk}A_k, \qquad (i, j=1, 2, \ldots, r), \tag{3}$$

$$(A_i, A_j)=\sum_{k=1}^{n^2}c_{ijk}A_k, \qquad (i, j=1, 2, \ldots, n^2),$$

where c_{ijk}, $(i, j, k=1, 2, \ldots, n^2)$, are the structural constants of G, and c_{ijk}, $(i, j, k=1, 2, \ldots, r)$, are the structural constants of G_r. Since there is no linear relation with constant coefficients between $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_r$, the corresponding matrices $A_1, A_2, \ldots, A_r$ are linearly independent; similarly, since there is no linear relation with constant coefficients between $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_{n^2}$, then $A_1, A_2, \ldots, A_{n^2}$ are linearly independent.

Let Γ denote the adjoint of the general linear homogeneous group G, and Γ_r the adjoint of G_r. The symbols of the infinitesimal transformations of the adjoint are differential operators which may be replaced for our present purpose by the matrices representing the square arrays of their coefficients. Thus, if $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_{n^2}$ are chosen as the symbols of the infinitesimal transformations of G, the corresponding symbols of the infinitesimal transformations of the adjoint are $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_{n^2}$, where $\mathscr{E}_i=\sum_{j,k=1}^{n^2}c_{ijk}\alpha_j\frac{\partial}{\partial\alpha_k}$, $(i=1, 2, \ldots, n^2)$. The equations, $\alpha'_\mu=\alpha_\mu+\delta t\mathscr{E}_i\alpha_\mu=\alpha_\mu+\delta t\sum_{j=1}^{n^2}c_{ij\mu}\alpha_j$, $(\mu=1, 2, \ldots, n^2)$, defining the infinitesimal transformation of Γ may be written in matrical form as follows: $(\alpha'_1, \alpha'_2, \ldots, \alpha'_{n^2})=(1+\delta tE_i)(\alpha_1, \alpha_2, \ldots, \alpha_{n^2})$, where E_i is the matrix defined by the equations

$$(E_i)_{\nu\mu}=c_{i\mu\nu}, \qquad (i, \mu, \nu=1, 2, \ldots, n^2); \tag{4}$$

and we may substitute for the symbols $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_{n^2}$, the respectively corresponding matrices $E_1, E_2, \ldots, E_{n^2}$. A similar remark applies to the group Γ_r, whose symbols of infinitesimal transformation are $\mathcal{E}'_1, \mathcal{E}'_2, \ldots, \mathcal{E}'_r$, where $\mathcal{E}'_i = \sum_{j,k=1}^{r} c_{ijk}\alpha_j \frac{\partial}{\partial \alpha_k}$, $(i=1, 2, \ldots, r)$, and these may be replaced by the respectively corresponding matrices $E'_1, E'_2, \ldots, E'_r$, where

$$(E'_i)_{\nu\mu} = c_{i\mu\nu}, \qquad (i, \mu, \nu = 1, 2, \ldots, r). \tag{5}$$

We have

$$(E_i, E_j) = \sum_{k=1}^{n^2} c_{ijk}E_k, \qquad (i, j = 1, 2, \ldots, n^2),$$

$$(E'_i, E'_j) = \sum_{k=1}^{r} c_{ijk}E'_k, \qquad (i, j = 1, 2, \ldots, r).$$

When $A_1, A_2, \ldots, A_{n^2}$ are chosen as the symbols of the infinitesimal transformations of G, the determination of the structural constants c_{ijk}, and hence of the matrices $E_1, E_2, \ldots, E_{n^2}$, may be accomplished as follows. The matrices $A_1, A_2, \ldots, A_{n^2}$, constitute a hypercomplex number system. Let $A_iA_j = \sum_{k=1}^{n^2} \gamma_{ijk}A_k$; by a theorem of Lie's (*Transformationsgruppen*, Vol. III, § 140)

$$c_{ijk} = \gamma_{ijk} - \gamma_{jik}. \tag{6}$$

To find γ_{ijk} we note that $e_{\mu\nu}$, $(\mu, \nu = 1, 2, \ldots, n)$, constitute a number system equivalent to $A_1, A_2, \ldots, A_{n^2}$, with a multiplication table determined by the equations $e_{\mu\nu}e_{\nu\sigma} = e_{\mu\sigma}$ and $e_{\mu\nu}e_{\lambda\sigma} = 0$, $(\mu, \nu, \lambda, \sigma = 1, 2, \ldots, n;\ \lambda \neq \nu)$. We may put $e_{\mu\nu}e_{\lambda\sigma} = \sum_{\rho,\tau=1}^{n} g_{\mu,\nu,\lambda,\sigma,\rho,\tau}\, e_{\rho\tau}$, provided $g_{\mu,\nu,\lambda,\sigma,\rho,\tau} = 0$ for all values of the subscripts except that

$$g_{\mu,\nu,\nu,\sigma,\mu,\sigma} = 1, \qquad (\mu, \nu, \sigma = 1, 2, \ldots, n).$$

From equation (1), or rather the equation

$$(A_1, A_2, \ldots, A_{n^2}) = T(e_{11}, e_{12}, \ldots, e_{n1}, e_{n2}, \ldots, e_{nn}),$$

we derive

$$(e_{11}, e_{12}, \ldots, e_{n1}, e_{n2}, \ldots, e_{nn}) = T^{-1}(A_1, A_2, \ldots, A_{n^2}). \tag{7}$$

Then $(T)_{i,\,(\mu-1)n+\nu} = a_{i\mu\nu}$,

$$(T^{-1})_{(\mu-1)n+\nu,\,i} = (-1)^{(\mu-1)n+\nu+i}\frac{1}{|T|}\frac{\partial}{\partial a_{i\mu\nu}}|T|,$$

$$(i = 1, 2, \ldots, n^2;\ \mu, \nu = 1, 2, \ldots, n);$$

and equations (1) and (7) are respectively equivalent to the two systems of equations

$$A_i = \sum_{\mu,\nu=1}^{n} a_{i\mu\nu}e_{\mu\nu} = \sum_{\mu,\nu=1}^{n} (T)_{i,\,(\mu-1)n+\nu}e_{\mu\nu}, \qquad (i = 1, 2, \ldots, n^2),$$

$$e_{\mu\nu} = \sum_{i=1}^{n^2} (T^{-1})_{(\mu-1)n+\nu,\,i}A_i, \qquad (\mu, \nu = 1, 2, \ldots, n).$$

Therefore, $$A_iA_j=\sum_{\mu,\nu=1}^{n}(T)_{i,\,(\mu-1)n+\nu}e_{\mu\nu}\sum_{\lambda,\sigma=1}^{n}(T)_{j,\,(\lambda-1)n+\sigma}e_{\lambda\sigma}$$
$$=\sum_{\mu,\nu,\lambda,\sigma=1}^{n}(T)_{i,\,(\mu-1)n+\nu}(T)_{j,\,(\lambda-1)n+\sigma}e_{\mu\nu}e_{\lambda\sigma}$$
$$=\sum_{\mu,\nu,\lambda,\sigma,\rho,\tau=1}^{n}(T)_{i,\,(\mu-1)n+\nu}(T)_{j,\,(\lambda-1)n+\sigma}g_{\mu,\nu,\lambda,\sigma,\rho,\tau}e_{\rho,\tau}$$
$$=\sum_{\mu,\nu,\lambda,\sigma,\rho,\tau=1}^{n}(T)_{i,\,(\mu-1)n+\nu}(T)_{j,\,(\lambda-1)n+\sigma}g_{\mu,\nu,\lambda,\sigma,\rho,\tau}\sum_{k=1}^{n^2}(T^{-1})_{(\rho-1)n+\tau,k}A_k,$$
$$(i,\,j=1,\,2,\,\ldots,\,n^2).$$

If the matrix M_i, $(1\leqq i\leqq n^2)$ of order n^2 be defined by the equations

$$(M_i)_{(\rho-1)n+\tau,\,(\lambda-1)n+\sigma}=\sum_{\mu,\nu=1}^{n}(T)_{i,\,(\mu-1)n+\nu}g_{\mu,\nu,\lambda,\sigma,\rho,\tau},\quad(\lambda,\,\sigma,\,\rho,\,\tau=1,\,2,\,\ldots,\,n),\tag{8}$$

we may write

$$A_iA_j=\sum_{k=1}^{n^2}\sum_{\rho,\tau,\lambda,\sigma=1}^{n}(T)_{j,\,(\lambda-1)n+\sigma}(M_i)_{(\rho-1)n+\tau,\,(\lambda-1)n+\sigma}(T^{-1})_{(\rho-1)n+\tau,\,k}A_k$$
$$=\sum_{k=1}^{n^2}\sum_{\lambda,\sigma,\rho,\tau=1}^{n}(\breve{T}^{-1})_{k,\,(\rho-1)n+\tau}(M_i)_{(\rho-1)n+\tau,\,(\lambda-1)n+\sigma}(\breve{T})_{(\lambda-1)n+\sigma,\,j}A_k$$
$$=\sum_{k=1}^{n^2}(\breve{T}^{-1}M_i\breve{T})_{kj}A_k,\qquad(i,\,j=1,\,2,\,\ldots,\,n^2),\tag{9}$$

where $\breve{T}$ and $\breve{T}^{-1}$ denote respectively the transverse or conjugate of T and T^{-1}.* Therefore, since the A's are linearly independent,

$$\gamma_{ijk}=(\breve{T}^{-1}N_i\breve{T})_{kj},\qquad(i,\,j,\,k=1,\,2,\,\ldots,\,n^2).\tag{10}$$

Interchanging A_i and A_j in (9) we obtain

$$A_jA_i=\sum_{k=1}^{n^2}\sum_{\mu,\nu,\lambda,\sigma,\rho,\tau=1}^{n}(T)_{j,\,(\lambda-1)n+\sigma}(T)_{i,\,(\mu-1)n+\nu}g_{\lambda,\sigma,\mu,\nu,\rho,\tau}(T^{-1})_{(\rho-1)n+\tau,\,j}A_k,$$
$$(i,\,j=1,\,2,\,\ldots,\,n^2).$$

and defining the matrix N_i by the equations

$$(N_i)_{(\rho-1)n+\tau,\,(\lambda-1)n+\sigma}=\sum_{\mu,\nu=1}^{n}(T)_{i,\,(\mu-1)n+\nu}g_{\lambda,\sigma,\mu,\nu,\rho,\tau},$$
$$(\lambda,\,\sigma,\,\rho,\,\tau=1,\,2,\,\ldots,\,n;\ i=1,\,2,\,\ldots,\,n^2),\tag{11}$$

$$A_jA_i=\sum_{k=1}^{n^2}\sum_{\lambda,\sigma,\rho,\tau=1}^{n}(T)_{j,\,(\lambda-1)n+\sigma}(N_i)_{(\rho-1)n+\tau,\,(\lambda-1)n+\sigma}(T^{-1})_{(\rho-1)n+\tau,\,k}A_k$$
$$=\sum_{k=1}^{n^2}(\breve{T}^{-1}N_i\breve{T})_{kj}A_k,\qquad(i,\,j=1,\,2,\,\ldots,\,n^2),$$

and therefore,

$$\gamma_{jik}=(\breve{T}^{-1}N_i\breve{T})_{kj},\qquad(i,\,j,\,k=1,\,2,\,\ldots,\,n^2).\tag{12}$$

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* The transverse of any matrix B is obtained by interchanging rows and columns so that $(B)_{ij}=b_{ij}=(\breve{B})_{ji}$. We have $(\breve{B})^{-1}=(\breve{B^{-1}})$; and if C is any second matrix of order n, $(\breve{B\pm C})=\breve{B}\pm\breve{C}$ and $(\breve{BC})=(\breve{C}\breve{B})$.

By virtue of the equations (6), (11), (12), we now have

$$c_{ijk}=(\breve{T}^{-1}M_i\breve{T})_{kj}-(\breve{T}^{-1}N_i\breve{T})_{kj}=(\breve{T}^{-1}(M_i-N_i)\breve{T})_{kj}, \qquad (i,\ j,\ k=1,\ 2,\ \ldots,\ n^2),$$

and therefore, by equation (4) which defines the matrices E_1, E_2, …, E_{n^2},

$$E_i=\breve{T}^{-1}(M_i-N_i)\breve{T}, \qquad (i=1,\ 2,\ \ldots,\ n^2). \tag{13}$$

The matrices M_i and N_i are respectively defined by the equations

$$(M_i)_{(\rho-1)n+\tau,\ (\lambda-1)n+\sigma}=\sum_{\mu,\ \nu=1}^{n} a_{i\mu\nu}g_{\mu,\ \nu,\ \lambda,\ \sigma,\ \rho,\ \tau}=\sum_{\mu,\ \nu=1}^{n}(A_i)_{\mu\nu}g_{\mu,\ \nu,\ \lambda,\ \sigma,\ \rho,\ \tau}, \tag{8}$$

$$(N_i)_{(\rho-1)n+\tau,\ (\lambda-1)n+\sigma}=\sum_{\mu,\ \nu=1}^{n} a_{i\mu\nu}g_{\lambda,\ \sigma,\ \mu,\ \nu,\ \rho,\ \tau}=\sum_{\mu,\ \nu=1}^{n}(A_i)_{\mu\nu}g_{\lambda,\ \sigma,\ \mu,\ \nu,\ \rho,\ \tau}, \tag{11}$$

$$(i=1,\ 2,\ \ldots,\ n^2),$$

where $g_{\mu,\ \nu,\ \lambda,\ \sigma,\ \rho,\ \tau}=0$ for all values of the subscripts except that $g_{\mu,\ \nu,\ \nu,\ \sigma,\ \mu,\ \sigma}=1$, $(\mu,\ \nu,\ \sigma=1,\ 2,\ \ldots,\ n)$. Whence it follows that

$$(M_i)_{(\mu-1)n+\sigma,\ (\nu-1)n+\sigma}=a_{i\mu\nu}=(A_i)_{\mu\nu}, \qquad (\mu,\ \nu,\ \sigma=1,\ 2,\ \ldots,\ n),$$
$$(N_i)_{(\lambda-1)n+\nu,\ (\lambda-1)n+\mu}=a_{i\mu\nu}=(A_i)_{\mu\nu}, \qquad (\lambda,\ \mu,\ \nu=1,\ 2,\ \ldots,\ n),$$

and that all other constituents of M_i and N_i $(1\leqq i\leqq n^2)$ are zero. Hence we find that the matrix M_i which is of order n^2, may be represented in the following way: $(M_i)_{\mu\nu}=M_{i\mu\nu}$, $(\mu,\ \nu=1,\ 2,\ \ldots,\ n)$, where, for $1\leqq i\leqq n^2$, the constituent $M_{i\mu\nu}$ is itself a matrix of order n defined as follows: $(M_{i\mu\nu})_{pp}=a_{i\mu\nu}$, $(M_{i\mu\nu})_{pq}=0$, $(p,\ q=1,\ 2,\ \ldots,\ n;\ p\neq q)$. If I denotes the identical substitution in n variables we may write

$$(M_i)_{\mu\nu}=(A_i)_{\mu\nu}I, \qquad (\mu,\ \nu=1,\ 2,\ \ldots,\ n;\ i=1,\ 2,\ \ldots,\ n^2). \tag{14}$$

Similarly, we find that N_i can be expressed in terms of constituent matrices as follows:

$$(N_i)_{pp}=\breve{A}_i,\ (N_i)_{pq}=0, \qquad (p,\ q=1,\ 2,\ \ldots,\ n;\ p\neq q;\ i=1,\ 2,\ \ldots,\ n^2). \tag{15}$$

If B, C, Q and R are any four matrices of order n, for two matrices W_b and W_c such that $(W_b)_{\mu\nu}=(B)_{\mu\nu}Q$ and $(W_c)_{\mu\nu}=(C)_{\mu\nu}R$, $(\mu,\ \nu=1,\ 2,\ \ldots,\ n)$, we have

$$(W_bW_c)_{\mu\nu}=\sum_{h=1}^{n}(B)_{\mu h}(C)_{h\nu}QR^{*}=(BC)_{\mu\nu}QR=(W_{bc})_{\mu\nu}, \quad (\mu,\nu=1,2,\ldots,n). \tag{16}$$

By this formula we obtain

$$(M_iM_j)_{\mu\nu}=(A_iA_j)_{\mu\nu}I, \tag{17}$$

$$(M_i^2)_{\mu\nu}=(A_i^2)_{\mu\nu}I, \quad (M_i^3)_{\mu\nu}=(M_iM_i^2)_{\mu\nu}=(A_i^3)_{\mu\nu}I, \tag{18}$$

and for any positive integer p

$$(M_i^p)_{\mu\nu}=(A_i^p)_{\mu\nu}I, \qquad (\mu,\ \nu=1,\ 2,\ \ldots,\ n;\ i,\ j=1,\ 2,\ \ldots,\ n^2); \tag{19}$$

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[*] The law of multiplication for matrices whose constituents are themselves matrices is the same as for matrices whose constituents are scalars.

again from (15) we have,

$$\left.\begin{aligned}(N_iN_j)_{\mu\nu}&=\sum_{h=1}^{n}(N_i)_{\mu h}(N_j)_{h\nu}=0,\\ (N_iN_j)_{\mu\mu}&=\sum_{h=1}^{n}(N_i)_{\mu h}(N_j)_{h\mu}=(N_i)_{\mu\mu}(N_j)_{\mu\mu}=\breve{A}_i\breve{A}_j,\end{aligned}\right\}\quad(20)$$

and for any positive integer p

$$(N_i^p)_{\mu\mu}=\breve{A}_i^p,\quad (N_i^p)_{\mu\nu}=0,\qquad (\mu,\nu=1,2,\ldots,n;\ \mu\neq\nu;\ i,j=1,2,\ldots,n^2).\quad(21)$$

Further, by equations (16), (19) and (21) for positive integers p, q, r, s, t, v we obtain

$$\begin{aligned}(M_i^sN_j^p)_{\mu\nu}&=\sum_{h=1}^{n}(M_i^s)_{\mu h}(N_j^p)_{h\nu}=(M_i^s)_{\mu\nu}(N_j^p)_{\nu\nu}=(A_i^s)_{\mu\nu}I\breve{A}_j^p=(N_j^p)_{\mu\mu}(M_i^s)_{\mu\nu}\\ &=\sum_{h=1}^{n}(N_j^p)_{\mu h}(M_i^s)_{h\nu}=(N_j^pM_i^s)_{\mu\nu},\end{aligned}\quad(22)$$

$$(M_i^pN_i^qM_j^rN_j^s)_{\mu\nu}=((M_i^pN_i^q)(M_j^rN_j^s))_{\mu\nu}=(A_i^pA_j^r)_{\mu\nu}\breve{A}_i^q\breve{A}_j^s,\quad(23)$$

$$\begin{aligned}(M_i^pN_i^qM_j^rN_j^sM_k^tN_k^v)_{\mu\nu}&=((M_i^pN_i^qM_j^rN_j^s)(M_k^tN_k^v))_{\mu\nu}=(A_i^pA_j^rA_k^t)_{\mu\nu}\breve{A}_i^q\breve{A}_j^s\breve{A}_k^v,\\ &(\mu,\ \nu=1,\ 2,\ \ldots,\ n;\ i,\ j,\ k=1,\ 2,\ \ldots,\ n^2).\end{aligned}\quad(24)$$

I shall employ the symbol S prefixed to a matrix M whose constituents are scalars, designating SM as the *scalar* of M, to denote the sum of the constituents in the principal diagonal of M. Then, if A and B are any two matrices,

$$S(A\pm B)=SA\pm SB,\quad SAB=SBA,\quad S\breve{A}=SA.\quad(25)$$

Further, if ρ is any scalar

$$S\rho A=\rho SA;\quad(26)$$

and if I is the identical substitution in n variables,

$$SI=n.\quad(27)$$

Moreover, SM is the sum of the roots of the characteristic equation of M, $|\omega-M|=0$, and SM^p is the sum of the *p-th* powers of the roots. In the case of a matrix $(M)_{\mu\nu}=M_{\mu\nu}$, $(\mu,\ \nu=1,\ 2,\ \ldots)$, whose constituents, the M's with double subscripts, are themselves matrices, those in the principal diagonal being square matrices, I shall define SM as follows:

$$SM=SM_{11}+SM_{22}+\ldots.$$

It may be readily seen that the properties of the scalar function given in equations (25), (26) and (27) hold in this case also.

We then have from equation (13), $SE_i = S\breve{T}^{-1}(M_i - N_i)\breve{T}$, and since the scalar product of two or more matrices is unchanged by an interchange of the factors leaving the cyclic order unaltered,

$$SE_i = S(M_i - N_i) = SM_i - SN_i, \qquad (i=1, 2, \ldots, n^2).$$

By equation (14), we have $SM_i = \sum_{\mu=1}^{n} (A_i)_{\mu\mu} SI = SA_i \cdot SI = nSA_i$, and by equation (15), $SN_i = nS\breve{A}_i = nSA_i$. Substituting these results in the above we obtain

$$SE_i = 0, \qquad (i=1, 2, \ldots, n^2). \tag{28}$$

Also

$$\begin{aligned} SE_iE_j &= S\breve{T}^{-1}(M_i - N_i)\breve{T} \cdot \breve{T}^{-1}(M_j - N_j)\breve{T} = S(M_i - N_i)(M_j - N_j) \\ &= SM_iM_j - SM_iN_j - SN_iM_j + SN_iN_j, \qquad (i, j=1, 2, \ldots, n^2). \end{aligned} \tag{29}$$

From equation (17) we obtain $SM_iM_j = \sum_{\mu=1}^{n} (A_iA_j)_{\mu\mu} SI = SA_iA_jSI = nSA_iA_j$; from equation (20), $SN_iN_j = nS\breve{A}_i\breve{A}_j = nSA_iA_j$; from equation (22), $SM_iN_j = SN_jM_i = \sum_{\mu=1}^{n} (A_i)_{\mu\mu} S\breve{A}_j = SA_iS\breve{A}_j = SA_iSA_j$, and substituting these values in equation (29) we find

$$SE_iE_j = 2(nSA_iA_j - SA_iSA_j), \qquad (i, j=1, 2, \ldots, n^2). \tag{30}$$

For any positive integer h we have

$$SE_i^h = (S\breve{T}^{-1}(M_i - N_i)\breve{T})^h = S(M_i - N_i)^h = \sum_{\mu=0}^{h} (-1)^\mu \binom{h}{\mu} SM_i^{h-\mu}N_i^\mu, \qquad (i=1, 2, \ldots, n^2).$$

Since for any odd integer, $h = 2p+1$, the number of terms in the expansion is even and the even terms have a negative sign, and the binomial coefficients $\binom{h}{j}$ and $\binom{h}{h-j}$ are equal, we may write

$$\begin{aligned} SE_i^{2p+1} &= \sum_{\mu=0}^{2p+1} (-1)^\mu \binom{2p+1}{\mu} SM_i^{2p-\mu+1}N_i^\mu \\ &= \sum_{\mu=0}^{p} (-1)^\mu \binom{2p+1}{\mu} (SM_i^{2p-\mu+1}N_i^\mu - SM_i^\mu N_i^{2p-\mu+1}), \\ &\qquad (i=1, 2, \ldots, n^2). \end{aligned} \tag{31}$$

From equations (19), (21) and (22), respectively, we obtain

$$SM_i^q = \sum_{\mu=1}^{n} (A_i^q)_{\mu\mu} SI = SA_i^q SI = nSA_i^q,$$

$$SN_i^q = nS\breve{A}_i^q = nSA_i^q, \quad SM_i^qN_i^s = \sum_{\mu=1}^{n} (A_i^q)_{\mu\mu} S\breve{A}_i^s = SA_i^qSA_i^s = SM_i^sN_i^q,$$

and substituting in equation (31)

$$SE_i^{2p+1}=0, \qquad (i=1, 2, \ldots, n^2). \tag{32}$$

For any even integer, $h=2p$, we have

$$SE_i^{2p}=\sum_{\mu=0}^{2p}(-1)^{\mu}\binom{2p}{\mu}SA_i^{2p-\mu}SA_i^{\mu}, \qquad (i=1, 2, \ldots, n^2), \tag{33}$$

where SA_i^0 is assigned the value n. For any positive integers p and q

$$\begin{aligned}SE_i^pE_j^q&=S(\breve{T}^{-1}(M_i-N_i)\breve{T})^p(\breve{T}^{-1}(M_j-N_j)\breve{T})^q=S(M_i-N_i)^p(M_j-N_j)^q\\&=\sum_{\mu=0}^{p}\sum_{\nu=0}^{q}(-1)^{\mu+\nu}\binom{p}{\mu}\binom{q}{\nu}SM_i^{p-\mu}N_i^{\mu}M_j^{q-\nu}N_j^{\nu}, \quad (i, j=1, 2, \ldots, n^2).\end{aligned} \tag{34}$$

From (23) we obtain $SM_i^{p-\mu}N_i^{\mu}M_j^{q-\nu}N_j^{\nu}=SA_i^{p-\mu}A_j^{q-\nu}S\breve{A}_i^{\mu}\breve{A}_j^{\nu}=SA_i^{p-\mu}A_j^{q-\nu}SA_i^{\mu}A_j^{\nu}$, and substituting in (34),

$$SE_i^pE_j^q=\sum_{\mu=0}^{p}\sum_{\nu=0}^{q}(-1)^{\mu+\nu}\binom{p}{\mu}\binom{q}{\nu}SA_i^{p-\mu}A_j^{q-\nu}SA_i^{\mu}A_j^{\nu}, \quad (i, j=1, 2, \ldots, n^2), \tag{35}$$

where $A_i^0=A_j^0=I$. Again, for positive integers p, q and r

$$\begin{aligned}SE_i^pE_j^qE_k^r&=S(M_i-N_i)^p(M_j-N_j)^q(M_k-N_k)^r\\&=\sum_{\mu=0}^{p}\sum_{\nu=0}^{q}\sum_{\sigma=0}^{r}(-1)^{\mu+\nu+\sigma}\binom{p}{\mu}\binom{q}{\nu}\binom{r}{\sigma}SM_i^{p-\mu}N_i^{\mu}M_j^{q-\nu}N_j^{\nu}M_k^{r-\sigma}N_k^{\sigma},\\&\qquad (i, j, k=1, 2, \ldots, n^2).\end{aligned} \tag{36}$$

From equation (24) $SM_i^{p-\mu}N_i^{\mu}M_j^{q-\nu}N_j^{\nu}M_k^{r-\sigma}N_k^{\sigma}=SA_i^{p-\mu}A_j^{q-\nu}A_k^{r-\sigma}S\breve{A}_i^{\mu}\breve{A}_j^{\nu}\breve{A}_k^{\sigma}$, and substituting in (36) we have

$$\begin{aligned}SE_i^pE_j^qE_k^r=\sum_{\mu=0}^{p}\sum_{\nu=0}^{q}\sum_{\sigma=0}^{r}(-1)^{\mu+\nu+\sigma}\binom{p}{\mu}\binom{q}{\nu}\binom{r}{\sigma}SA_i^{p-\mu}A_j^{q-\nu}A_k^{r-\sigma}S\breve{A}_i^{\mu}\breve{A}_j^{\nu}\breve{A}_k^{\sigma},\\(i, j, k=1, 2, \ldots, n^2),\end{aligned} \tag{37}$$

where $A_i^0=A_j^0=A_k^0=I$. In particular

$$\begin{aligned}SE_iE_jE_k=n(SA_iA_jA_k-S\breve{A}_i\breve{A}_j\breve{A}_k)=nS(A_i, A_j)A_k,\\(i, j, k=1, 2, \ldots, n^2).\end{aligned} \tag{38}$$

The equations (32) and (33) enable us to establish certain relations between the roots of the characteristic equation,

$$|\omega-E_i|=\omega^{n^2}-\Psi_1\omega^{n^2-1}+\Psi_2\omega^{n^2-2}-\ldots\pm\Psi_{n^2-1}\omega\mp\Psi_{n^2}=0, \quad (\Psi_{n^2}=0),$$ [*]

of E_i and the roots of the characteristic equation,

$$|\rho-A_i|=\rho^n-P_1\rho^{n-1}+P_2\rho^{n-2}-\ldots\mp P_{n-1}\rho\pm P_n=0,$$

of A_i. Let $\omega_{11}, \omega_{12}, \ldots, \omega_{nn}$ denote the roots of the equation $|\omega-E_i|=0$, and $\mathscr{S}_q$ the sum of the *q-th* powers of the roots; let $\rho_1, \rho_2, \ldots, \rho_n$ denote the roots of $|\rho - A_i| = 0$, and s_q the sum of their *q-th* powers. By equation (32) $\mathscr{S}_{2p+1} = SE_i^{2p+1} = 0$, $(p=0, 1, 2, \dots)$, wherefore $\Psi_{2p+1} = 0$, $(p=0, 1, 2, \dots)$, and the non-zero roots of $|\omega - E_i| = 0$ occur in pairs of equal value but opposite sign;* from equation (33)

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[*] Killing has shown that the characteristic equation of the general infinitesimal transformation of the adjoint has at least one zero root (*Math. Ann.*, Vol. XXXI, p. 260).

$$\mathscr{S}_{2p} = \sum_{\mu=0}^{2p} (-1)^\mu \binom{2p}{\mu} s_{2p-\mu} s_\mu , \qquad (p=0, 1, 2, \dots),$$

from which relation it then follows that the non-zero roots of $|\omega - E_i| = 0$ not exceeding $n^2 - n$ in number, shall be related to the roots of $|\rho - A_i| = 0$ by the equation $\omega_{ij} = \rho_i - \rho_j$, $(i \neq j;\ i, j = 1, 2, \dots, n)$.†

For any linear function of the E's with constant coefficients, $\sum_{i=1}^{n^2} \alpha_i E_i$, we have

$$E = \sum_{i=1}^{n^2} \alpha_i E_i = \breve{T}^{-1}(M - N)\breve{T}, \tag{39}$$

where $M = \sum_{i=1}^{n^2} \alpha_i M_i$ and $N = \sum_{i=1}^{n^2} \alpha_i N_i$. If $A = \sum_{i=1}^{n^2} \alpha_i A_i$ we see from equation (14) that $(M)_{\mu\nu} = (A)_{\mu\nu} I$, $(\mu, \nu = 1, 2, \dots, n)$, and by the aid of equation (16) we can show that for any positive integer p, $(M^p)_{\mu\nu} = (A^p)_{\mu\nu} I$, $(\mu, \nu = 1, 2, \dots, n)$. It is evident from equation (15) that $(N)_{\mu\mu} = \breve{A}$ and $(N^p)_{\mu\mu} = \breve{A}^p$ for any positive integer p $(\mu = 1, 2, \dots, n)$ and that all other constituent matrices of these matrices are zero. Also corresponding to equation (22) we have $(M^p N^q)_{\mu\nu} = (A^p)_{\mu\nu} \breve{A}^q = (N^q M^p)_{\mu\nu}$, $(\mu, \nu = 1, 2, \dots, n)$ for any positive integers p and q. Hence

$$\left.\begin{aligned} SM^p &= \sum_{\mu=1}^{n} (A^p)_{\mu\mu} SI = nSA^p, \quad SN^p = \sum_{\mu=1}^{n} (N^p)_{\mu\mu} = nSA^p, \\ SM^p N^q &= \sum_{\mu=1}^{n} (A^p)_{\mu\mu} S\breve{A}^q = SA^p SA^q = SM^q N^p. \end{aligned}\right\} \tag{40}$$

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[*] Substituting these values of $\mathscr{S}_{2p+1}$ in Newton's formula,

$$\mathscr{S}_k - \Psi_1 \mathscr{S}_{k-1} + \Psi_2 \mathscr{S}_{k-2} - \dots + (-1)^{k-1} \Psi_{k-1} \mathscr{S}_1 + (-1)^k k \Psi_k = 0,$$

and considering only $k = 1, 3, 5, \dots$, we obtain $\Psi_{2p+1} = 0$ for $p = 0, 1, 2, \dots$.

[†] Let $\mathscr{S}_h$ be the sum of the *h-th* powers of the $n(n-1)$ non-zero roots $\rho_i - \rho_j$, $(i, j = 1, 2, \dots, n;\ i \neq j)$. Each root is paired with its negative and hence $\mathscr{S}_{2p+1} = 0$, $(p = 1, 2, \dots)$. Let

$$\phi(\rho) = \sum_{i=1}^{n} (\rho - \rho_i)^{2p} = \sum_{i=1}^{n} \sum_{\mu=0}^{2p} (-1)^\mu \binom{2p}{\mu} \rho^{2p-\mu} \rho_i^{\mu} = \sum_{\mu=0}^{2p} (-1)^\mu \binom{2p}{\mu} \rho^{2p-\mu} s_\mu, \quad (p = 1, 2, \dots).$$

Then $$\mathscr{S}_{2p} = \sum_{j=1}^{n} \phi(\rho_j) = \sum_{j=1}^{n} \sum_{\mu=0}^{2p} (-1)^\mu \binom{2p}{\mu} \rho_j^{2p-\mu} s_\mu = \sum_{\mu=0}^{2p} (-1)^\mu \binom{2p}{\mu} s_{2p-\mu} s_\mu , \quad (p = 1, 2, \dots).$$

From equation (39) we obtain

$$\begin{aligned} SE^{2p+1}&=S(\breve{T}^{-1}(M-N)\breve{T})^{2p+1}=S(M-N)^{2p+1}\\ &=\sum_{\mu=0}^{2p+1}(-1)^{\mu}\binom{2p+1}{\mu}SM^{2p-\mu+1}N^{\mu}\\ &=\sum_{\mu=0}^{p}(-1)^{\mu}\binom{2p+1}{\mu}(SM^{2p-\mu+1}N^{\mu}-SM^{\mu}N^{2p-\mu+1}),\quad (p=0,1,2,\ldots\ldots), \end{aligned}$$

and $SE^{2p}=S(M-N)^{2p}=\sum_{\mu=0}^{2p}(-1)^{\mu}\binom{2p}{\mu}SM^{2p-\mu}N^{\mu}$, $(p=1,2,\ldots\ldots)$, and substituting from equations (40) we have

$$SE^{2p+1}=0,\quad SE^{2p}=\sum_{\mu=0}^{2p}(-1)^{\mu}\binom{2p}{\mu}SA^{2p-\mu}SA^{\mu},\qquad (p=1,2,\ldots\ldots).\quad (41)$$

The characteristic equation of E, where $E=\sum_{i=1}^{n^2}\alpha_iE_i$ for an arbitrary choice of the α's, is termed by Killing the "characteristic equation" of the group G. The characteristic equation of $\sum_{i=1}^{r}\alpha_iE_i$, for an arbitrary choice of the α's, is termed by Cartan the "characteristic equation" of G relative to the subgroup G_r. It follows from equations (41) that, for $1\leqq r\leqq n^2$, the roots of $|\omega-\sum_{i=1}^{r}\alpha_iE_i|=0$ are the n^2 differences of the roots of the equation $|\rho-\sum_{i=1}^{r}\alpha_iA_i|=0$, and hence we have the following theorem:

The roots of the characteristic equation of the general linear homogeneous group relative to any given subgroup can be obtained by taking all possible differences of the roots of the characteristic equation of the general infinitesimal transformation of this subgroup of the general linear homogeneous group. The non-zero roots do not exceed n^2-n in number and occur in pairs of equal absolute value but opposite sign.

Section 2.

The special linear homogeneous group in n variables, denoted in what follows by G', is constituted by the aggregate of transformations with determinant $+1$ of the general linear homogeneous group, G, in n variables. I shall now assume that G_r is also a subgroup of the special linear homogeneous group, G'. For $1\leqq i\leqq r$ let $\rho_1,\rho_2,\ldots,\rho_n$ be the roots of the characteristic equation, $|\rho-A_i|=0$, of A_i. The finite transformations of G' are given by e^{tA_i} for all possible values of t. The roots of the characteristic equation of e^{tA_i} are $e^{t\rho_1}, e^{t\rho_2},\ldots, e^{t\rho_n}$; and since $e^{t(\rho_1+\rho_2+\cdots\cdots+\rho_n)}=|e^{tA_i}|=1$ for all values of t, $\rho_1+\rho_2+\cdots\cdots+\rho_n=0$. Therefore, $SA_i=0$, $(i=1,2,\ldots,r)$. Therefore, $S\sum_{i=1}^{r}a_iA_i=\sum_{i=1}^{r}a_iSA_i=0$.

We may take as the symbols of the infinitesimal transformations of G', any n^2-1 linearly independent linear homogeneous functions of the differential operators ε_{ij}, $(i, j=1, 2, \ldots, n;\ i\neq j)$, and $\varepsilon_{ii}-\varepsilon_{nn}$, $(i=1, 2, \ldots, n-1)$, or of the aggregate of respectively corresponding matrices. Thus we may take as the symbols of the infinitesimal transformations of G', $A_1, A_2, \ldots, A_r$ together with any n^2-r-1 other matrices linear in those above mentioned, the n^2-1 A's defined by equation (1), being linearly independent and subject to the condition $\sum\limits_{\mu=1}^{n} a_{i\mu\mu}=SA_i=0$, $(i=1, 2, \ldots, n^2-1)$. For the symbols of the infinitesimal transformations of G we may take the n^2-1 linearly independent linear homogeneous functions of ε_{ij}, $(i, j=1, 2, \ldots, n;\ i\neq j)$, and $\varepsilon_{ii}-\varepsilon_{nn}$, $(i=1, 2, \ldots, n-1)$, which constitute the symbols of the infinitesimal transformations of G' together with the operator $\sum\limits_{\mu=1}^{n}\varepsilon_{\mu\mu}$, or the aggregate of matrices corresponding to these; and therefore, we may choose as the aggregate of matrices representing the infinitesimal transformations of G the matrices $A_1, A_2, \ldots, A_{n^2-1}$ and $A_{n^2}=\sum\limits_{\mu=1}^{n} e_{\mu\mu}=I$.

Since G' is an invariant subgroup of G, we have

$$(A_i, A_j)=\sum_{k=1}^{n^2-1} c_{ijk}A_k, \qquad (i, j=1, 2, \ldots, n^2-1);$$

moreover, $(A_i, A_{n^2})=0$, $(i=1, 2, \ldots, n^2)$, and therefore,

$$c_{ijk}=0, \qquad (i, j \text{ or } k=n^2). \tag{42}$$

Let Γ' denote the adjoint group of the special linear homogeneous group and let the symbols of the infinitesimal transformations be the differential operators $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{n^2-1}$ where

$$\mathcal{H}_i=\sum_{j=1}^{n^2-1} c_{ijk}\alpha_j\frac{\partial}{\partial\alpha_k}, \qquad (i=1, 2, \ldots, n^2-1),$$

or the respectively corresponding matrices $H_1, H_2, \ldots, H_{n^2-1}$ where

$$(H_i)_{\nu\mu}=c_{i\mu\nu}, \qquad (\mu, \nu, i=1, 2, \ldots, n^2-1). \tag{43}$$

With the infinitesimal transformations of G chosen as above, we see from (4), (42) and (43) that the aggregate of matrices representing the infinitesimal transformations of Γ have the following form:

$$E_i=\begin{pmatrix} H_i & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{n^2}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad (i=1, 2, \ldots, n^2-1). \tag{44}$$

Then since Γ and Γ' have the same structure as G and G', respectively,

$$(H_i, H_j) = \sum_{k=1}^{n^2-1} c_{ijk} H_k, \quad (E_i, E_j) = \sum_{k=1}^{n^2-1} c_{ijk} E_k, \qquad (i, j = 1, 2, \ldots, n^2-1),$$

$$(E_i, E_{n^2}) = 0, \qquad (i = 1, 2, \ldots, n^2).$$

By the aid of equation (44) we obtain the relations:

$$SE_i = SH_i, \quad SE_iE_j = SH_iH_j, \qquad (i, j = 1, 2, \ldots, n^2-1), \tag{45}$$

and for any natural number p

$$SE_i^p = SH_i^p, \qquad (i = 1, 2, \ldots, n^2-1). \tag{46}$$

It now follows from equations (28) and (45) that

$$SH_i = 0, \qquad (i = 1, 2, \ldots, n^2-1);$$

and since $SA_i = 0$, $(i = 1, 2, \ldots, n^2-1)$, we obtain from equations (30) and (45)

$$SH_iH_j = 2nSA_iA_j, \qquad (i, j = 1, 2, \ldots, n^2-1). \tag{47}$$

Further, we obtain for $p = 0, 1, 2, \ldots$ from equations (32), (33) and (46),

$$SH_i^{2p+1} = 0, \quad SH_i^{2p} = \sum_{\mu=0}^{2p} (-1)^\mu \binom{2p}{\mu} SA_i^{2p-\mu} SA_i^\mu, \qquad (i = 1, 2, \ldots, n^2-1). \tag{48}$$

We can thereby show that the non-zero roots of the characteristic equation of H_i, $|\omega - H_i| = 0$, occur in pairs of equal value but opposite sign which can be obtained by taking all possible differences of the roots of the equation $|\rho - A_i| = 0$ for $1 \leq i \leq n^2-1$.

From equation (44) it follows that for any linear function of the H's with constant coefficients, $H = \sum_{i=1}^{n^2-1} \alpha_i H_i$,

$$\sum_{i=1}^{n^2-1} \alpha_i E_i = \begin{pmatrix} H, & 0 \\ 0, & 0 \end{pmatrix}. \quad \text{Hence } \Big(\sum_{i=1}^{n^2-1} \alpha_i E_i\Big)^p = \begin{pmatrix} H^p, & 0 \\ 0, & 0 \end{pmatrix} \text{ and } S\Big(\sum_{i=1}^{n^2-1} \alpha_i E_i\Big)^p = SH^p$$

for any positive integer p. From this result and equations (41) we see that $SH^{2p+1} = 0$, $SH^{2p} = \sum_{\mu=0}^{2p} (-1)^\mu \binom{2p}{\mu} SA^{2p-\mu} SA^\mu$, $(p = 1, 2, \ldots)$, where $A = \sum_{i=1}^{n^2-1} \alpha_i A_i$. Therefore, the roots of $|\omega - \sum_{i=1}^{n^2-1} \alpha_i H_i| = 0$ are related to the roots of $|\rho - \sum_{i=1}^{n^2-1} \alpha_i A_i| = 0$ in the way described above. For an arbitrary choice of the α's $|\omega - \sum_{i=1}^{r} \alpha_i H_i| = 0$ is the characteristic equation of the special linear homogeneous group relative to the subgroup G_r—when $r = n^2-1$, it is the characteristic equation of the special linear homogeneous group—and $\sum_{i=1}^{r} \alpha_i A_i$ is the

An important classification of finite continuous groups due to Lie consists in the division into integrable and non-integrable groups. An integrable group is a group some one of whose derived groups contains only the identical transformation ("Continuierliche Gruppen," p. 58). Finite continuous groups are also classified as "simple" and "non-simple" groups. A "simple" group is a group that contains no invariant subgroup. A group is "semi-simple" if no invariant subgroup is integrable (Cartan, *Thèses*, p. 51).

We have the following theorems due to Lie, Engel, Killing and Cartan:

(i). Groups of rank zero are integrable and the derived groups of an integrable group are of rank zero (Umlauf, *These*, Leipzig).

(ii). If the first derived of a group is integrable, then the group itself is integrable (Lie, "Continuierliche Gruppen," p. 547).

(iii). The necessary and sufficient condition that a group of order r be integrable is that the transformations of its first derived annul identically the coefficient $\Psi_2(\alpha)$ in its characteristic equation (Cartan, *Thèses*, p. 47).

(iv). If an infinitesimal transformation annuls $\Psi_{r-1}(\alpha)$ without annulling all the first minors of $\sum_{i=1}^{r}\alpha_i E_i$, it annuls all the invariants of the adjoint which are homogeneous and of degree non-zero (Cartan, *Thèses*, p. 31).

(v). The characteristic equation of a simple or semi-simple group with r parameters and of rank k has $r-k$ non-zero unequal roots which occur in pairs of equal value but opposite sign, and the transformations belonging to such a pair of roots are not commutable (Killing, *Math. Ann.*, Vol. XXXIV, p. 102).

Let $\rho_1, \rho_2, \ldots, \rho_n$ be the roots of the characteristic equation $|\rho-\sum_{i=1}^{r}\alpha_i A_i|=0$ of the general infinitesimal transformation of G_r. It was shown in § 1 and § 2 that the roots of the characteristic equation $|\omega-\sum_{i=1}^{r}\alpha_i E_i|=0$ of G' (or G) relative to G_r are given by $\rho_i-\rho_j$ for $i, j=1, 2, \ldots, n$. Whence it follows by (49) that the roots of the characteristic equation $|\omega-\sum_{i=1}^{r}\alpha_i E'_i|=0$ of G_r are numbers contained in the aggregate of differences of the roots of the characteristic equation $|\rho-\sum_{i=1}^{r}\alpha_i A_i|=0$ of the general infinitesimal transformation of G_r.

I shall denote by l in what follows the maximum number of distinct roots of the equation $|\rho-\sum_{i=1}^{r}\alpha_i A_i|=0$ for any infinitesimal transformation of G_r.

Let $\rho_1, \rho_2, \ldots, \rho_l$ be the distinct roots of the equation $|\rho-\sum_{i=1}^{r}\alpha_i A_i|=0$ relative to the general infinitesimal transformation of G_r. The distinct non-zero roots of $|\omega-\sum_{i=1}^{r}\alpha_i E_i|=0$ corresponding to the general infinitesimal transformation of G' (or G) relative to the subgroup G_r and, therefore, the distinct non-zero roots of $|\omega-\sum_{i=1}^{r}\alpha_i E'_i|=0$ relative to the general infinitesimal transformation of G_r are contained in the aggregate of l^2-l functions $\rho_i-\rho_j$ of the α's for $i, j=1, 2, \ldots, l$ and $j \neq i$. But these l^2-l functions of the α's are themselves functions of the $l-1$ differences $\rho_i-\rho_l$, $(i=1, 2, \ldots, l-1)$. Therefore, the non-zero roots of $|\omega-\sum_{i=1}^{r}\alpha_i E'_i|=0$ relative to the general infinitesimal transformation of G_r are functions of at most $l-1$ functions of the α's, and thus at most $l-1$ of the roots of the equation $|\omega-\sum_{i=1}^{r}\alpha_i E_i|=0$ are independent functions of the α's. We have, therefore, the following theorems:

I. *If the maximum number of distinct roots of the characteristic equation of the general infinitesimal transformation of* G_r *is* l, *the rank of* G_r *can not exceed* $l-1$.

II. *If the roots of the characteristic equation of the general infinitesimal transformation of* G_r *are all equal the rank of* G_r *is zero; therefore,* G_r *is integrable.*

III. *If the roots of the characteristic equation of the general infinitesimal transformation of the first or any subsequent derived group of* G_r *are all equal, the rank of the derived group is zero; therefore,* G_r *is integrable.*

The roots of the characteristic equation of any infinitesimal transformation of the following group

$$A_1=x_3\frac{\partial}{\partial x_1},\quad A_2=x_2\frac{\partial}{\partial x_1},\quad A_3=x_3\frac{\partial}{\partial x_2},\quad A_4=x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}$$

are all equal; this group illustrates theorems II and III, for the characteristic equation of the general infinitesimal transformation is

$$|\rho-\sum_{i=1}^{4}\alpha_i A_i| = (\rho-\alpha_4)^3=0.$$

The first derived of this group is $A_1=x_3\frac{\partial}{\partial x_1}$. The group contains two exceptional infinitesimal transformations, viz.,

$$A_1=x_3\frac{\partial}{\partial x_1},\quad A_4=x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}.$$

From what was stated above it is clear that if $l' \leqq l$ is the number of independent roots of the equation $|\rho - \sum_{i=1}^{r} \alpha_i A_i| = 0$, or the number of coefficients of this equation which are independent functions of the α's, the number of independent roots of the equation $|\omega - \sum_{i=1}^{r} \alpha_i E'_i| = 0$ can not exceed l'. Whence it follows that:

IV. *If l' is the number of independent coefficients of the characteristic equation*

$$|\rho - \sum_{i=1}^{r} \alpha_i A_i| = \rho^n - P_1 \rho^{n-1} + \ldots \mp P_{n-1}\rho \pm P_n = 0,$$

of the general infinitesimal transformation G_r, the rank of G_r can not exceed l'.

The maximum number of distinct roots of the equation $|\rho - \sum_{i=1}^{r} \alpha_i A_i| = 0$ for any group G_r in n variables is n. Therefore, from *I* follows the theorem:

V. *The rank of no subgroup of the general linear homogeneous group in n variables can exceed $n-1$.*

By the preceding theorem the rank of the special linear homogeneous group, G', can not exceed $n-1$. Suppose the rank of G' to be $k < n-1$. G' is a simple group (*Transformationsgruppen*, Vol. I, p. 560); and hence by theorem (v) the number of non-zero roots of the characteristic equation of G' is $n^2 - k - 1$. If $k < n-1$ then $n^2 - k - 1 > n^2 - n$ or the number of non-zero roots exceeds $n^2 - n$. But in § 2 it was shown that the number of non-zero roots does not exceed $n^2 - n$, hence $k \nless n-1$, and we must have $k = n-1$.

Killing has shown that the number of independent invariants of the transformations of the adjoint of any group can not be less than the rank of the group (*Math. Ann.*, Vol. XXXI, p. 266). Thus the number of independent invariants of the adjoint of G' can not be less than $n-1$. Moreover, the number of invariants of the adjoint of any group G_r is the same as the nullity of the matrix $\sum_{i=1}^{r} \alpha_i E'_i$ which can not exceed the number of zero roots of the characteristic equation $|\omega - \sum_{i=1}^{r} \alpha_i E_i| = 0$. Since the rank of G' is $n-1$ it follows that the characteristic equation of G' has exactly $n-1$ roots zero and therefore the number of invariants can not exceed $n-1$ and must, therefore, be exactly $n-1$. Hence:

VI. *The rank of the special linear homogeneous group in n variables is $n-1$, and is the same as the number of independent invariants of the transformations of its adjoint group.*

From equations (44) it follows that the characteristic equation,

$$|\omega-\sum_{i=1}^{n^2}\alpha_i E_i|=\omega|\omega-\sum_{i=1}^{n_2-1}\alpha_i H_i|=0,$$

of the general linear homogeneous group, G, has the same number of independent coefficients as the characteristic equation $|\omega-\sum_{i=1}^{n^2-1}\alpha_i H_i|=0$ of G'. Thus the rank of G is equal to the rank of G' and is $n-1$. Also since the nullity of $\sum_{i=1}^{n^2-1}\alpha_i H_i$, a matrix of order n^2-1, is $n-1$ it is evident that the nullity of $\sum_{i=1}^{n^2}\alpha_i E_i=\begin{pmatrix}\sum_{i=1}^{n^2-1}\alpha_i H_i, & 0\\ & 0, & 0\end{pmatrix}$, a matrix of order n^2, is exactly n. We have then:

VII. *The rank of the general linear homogeneous group in n variables is* $n-1$ *and the number of independent invariants of the transformations of its adjoint group is n.*

Let the l distinct roots of $|\rho-\sum_{i=1}^{r}\alpha_i A_i|=0$ be respectively of multiplicity $\mu_1, \mu_2, \ldots, \mu_l$. The equation $|\omega-\sum_{i=1}^{r}\alpha_i E_i|=0$ has then at least $\mu_1^2+\mu_2^2+\ldots+\mu_l^2$ zero roots. If this equation has as many as n^2-r+2 zero roots it follows from (49) that $\Psi_{r-1}(\alpha)=0$. Therefore, $\Psi_{r-1}(\alpha)=0$ if $\mu_1^2+\mu_2^2+\ldots+\mu_l^2\geqq n^2-r+2$. But Cartan has shown that, when the coefficient $\Psi_{r-1}(\alpha)$ of the characteristic equation of G_r vanishes identically, every transformation of G_r is commutative with at least one other transformation of G_r (*Thèses*, p. 32). Whence we have the following theorem:

VIII. *Let G_r be a subgroup with r parameters of the general linear homogeneous group in n variables; and let the l distinct roots of the characteristic equation of the general infinitesimal transformation of G_r be of multiplicity $\mu_1, \mu_2, \ldots, \mu_l$. If $\mu_1^2+\mu_2^2+\ldots+\mu_l^2\geqq n^2-r+2$, then every infinitesimal transformation of G_r is commutative with at least one other infinitesimal transformation of this group.*

The group

$$A_1=x_1\frac{\partial}{\partial x_1},\quad A_2=x_1\frac{\partial}{\partial x_2},\quad A_3=x_1\frac{\partial}{\partial x_3},\quad A_4=x_1\frac{\partial}{\partial x_4},\quad A_5=x_3\frac{\partial}{\partial x_2},$$

$$A_6=x_3\frac{\partial}{\partial x_4},\quad A_7=x_4\frac{\partial}{\partial x_2},\quad A_8=x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}+x_4\frac{\partial}{\partial x_4},$$

illustrates the above theorem. The characteristic equation of the general infinitesimal transformation is $(\rho-\alpha_1)(\rho-\alpha_4)^3=0$. Thus $\mu_1^2+\mu_2^2=10=n^2-r+2$.

This group contains no exceptional transformation but $(A_1, A_8)=(A_2, A_7)=(A_3, A_4)=(A_5, A_6)=0$. The group given on page 446 also illustrates this theorem. In this case we have $\mu_1^2=9>7=n^2-r+2$, and the group contains two exceptional transformations.

By theorem (v) the characteristic equation of a simple or semi-simple group with r parameters and of rank k has $r-k$ distinct non-zero roots. I have shown that the number of distinct non-zero roots of the characteristic equation of G_r can not exceed l^2-l, and that the rank of G_r can not exceed $l-1$ or exceed l'. Suppose G is a simple group of rank k, then from Killing's theorem and what has just been stated $k\leqq l-1$, $k\leqq l'$ and $r-k\leqq l^2-l$. Therefore, $r\leqq l^2-1$ and $r\leqq l^2-l+l'$. We have then the following theorem:

IX. *Let G_r be a subgroup with r parameters of the general linear homogeneous group; let l be the maximum number of distinct roots of the characteristic equation of the general infinitesimal transformation of G_r and let l' be the number of these roots which are independent. If either l^2-1 or l^2-l+l' is less than r then the group can be neither simple nor semi-simple.*

The example given on page 448 illustrates this theorem for $l^2-1=3<8=r$ and the group contains the invariant subgroup A_2, A_3, A_4, A_5, and by III is integrable.

Since $\Psi_r(\alpha)=|\sum_{i=1}^{r}\alpha_i E'_i|$ is identically zero, the nullity of the matrix $\sum_{i=1}^{r}\alpha_i E'_i$ is at least 1. For a given infinitesimal transformation $\sum_{i=1}^{r}\alpha_i A_i$ of G_r, let the nullity of $\sum_{i=1}^{r}\alpha_i E'_i$ be unity, and let the equation $|\omega-\sum_{i=1}^{r}\alpha_i E'_i|=0$ have two roots zero. Then, for the given infinitesimal transformation $\sum_{i=1}^{r}\alpha_i A_i$ of G_r, $\Psi_{r-1}(\alpha)$ is annulled, but not all the first minors of $\sum_{i=1}^{r}\alpha_i E'_i$, and hence by theorem (iv) all the invariants of the adjoint, homogeneous and of degree non-zero, are annulled by the given infinitesimal transformation. Let $\mu_1, \mu_2, \ldots, \mu_l$ be the multiplicities of the distinct roots of the characteristic equation of $\sum_{i=1}^{r}\alpha_i A_i$. From (49) it follows that $\Psi_{r-1}(\alpha)=0$ if $|\omega-\sum_{i=1}^{r}\alpha_i E_i|=0$ has at least n^2-r+2 zero roots, which will be the case if $\sum_{i=1}^{l}\mu_i^2\geqq n^2-r+2$ (cf. p. 448); and the nullity of $\sum_{i=1}^{r}\alpha_i E'_i$ will not exceed unity if the nullity of $\sum_{i=1}^{r}\alpha_i E_i$ does not exceed n^2-r+1.

From (13) it follows that the nullity of $\sum_{i=1}^{r} \alpha_i E_i$ is equal to the nullity of $\sum_{i=1}^{r} \alpha_i (M_i - N_i)$. We have then the following theorem:

X. *Let* G_r *be a subgroup with* r *parameters of the general linear homogeneous group in* n *variables. Let the* $l \leqq n$ *distinct roots of the characteristic equation of any given infinitesimal transformation* $\sum_{i=1}^{r} \alpha_i A_i$ *of* G_r *be respectively of multiplicity* $\mu_1, \mu_2, \ldots, \mu_l$. *If* $\sum_{i=1}^{l} \mu_i^2 \geqq n^2 - r + 2$ *and the nullity of* $\sum_{i=1}^{r} \alpha_i (M_i - N_i)$ *(see p. 437) does not exceed* $n^2 - r + 1$, *the invariants of the adjoint, homogeneous and of degree non-zero, are then all annulled for this infinitesimal transformation.*